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THE MOMENT-DISTRIBUTION METHOD FOR STATICALLY INDETERMINATE BEAMS USING THREE DIFFERENT MODELS

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Abstract. *This paper presents the Moment-Distribution Method (method of successive approximations) for statically indeterminate beams using three different models: the Model 1 considers the bending deformations and shear in the methodology, and also in the fixed-end moments, and this is the innovative part of this paper; the Model 2 takes into account the bending deformations and shear in the methodology, and in the fixed-end moments considers only the bending deformations; the Model 3 is the classical model called Hardy Cross method, which considers the bending deformations in the methodology and also in the fixed-end moments. Also a comparison is made between the three models through three different problems for lengths of 5.00, 7.50, 10.00 m to observe differences. The results show the differences between the three models, and when members tend to be shorter, the differences are increased in proposed model (Model 1) with respect to the other two models. Therefore, for the normal practice of using the Model 2 and Model 3, these are not a recommended solution, when lengths are short between supports. Then, the Model 1 (proposed model) passes to be the more appropriate model for structural analysis of continuous beams using the Moment-Distribution Method (method of successive approximations) and also is adjusted to the real conditions, since the shear forces and moments are presented in any analysis of continuous beams, and the bending deformations and shear appear.*
Keywords: Method of successive approximations, Statically indeterminate beams, Three different models, Bending deformations and shear

1. **Introduction.** Structural analysis is the study of structures such as discrete systems. The structures theory is based essentially on the mechanics fundamentals with which are formulated the different structural members. The laws or rules that define the balance and continuity of a structure can be expressed in different ways, including partial differential equations of three-dimensional continuous mediums, ordinary differential equations that define a member or the various theories of beams, or simply, algebraic equations for a discretized structure [1].

Structural analysis can be addressed using three main approaches: a) Tensor formulation (Newtonian mechanics and vector), b) Formulation based on the principles of virtual work, c) Formulation based on the classical mechanics [2].

In the design of steel structures, reinforced concrete and prestressed concrete, the study of the structural analysis is a crucial stage in its design, since the axial forces, shear forces and bending moments are those who govern the design of rigid frames and for the case of beams only the shear forces and the bending moments.

Structural systems analysis has been studied by diverse researchers in the past. Make a brief historical review of progress in this subject.

Benoit Paul Emile Clapeyron in 1857 presented to the French Academy his "Theorem of Three Moments" for analysis of continuous beams, and in the same way Bertot had published two years ago in the Memories of the Society of Civil Engineers of France, but without giving credit. It can be said that from this time begin the development true of "Theory of Structures" [3,4].

French Engineer Jacques Antoine Charles Bresse in 1854 published his book "Recherches Analytiques sur la Flexion et la Résistance de Pieces Courbés" in which he presented practical methods for the analysis of curved beams and arcs [3,4].

In 1867 the "Influence Line" was introduced by the German Emil Winkler (1835-1888). He also made important contributions to the Resistance of materials, especially in the bending theory of curved beams, bending of beams resting on elastic medium [3,4].

James Clerk Maxwell (1830-1879) University of Cambridge published what might be called the first systematic method of analysis for statically indeterminate structures based on the equality of the internal energy of deformation of a loaded structure and the external work done by applied loads, and equality had been established by Clapeyron. In his analysis, it presents the Theorem of the Reciprocal Deformations, which by in its brevity was not appreciated at the time. Another publication later presented his diagram of internal forces to trusses, which combine in a single figure all the polygons of forces. The diagram was extended by Cremona, what is known as the Maxwell-Cremona diagram [3,4].

The Italian Betti in 1872 published a generalized form, the Maxwell's theorem, known as the reciprocal theorem of Maxwell-Betti [3,4].

The German Otto Mohr (1835-1918) made great contributions to the Structures Theory. He developed the method for determining the deflections in beams, known as the elastic loads method or the conjugate beam. He also presented a simple derivation and more extensive with the general method of Maxwell for indeterminate structures analysis using the principles of virtual work. He made contributions in the graphical analysis of deflections in trusses, complemented by Williot diagram, known as the Mohr-Williot diagram of great practical utility. He also earned his famous Mohr Circle for the graphical representation of the stresses in a biaxial stress state [3,4].

Alberto Castigliano (1847-1884) in 1873 introduced the principle of the minimum work, which had been previously suggested by Menabrea, and is known as the First Theorem of Castigliano. Later, it presented the second Theorem of Castigliano to find deflections, as a corollary of the first. Published in Paris in 1879 his famous book "Thèoreme de l'Equilibre de Systèmes Elastiques et ses Applications", remarkable by its originality and very important in the development of the hyperstatic structures analysis [3,4].

Heinrich Müller-Breslau (1851-1925), published in 1886 a basic method for indeterminate structures analysis, but was essentially a variation of those presented by Maxwell and Mohr. He gave great importance to Maxwell's Theorem, of the Reciprocal Deflections in the assessment of displacements. He discovered that the "influence line" for the reaction or an inner strength of a structure was, on some scale, the elastic curve produced by an action similar to that reaction, or inner strength. Known as the theorem's Müller-Breslau is the basis for other indirect methods of structural analysis using models [3,4].

Hardy Cross (1885-1959), professor at the University of Illinois, published in 1930 his famous moments-distribution method, which can be said that revolutionized the structures analysis of reinforced concrete for continuous frames and can be considered one of the greatest contributions for indeterminate structures analysis. This successive approximations method evades solution of equations systems, as presented in the Mohr and Maxwell methods. Method's popularity is declined with the availability of computers, in which resolution of equations systems is not a problem. The general concepts of the method were extended in the study of pipes flow. Later the methods of Kani and Takabeya became in the most popular, also of iterative type [3,4].

In the early 50's, Turner, Clough, Martin and Topp present the stiffness matrix method, that may be called as the beginning of the application to structures, which have gained popularity today. Subsequently, the finite element methods are developed, which have allowed the systematic analysis of large numbers of structures and obtain the forces and deformations in complex systems, such as concrete dams used in hydroelectric plants. Among its authors are: Clough, Wilson, Zienkiewics and Gallagher [5].

Recently, a method of structural analysis for statically indeterminate beams and rigid frames was developed, and the method takes into account the bending deformations and shear to generate a system of equations in function of rotations and displacements [6-8]. Also a moments-distribution method considering the bending deformations and shear was presented [9]. These methods do not consider shear deformations in the fixed-end moments.

Later, a mathematical model is presented to obtain the fixed-end moments of a beam subjected to a uniformly distributed load and also to a triangularly distributed load taking into account the bending deformations and shear [10,11].

As regards the conventional techniques of structural analysis of continuous beams, the common practice is to neglect the shear deformations [12,13].

This paper presents the Moment-Distribution Method (method of successive approximations) for statically indeterminate beams using three different models: the Model 1 considers the bending deformations and shear in the methodology, and also in the fixed-end moments, and this is the innovative part of this paper; the Model 2 takes into account the bending deformations and shear in the methodology, and in the fixed-end moments considers only the bending deformations; the Model 3 is the classical model called Hardy Cross method, which considers the bending deformations in the methodology and also in the fixed-end moments. Also a comparison is made between the three models through three different problems for lengths of 5.00, 7.50, 10.00 m to observe differences.

2. Methodology.

2.1. Model 1 (proposed model).

This model considers the bending deformations and shear in methodology and in the fixed-end moments.

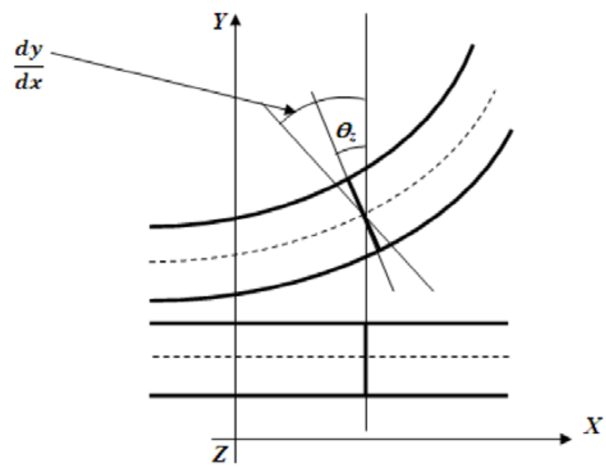


FIGURE 1. Deformation of a beam member

2.1.1. *Theoretical principles.* A deformed structure member is presented in Figure 1, and shows the difference between the Timoshenko theory and Euler-Bernoulli theory: the first "θ_z" and "dy/dx" do not coincide necessarily, while in the second are equal [6-11].

The main difference between the Euler-Bernoulli theory and Timoshenko theory is that in the first the relative rotation of the section is approximated by the derivative of vertical displacement, this is an approximation valid only for long members in relation to the dimensions of cross section, and then it happens that due to shear deformations are negligible in comparison with the deformations caused by moment. On the Timoshenko theory, which considers the deformation due to shear, i.e., and is valid for the short members and long, the equations of the elastic curve are given by the complex system of equations:

$$G\left(\frac{dy}{dx}-\theta_z\right)=\frac{V_y}{A_s} \tag{1}$$

$$E\left(\frac{d\theta_z}{dx}\right)=\frac{M_z}{I_z} \tag{2}$$

where: G = shear modulus, dy/dx = total rotation around axis "Z", θ_z = rotation around axis "Z", due to the bending, V_y = shear force in direction "Y", A_s = shear area, $d\theta_z/dx = d^2y/dx^2$, E = modulus of elasticity, M_z = bending moment around axis "Z", I_z = moment of inertia around axis "Z".

Deriving Equation (1) and substituting into Equation (2), it is arrived at the equation of the elastic curve including the effect of shear stress:

$$\frac{d^2y}{dx^2}=\frac{1}{GA_s}\frac{dV_y}{dx}+\frac{M_z}{EI_z} \tag{3}$$

Equation (3) is integrated to obtain the rotation in any point:

$$\frac{dy}{dx}=\frac{V_y}{GA_s}+\int\frac{M_z}{EI_z}dx \tag{4}$$

2.1.2. *General description of moment-distribution method.* The moment-distribution method can be used to analyze all types of statically indeterminate beams. Essentially it consists in solving the simultaneous equations in the slope-deflection method by successive approximations. In order to develop the method, it will be helpful to consider the following problem: if a clockwise moment of "M_{AB}" is applied at the simple support of a straight member of constant cross section simply supported at one end and fixed at the other to find the rotation "θ_A" at the simple support and the moment "M_{BA}" at the fixed end, as shown in Figure 2.

The additional end moments, "M_{AB}" and "M_{BA}", should be such as to cause rotations of "θ_A" and "θ_B", respectively. If "θ_{A1}" and "θ_{B1}" are the rotations caused by "M_{AB}", according to Figure 2(b), and "θ_{A2}" and "θ_{B2}" by "M_{BA}", these are observed in Figure 2(c).

The conditions required of geometry are [6-11,14-16]:

$$\theta_A=\theta_{A1}-\theta_{A2} \tag{5}$$

$$0=\theta_{B1}-\theta_{B2} \tag{6}$$

The beam of Figure 2(b) is analyzed to find "θ_{A1}" and "θ_{B1}" in function of "M'_{AB}":

It is considered that "$V_A = V_B$", the sum of moments in B is realized to find value of "M_{AB}" in function of "V_A":

$$M_{AB}=V_AL \tag{7}$$

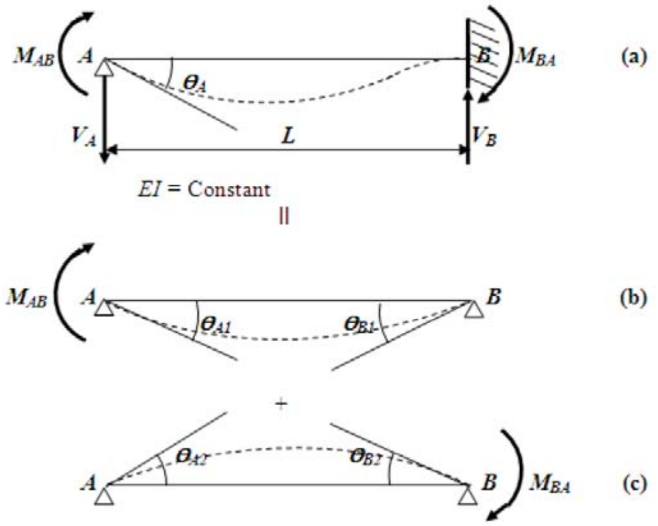


FIGURE 2. Derivation of moment-distribution equations

Therefore, the shear forces and moments at a distance "x" are:

$$V_x = \frac{M_{AB}}{L} \tag{8}$$

$$M_x = \frac{M_{AB}}{L}(L - x) \tag{9}$$

where: L = beam length, "$V_x = V_y$" and "$M_x = M_z$".

Substituting "M_x" and "V_x" in function of "M'_{AB}" in Equation (4), and is separated the shear deformation and bending to obtain the stiffness, is presented as follows:

- Shear deformation:

$$\frac{dy}{dx} = \frac{M'_{AB}}{GA_sL} \tag{10}$$

The integral of Equation (10) is presented as follows:

$$y = \frac{M'_{AB}}{GA_sL}x + C_1 \tag{11}$$

The boundary conditions are substituted into Equation (11), when $x = 0$ and $y = 0$ to obtain $C_1 = 0$. Then Equation (11) is presented:

$$y = \frac{M'_{AB}}{GA_sL}x \tag{12}$$

- Bending deformation:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_zL}\int (L - x)dx \tag{13}$$

The integral of Equation (13) is developed:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_zL}\left(Lx - \frac{x^2}{2} + C_2\right) \tag{14}$$

Subsequently Equation (14) is integrated to obtain displacement in the beam:

$$y = \frac{M'_{AB}}{EI_zL}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_2x + C_3\right) \tag{15}$$

The boundary conditions are substituted into Equation (15), when $x = 0$ and $y = 0$ to find $C_3 = 0$. Then Equation (15) is shown:

$$y = \frac{M'_{AB}}{EI_z L}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_2 x\right) \tag{16}$$

Now the boundary conditions are substituted into Equation (16), when $x = 0$ and $y = L$ to obtain $C_2 = -L^2/3$. Then Equations (14) and (16) are presented:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_z L}\left(Lx - \frac{x^2}{2} - \frac{L^2}{3}\right) \tag{17}$$

$$y = \frac{M'_{AB}}{EI_z L}\left(\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^2}{3}x\right) \tag{18}$$

Substituting $x = 0$ into Equation (17) to find the rotation in support "A" due to bending deformation "θ_{A1b}", we give:

$$\theta_{A1b} = -\frac{M'_{AB}L}{3EI_z} \tag{19}$$

Now substituting $x = L$ in Equation (17) to obtain the rotation in support "B" due to bending deformation "θ_{B1b}", we find:

$$\theta_{B1b} = \frac{M'_{AB}L}{6EI_z} \tag{20}$$

The rotations are positive when the curvature radius is found in the following part of beam, and negative when the curvature radius is presented in the top of beam. The rotations are:

$$\theta_{A1b} = +\frac{M'_{AB}L}{3EI_z} \tag{21}$$

$$\theta_{B1b} = +\frac{M'_{AB}L}{6EI_z} \tag{22}$$

The rotation due to shear deformation "θ_{A1s}" and "θ_{B1s}" taking into account the curvature radius is:

$$\theta_{A1s} = \frac{dy}{dx} = \frac{M'_{AB}}{GA_s L} \tag{23}$$

$$\theta_{B1s} = \frac{dy}{dx} = -\frac{M'_{AB}}{GA_s L} \tag{24}$$

The sum of the shear rotation and bending rotation in the support "A" is obtained:

$$\theta_{A1} = \theta_{A1b} + \theta_{A1s} \tag{25}$$

Equations (21) and (23) are substituted into Equation (25):

$$\theta_{A1} = +\frac{M'_{AB}L}{3EI_z} + \frac{M'_{AB}}{GA_s L} \tag{26}$$

The common factor of the moment "M'_{AB}" is obtained of Equation (26), this is as follows:

$$\theta_{A1} = \frac{M_{AB}L}{12EI_z}\left(4 + \frac{12EI_z}{GA_s L^2}\right) \tag{27}$$

and $\varnothing$ is [6-11,17]:

$$\varnothing = \frac{12EI_z}{GA_s L^2} \tag{28}$$

The value of "G" is obtained as follows:

$$G = \frac{E}{2(1+v)} \tag{29}$$

where: $\varnothing$ is form factor, ν is Poisson's ratio.

Substituting Equation (28) into Equation (27) is obtained:

$$\theta_{A1} = \frac{M'_{AB}L}{12EI_z}(4+\varnothing) \tag{30}$$

The sum of the shear rotation and bending rotation in the support "B" is presented:

$$\theta_{B1} = \theta_{B1b} + \theta_{B1s} \tag{31}$$

Equations (22) and (24) are substituted into Equation (31):

$$\theta_{B1} = +\frac{M'_{AB}L}{6EI_z} - \frac{M'_{AB}}{GA_sL} \tag{32}$$

The common factor of the moment "M'_{AB}" is obtained of Equation (32), and this is as follows:

$$\theta_{B1} = \frac{M'_{AB}L}{12EI_z}\left(2 - \frac{12EI_z}{GA_sL^2}\right) \tag{33}$$

Substituting Equation (28) into Equation (33) is presented:

$$\theta_{B1} = \frac{M'_{AB}L}{12EI_z}(2-\varnothing) \tag{34}$$

Analyzing the beam of Figure 2(c) to find "θ_{A2}" and "θ_{B2}" in function of "M'_{BA}" of the same way as was done in Figure 2(b), we obtain the following:

$$\theta_{A2} = \frac{M'_{BA}L}{12EI_z}(2-\varnothing) \tag{35}$$

$$\theta_{B2} = \frac{M'_{BA}L}{12EI_z}(4+\varnothing) \tag{36}$$

Now, substituting Equations (34) and (36) into Equation (6), it is as follows:

$$0 = \frac{M_{AB}L}{12EI_z}(2-\varnothing) - \frac{M_{BA}L}{12EI_z}(4+\varnothing) \tag{37}$$

The moment "M_{BA}" is found in function of "M_{AB}":

$$M_{BA} = \left(\frac{2-\varnothing}{4+\varnothing}\right)M_{AB} \tag{38}$$

Also, Equations (30) and (35) are substituted into Equation (5):

$$\theta_A = \frac{M_{AB}L}{12EI_z}(4+\varnothing) - \frac{M_{BA}L}{12EI_z}(2-\varnothing) \tag{39}$$

Substituting Equation (38) into Equation (39), it is obtained:

$$\theta_A = \frac{M_{AB}L}{12EI_z}(4+\varnothing) - \left\{\left(\frac{2-\varnothing}{4+\varnothing}\right)M_{AB}\right\}\frac{L(2-\varnothing)}{12EI_z} \tag{40}$$

The moment "M_{AB}" is found in function of "θ_A":

$$M_{AB} = \left(\frac{4+\varnothing}{1+\varnothing}\right)\frac{EL_z}{L}\theta_A \tag{41}$$

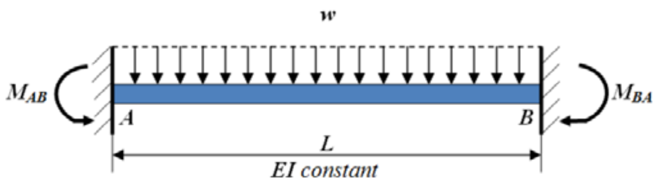
Thus, for a span "AB" which is simply supported at "A" and fixed at "B", a clockwise rotation of "θ_A" can be affected by applying a clockwise moment of $M_{AB} = [(4+\varnothing)/(1+\varnothing)](EI_z/L)\theta_A$ at "A", which in turn induces a clockwise moment of $M_{BA} = [(2-\varnothing)/(4+$

$\varnothing)]M_{AB}$ on the member at "B". The expression, $[(4+\varnothing)/(1+\varnothing)](EI_z/L)$ is usually called the stiffness factor, which is defined as the moment required to be applied at "A" to cause a rotation of 1 rad at "A" of a span "AB" simply supported at "A" and fixed at "B"; the number $[(2-\varnothing)/(4+\varnothing)]$ is the carry-over factor, which is the radio of the moment induced at "B" to the moment applied at "A".

2.1.3. *Fixed-end moments.* The fixed-end moments for beams considering the bending deformations and shear are presented in Figure 3 [10,11].

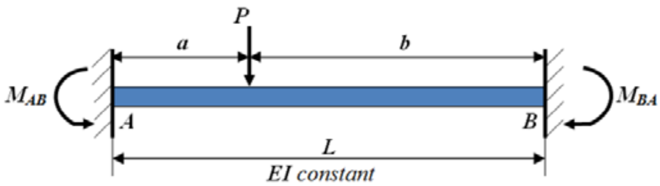
2.2. **Model 2.** This model takes into account the bending deformations and shear in the methodology, and in the fixed-end moments considers the bending deformations.

The equations for this model are same as that case previous, i.e., the stiffness factor is $[(4+\varnothing)/(1+\varnothing)](EI_z/L)$ and the carry-over factor is $[(2-\varnothing)/(4+\varnothing)]$ [9], but the fixed-end moments are shown in Figure 4 [12-16].



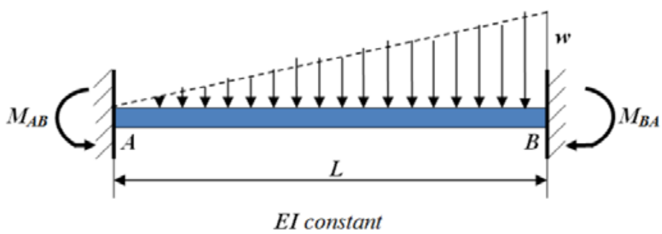
$$M_{FAB} = +\frac{wL^2}{12}(1-\varnothing) \quad M_{FBA} = -\frac{wL^2}{12}(1-\varnothing) \tag{42}$$

(a)



$$M_{FAB} = +\frac{Pab(2b+L\varnothing)}{2L^2(1+\varnothing)} \quad M_{FBA} = -\frac{Pab(2a+L\varnothing)}{2L^2(1+\varnothing)} \tag{43}$$

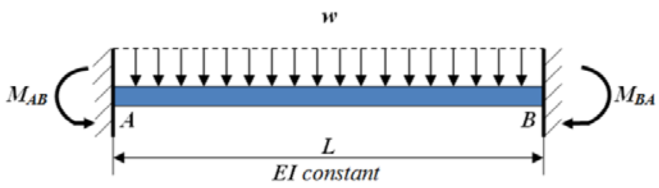
(b)



$$M_{FAB} = +\frac{wL^2(4+5\varnothing-5\varnothing^2)}{120(1+\varnothing)} \quad M_{FBA} = -\frac{wL^2(6-5\varnothing-5\varnothing^2)}{120(1+\varnothing)} \tag{44}$$

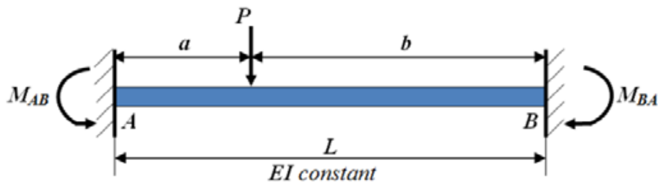
(c)

FIGURE 3. Fixed-end moments: (a) uniformly distributed load, (b) concentrated load, (c) triangularly distributed load



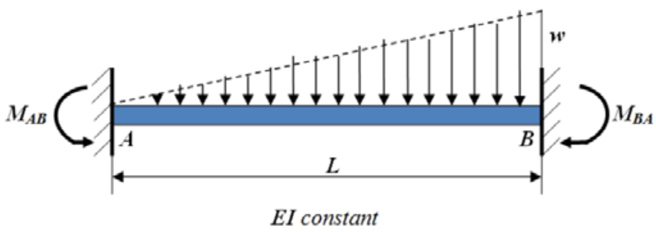
$$M_{FAB} = +\frac{wL^2}{12} \quad M_{FBA} = -\frac{wL^2}{12} \tag{45}$$

(a)



$$M_{FAB} = +\frac{Pab^2}{L^2} \quad M_{FBA} = -\frac{Pa^2b}{L^2} \tag{46}$$

(b)



$$M_{FAB} = +\frac{wL^2}{30} \quad M_{FBA} = -\frac{wL^2}{20} \tag{47}$$

(c)

FIGURE 4. Fixed-end moments: (a) uniformly distributed load, (b) concentrated load, (c) triangularly distributed load

2.3. **Model 3.** This model is the classical model called Hardy Cross method, which considers the bending deformations in the methodology and in the fixed-end moments, i.e., shear deformations are neglected totally.

The equations for this model are: the stiffness factor is "$4EI_z/L$" and the carry-over factor is 0.5, but the fixed-end moments are shown in Figure 4.

2.4. **General procedure of method.** Procedure to the analysis of statically indeterminate beams is presented:

1. Determine the fixed-end moments at the ends of each span, using the expressions as shown in Figures 3 and 4, according to the case.
2. Determine the carry-over factors and stiffness factors at the ends of each span, using the equations corresponding, according to the model.
3. Determine the distribution factors "DF_{ij}" at the ends of each span, using the following equation:

$$DF_{ij} = \frac{K_{ij}}{\sum K_i}$$

where: K_{ij} is stiffness factor of span "ij", $\sum K_i$ is the sum of all rigidities that arrive to the joint "i".

4. Establish the table: The Moment-Distribution Method consists in successively locking and releasing the joints; the first locking moments are the fixed-end moments due to the applied loading, after the first balancing; the successive locking moments are the carry-over moments which are induced to act at the other ends of the respective spans by the balancing moments.

5. The same process can be repeated for as many cycles as desired to bring the balancing or carry-over moments to very small magnitudes. Thus, any degree of accuracy can be obtained, and the work required decreases as the required accuracy decreases. The final end moments or total are obtained by adding all numbers in the respective columns.

6. Through the free-body diagram of each beam is performed static balance to obtain shear forces.

7. Draw the diagrams shear forces and moments.

3. **Application.** The structural analysis of a steel beam in three different problems is presented in Figure 5, and the example is developed for three different models: the Model 1 considers the bending deformations and shear in the methodology and in the fixed-end moments; the Model 2 takes into account the bending deformations and shear in the methodology, and in the fixed-end moments considers the bending deformations; the Model 3 is the moment-distribution classical method, which considers the bending deformations in the methodology and in the fixed-end moments. The data considered are: $w = 34.335$ kN/m; $L = 10.00$ m, 7.50 m, 5.00 m; $E = 20019.6$ kN/cm^2, $\nu = 0.32$. Properties of the beam W24X94 are: $A = 178.71$ cm^2; $A_s = 80.83$ cm^2; $I_z = 111966$ cm^4.

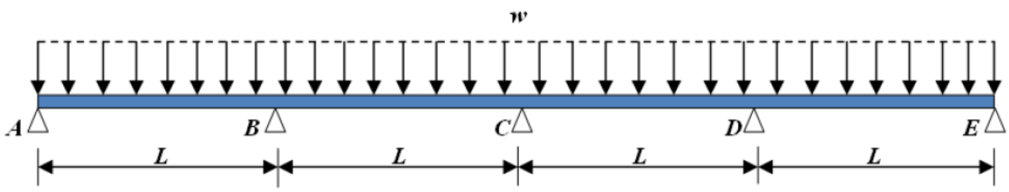


FIGURE 5. Continuous beam on four equal lengths with uniformly distributed load

Use Equation (29) to obtain the shear modulus as follows:

$$G = \frac{E}{2(1+v)} = \frac{20019.6}{2(1+0.32)} = 7583.182 \text{ kN/cm}^2$$

By Equation (28) is found the form factor:
For beams of $L = 10.00$ m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \varnothing_{DE} = \frac{12EI_z}{GA_sL^2} = \frac{12(20019.6)(111966)}{(7583.182)(80.83)(1000)^2} = 0.04388325783$$

For beams of $L = 7.50$ m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \varnothing_{DE} = \frac{12EI_z}{GA_sL^2} = \frac{12(20019.6)(111966)}{(7583.182)(80.83)(750)^2} = 0.07801468059$$

For beams of $L = 5.00$ m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \varnothing_{DE} = \frac{12EI_z}{GA_sL^2} = \frac{12(20019.6)(111966)}{(7583.182)(80.83)(500)^2} = 0.1755330313$$

The fixed-end moments for beams with uniformly distributed load considering the bending deformations by Equation (45) are:

For beams of $L = 10.00$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12} = +\frac{(34.335)(10.00)^2}{12} = +286.125 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12} = -\frac{(34.335)(10.00)^2}{12} = -286.125 \text{ kN-m}$$

For beams of $L = 7.50$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12} = +\frac{(34.335)(7.50)^2}{12} = +160.945 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12} = -\frac{(34.335)(7.50)^2}{12} = -160.945 \text{ kN-m}$$

For beams of $L = 5.00$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12} = +\frac{(34.335)(5.00)^2}{12} = +71.531 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12} = -\frac{(34.335)(5.00)^2}{12} = -71.531 \text{ kN-m}$$

The fixed-end moments for beams with uniformly distributed load, taking into account the bending deformations and shear by Equation (42) are:

For beams of $L = 10.00$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12}(1 - \varnothing)$$

$$= +\frac{(34.335)(10.00)^2}{12}(1 - 0.04388325783) = +273.569 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12}(1 - \varnothing)$$

$$= -\frac{(34.335)(10.00)^2}{12}(1 - 0.04388325783) = -273.569 \text{ kN-m}$$

For beams of $L = 7.50$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12}(1 - \varnothing)$$

$$= +\frac{(34.335)(7.50)^2}{12}(1 - 0.07801468059) = +148.389 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12}(1 - \varnothing)$$

$$= -\frac{(34.335)(7.50)^2}{12}(1 - 0.07801468059) = -148.389 \text{ kN-m}$$

For beams of $L = 5.00$ m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = M_{FDE} = \frac{wL^2}{12}(1 - \varnothing)$$

$$= +\frac{(34.335)(5.00)^2}{12}(1 - 0.1755330313) = +58.975 \text{ kN-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = M_{FED} = -\frac{wL^2}{12}(1 - \varnothing)$$

$$= -\frac{(34.335)(5.00)^2}{12}(1 - 0.1755330313) = -58.975 \text{ kN-m}$$

Calculation of the value of "EI_z" for all beams is:

$$EI_z = (20019.6)(111966) = 2241514534 \text{ kN-cm}^2 = 224151.453 \text{ kN-m}^2$$

The stiffness factors of each beam considering the bending deformations for all cases are:

$$K_{ij} = \frac{4EI_z}{L}$$

The stiffness factors of each beam taking into account the bending deformations and shear by Equation (41) are:

To 10.00 m is:

$$K_{ij} = \left(\frac{4 + 0.04388325783}{1 + 0.04388325783}\right)\frac{EI_z}{L} = \frac{3.873884582EI_z}{L}$$

To 7.50 m is:

$$K_{ij} = \left(\frac{4 + 0.07801468059}{1 + 0.07801468059}\right)\frac{EI_z}{L} = \frac{3.782893456EI_z}{L}$$

To 5.00 m is:

$$K_{ij} = \left(\frac{4 + 0.1755330313}{1 + 0.1755330313}\right)\frac{EI_z}{L} = \frac{3.552033775EI_z}{L}$$

The distribution factors of each beam considering the bending deformations are:

$$DF_{AB} = DF_{ED} = 1.0$$

$$DF_{BA} = DF_{BC} = DF_{CB} = DF_{CD} = DF_{DC} = DF_{DE} = 0.5$$

The distribution factors of each beam taking into account the bending deformations and shear are:

$$DF_{AB} = DF_{ED} = 1.0$$

$$DF_{BA} = DF_{BC} = DF_{CB} = DF_{CD} = DF_{DC} = DF_{DE} = 0.5$$

The carry-over factors of each beam considering the bending deformations for all cases are:

$$M_{ji} = 0.5M_{ij}$$

The carry-over factors of each beam taking into account the bending deformations and shear by Equation (38) are:

To 10.00 m is:

$$M_{ji} = \left(\frac{2 - 0.04388325783}{4 + 0.04388325783}\right)M_{ij} = 0.4837223573M_{ij}$$

To 7.50 m is:

$$M_{ji} = \left(\frac{2 - 0.07801468059}{4 + 0.07801468059}\right)M_{ij} = 0.4713041688M_{ij}$$

To 5.00 m is:

$$M_{ji} = \left(\frac{2 - 0.1755330313}{4 + 0.1755330313}\right)M_{ij} = 0.436942291M_{ij}$$

Tables 1 to 9 presented the results by the moment-distribution method.

TABLE 1. Model 1 for $L = 10.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236
Cycle 1	FEM	+273.569	−273.569	+273.569	−273.569	+273.569	−273.569	+273.569	−273.569
	Bal.	−273.569	0	0	0	0	0	0	+273.569
Cycle 2	CO	0	−132.331442	0	0	0	0	+132.331442	0
	Bal.	0	+66.165721	+66.165721	0	0	−66.165721	−66.165721	0
Cycle 3	CO	+32.005839	0	0	+32.005839	−32.005839	0	0	−32.005839
	Bal.	−32.005839	0	0	0	0	0	0	+32.005839
Cycle 4	CO	0	−15.481940	0	0	0	0	+15.481940	0
	Bal.	0	+7.740970	+7.740970	0	0	−7.740970	−7.740970	0
Cycle 5	CO	+3.744480	0	0	+3.744480	−3.744480	0	0	−3.744480
	Bal.	−3.744480	0	0	0	0	0	0	+3.744480
Cycle 6	CO	0	−1.811289	0	0	0	0	+1.811289	0
	Bal.	0	+0.905644	+0.905644	0	0	−0.905644	−0.905644	0
Cycle 7	CO	+0.438080	0	0	+0.438080	−0.438080	0	0	−0.438080
	Bal.	−0.438080	0	0	0	0	0	0	+0.438080
Total Moments		0	−348.381336	+348.381336	−237.380601	+237.380601	−348.381336	+348.381336	0

TABLE 2. Model 2 for $L = 10.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458	3.87388458
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236	0.48372236
Cycle 1	FEM	+286.125	−286.125	+286.125	−286.125	+286.125	−286.125	+286.125	−286.125
	Bal.	−286.125	0	0	0	0	0	0	+286.125
Cycle 2	CO	0	−138.405061	0	0	0	0	+138.405061	0
	Bal.	0	+69.202530	+69.202530	0	0	−69.202530	−69.202530	0
Cycle 3	CO	+33.474811	0	0	+33.474811	−33.474811	0	0	−33.474811
	Bal.	−33.474811	0	0	0	0	0	0	+33.474811
Cycle 4	CO	0	−16.192515	0	0	0	0	+16.192515	0
	Bal.	0	+8.096257	+8.096257	0	0	−8.096257	−8.096257	0
Cycle 5	CO	+3.916341	0	0	+3.916341	−3.916341	0	0	−3.916341
	Bal.	−3.916341	0	0	0	0	0	0	+3.916341
Cycle 6	CO	0	−1.894422	0	0	0	0	+1.894422	0
	Bal.	0	+0.947211	+0.947211	0	0	−0.947211	−0.947211	0
Cycle 7	CO	+0.458187	0	0	+0.458187	−0.458187	0	0	−0.458187
	Bal.	−0.458187	0	0	0	0	0	0	+0.458187
Total Moments		0	−364.371	+364.370998	−248.275661	+248.275661	−364.370998	+364.371	0

TABLE 3. Model 3 for $L = 10.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		4	4	4	4	4	4	4	4
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
Cycle 1	FEM	+286.125	−286.125	+286.125	−286.125	+286.125	−286.125	+286.125	−286.125
	Bal.	−286.125	0	0	0	0	0	0	+286.125
Cycle 2	CO	0	−143.0625	0	0	0	0	+143.0625	0
	Bal.	0	+71.53125	+71.53125	0	0	−71.53125	−71.53125	0
Cycle 3	CO	+35.765625	0	0	+35.765625	−35.765625	0	0	−35.765625
	Bal.	−35.765625	0	0	0	0	0	0	+35.765625
Cycle 4	CO	0	−17.882812	0	0	0	0	+17.882812	0
	Bal.	0	+8.941406	+8.941406	0	0	−8.941406	−8.941406	0
Cycle 5	CO	+4.470703	0	0	+4.470703	−4.470703	0	0	−4.470703
	Bal.	−4.470703	0	0	0	0	0	0	+4.470703
Cycle 6	CO	0	−2.235352	0	0	0	0	+2.235352	0
	Bal.	0	+1.117676	+1.117676	0	0	−1.117676	−1.117676	0
Cycle 7	CO	+0.558838	0	0	+0.558838	−0.558838	0	0	−0.558838
	Bal.	−0.558838	0	0	0	0	0	0	+0.558838
Total Moments		0	−367.715332	+367.715332	−245.329834	+245.329834	−367.715332	+367.715332	0

Table 4. Model 1 for $L = 7.50$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417
Cycle 1	FEM	+148.389	−148.389	+148.389	−148.389	+148.389	−148.389	+148.389	−148.389
	Bal.	−148.389	0	0	0	0	0	0	+148.389
Cycle 2	CO	0	−69.936354	0	0	0	0	+69.936354	0
	Bal.	0	+34.968177	+34.968177	0	0	−34.968177	−34.968177	0
Cycle 3	CO	+16.480678	0	0	+16.480678	−16.480678	0	0	−16.480678
	Bal.	−16.480678	0	0	0	0	0	0	+16.480678
Cycle 4	CO	0	−7.767398	0	0	0	0	+7.767398	0
	Bal.	0	+3.883699	+3.883699	0	0	−3.883699	−3.883699	0
Cycle 5	CO	+1.830404	0	0	+1.830404	−1.830404	0	0	−1.830404
	Bal.	−1.830404	0	0	0	0	0	0	+1.830404
Cycle 6	CO	0	−0.862677	0	0	0	0	+0.862677	0
	Bal.	0	+0.431338	+0.431338	0	0	−0.431338	−0.431338	0
Cycle 7	CO	+0.203292	0	0	+0.203292	−0.203292	0	0	−0.203292
	Bal.	−0.203292	0	0	0	0	0	0	+0.203292
Total Moments		0	−187.672215	+187.672215	−129.874626	+129.874626	−187.672215	+187.672215	0

Table 5. Model 2 for $L = 7.50$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346	3.78289346
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417	0.47130417
Cycle 1	FEM	+160.945	−160.945	+160.945	−160.945	+160.945	−160.945	+160.945	−160.945
	Bal.	−160.945	0	0	0	0	0	0	+160.945
Cycle 2	CO	0	−75.854050	0	0	0	0	+75.854050	0
	Bal.	0	+37.927025	+37.927025	0	0	−37.927025	−37.927025	0
Cycle 3	CO	+17.875165	0	0	+17.875165	−17.875165	0	0	−17.875165
	Bal.	−17.875165	0	0	0	0	0	0	+17.875165
Cycle 4	CO	0	−8.424640	0	0	0	0	+8.424640	0
	Bal.	0	+4.212320	+4.212320	0	0	−4.212320	−4.212320	0
Cycle 5	CO	+1.985284	0	0	+1.985284	−1.985284	0	0	−1.985284
	Bal.	−1.985284	0	0	0	0	0	0	+1.985284
Cycle 6	CO	0	−0.935673	0	0	0	0	+0.935673	0
	Bal.	0	+0.467836	+0.467836	0	0	−0.467836	−0.467836	0
Cycle 7	CO	+0.220493	0	0	+0.220493	−0.220493	0	0	−0.220493
	Bal.	−0.220493	0	0	0	0	0	0	+0.220493
Total Moments		0	−203.552182	+203.552182	−140.864058	+140.864058	−203.552182	+203.552182	0

Table 6. Model 3 for $L = 7.50$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		4	4	4	4	4	4	4	4
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
Cycle 1	FEM	+160.945	−160.945	+160.945	−160.945	+160.945	−160.945	+160.945	−160.945
	Bal.	−160.945	0	0	0	0	0	0	+160.945
Cycle 2	CO	0	−80.4725	0	0	0	0	+80.4725	0
	Bal.	0	+40.23625	+40.23625	0	0	−40.23625	−40.23625	0
Cycle 3	CO	+20.118125	0	0	+20.118125	−20.118125	0	0	−20.118125
	Bal.	−20.118125	0	0	0	0	0	0	+20.118125
Cycle 4	CO	0	−10.059062	0	0	0	0	+10.059062	0
	Bal.	0	+5.029531	+5.029531	0	0	−5.029531	−5.029531	0
Cycle 5	CO	+2.514766	0	0	+2.514766	−2.514766	0	0	−2.514766
	Bal.	−2.514766	0	0	0	0	0	0	+2.514766
Cycle 6	CO	0	−1.257383	0	0	0	0	+1.257383	0
	Bal.	0	+0.628691	+0.628691	0	0	−0.628691	−0.628691	0
Cycle 7	CO	+0.314346	0	0	+0.314346	−0.314346	0	0	−0.314346
	Bal.	−0.314346	0	0	0	0	0	0	+0.314346
Total Moments		0	−206.839473	+206.839473	−137.997763	+137.997763	−206.839473	+206.839473	0

Table 7. Model 1 for $L = 5.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229
Cycle 1	FEM	+58.975	−58.975	+58.975	−58.975	+58.975	−58.975	+58.975	−58.975
	Bal.	−58.975	0	0	0	0	0	0	+58.975
Cycle 2	CO	0	−25.768672	0	0	0	0	+25.768672	0
	Bal.	0	+12.884336	+12.884336	0	0	−12.884336	−12.884336	0
Cycle 3	CO	+5.629711	0	0	+5.629711	−5.629711	0	0	−5.629711
	Bal.	−5.629711	0	0	0	0	0	0	+5.629711
Cycle 4	CO	0	−2.459859	0	0	0	0	+2.459859	0
	Bal.	0	+1.229929	+1.229929	0	0	−1.229929	−1.229929	0
Cycle 5	CO	+0.537408	0	0	+0.537408	−0.537408	0	0	−0.537408
	Bal.	−0.537408	0	0	0	0	0	0	+0.537408
Cycle 6	CO	0	−0.234816	0	0	0	0	+0.234816	0
	Bal.	0	+0.117408	+0.117408	0	0	−0.117408	−0.117408	0
Cycle 7	CO	+0.051301	0	0	+0.051301	−0.051301	0	0	−0.051301
	Bal.	−0.051301	0	0	0	0	0	0	+0.051301
Total Moments		0	−73.206674	+73.206674	−52.75658	+52.75658	−73.206674	+73.206674	0

Table 8. Model 2 for $L = 5.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378	3.55203378
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229	0.43694229
Cycle 1	FEM	+71.53125	−71.53125	+71.53125	−71.53125	+71.53125	−71.53125	+71.53125	−71.53125
	Bal.	−71.53125	0	0	0	0	0	0	+71.53125
Cycle 2	CO	0	−31.255029	0	0	0	0	+31.255029	0
	Bal.	0	+15.627515	+15.627515	0	0	−15.627515	−15.627515	0
Cycle 3	CO	+6.828322	0	0	+6.828322	−6.828322	0	0	−6.828322
	Bal.	−6.828322	0	0	0	0	0	0	+6.828322
Cycle 4	CO	0	−2.983583	0	0	0	0	+2.983583	0
	Bal.	0	+1.491791	+1.491791	0	0	−1.491791	−1.491791	0
Cycle 5	CO	+0.651827	0	0	+0.651827	−0.651827	0	0	−0.651827
	Bal.	−0.651827	0	0	0	0	0	0	+0.651827
Cycle 6	CO	0	−0.284811	0	0	0	0	+0.284811	0
	Bal.	0	+0.142405	+0.142405	0	0	−0.142405	−0.142405	0
Cycle 7	CO	+0.062223	0	0	+0.062223	−0.062223	0	0	−0.062223
	Bal.	−0.062223	0	0	0	0	0	0	+0.062223
Total Moments		0	−88.792962	+88.792962	−63.988878	+63.988878	−88.792962	+88.792962	0

Table 9. Model 3 for $L = 5.00$ m

Joint		A	B		C		D		E
Member		AB	BA	BC	CB	CD	DC	DE	ED
Stiffness Factor		4	4	4	4	4	4	4	4
Distribution Factor		1.000	0.500	0.500	0.500	0.500	0.500	0.500	1.000
Carry-Over Factor		0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
Cycle 1	FEM	+71.53125	−71.53125	+71.53125	−71.53125	+71.53125	−71.53125	+71.53125	−71.53125
	Bal.	−71.53125	0	0	0	0	0	0	+71.53125
Cycle 2	CO	0	−35.765625	0	0	0	0	+35.765625	0
	Bal.	0	+17.882812	+17.882812	0	0	−17.882812	−17.882812	0
Cycle 3	CO	+8.941406	0	0	+8.941406	−8.941406	0	0	−8.941406
	Bal.	−8.941406	0	0	0	0	0	0	+8.941406
Cycle 4	CO	0	−4.470703	0	0	0	0	+4.470703	0
	Bal.	0	+2.235352	+2.235352	0	0	−2.235352	−2.235352	0
Cycle 5	CO	+1.117676	0	0	+1.117676	−1.117676	0	0	−1.117676
	Bal.	−1.117676	0	0	0	0	0	0	+1.117676
Cycle 6	CO	0	−0.558838	0	0	0	0	+0.558838	0
	Bal.	0	+0.279619	+0.279619	0	0	−0.279619	−0.279619	0
Cycle 7	CO	+0.139709	0	0	+0.139709	−0.139709	0	0	−0.139709
	Bal.	−0.139709	0	0	0	0	0	0	+0.139709
Total Moments		0	−91.928633	+91.929033	−61.332459	+61.332459	−91.929033	+91.928633	0

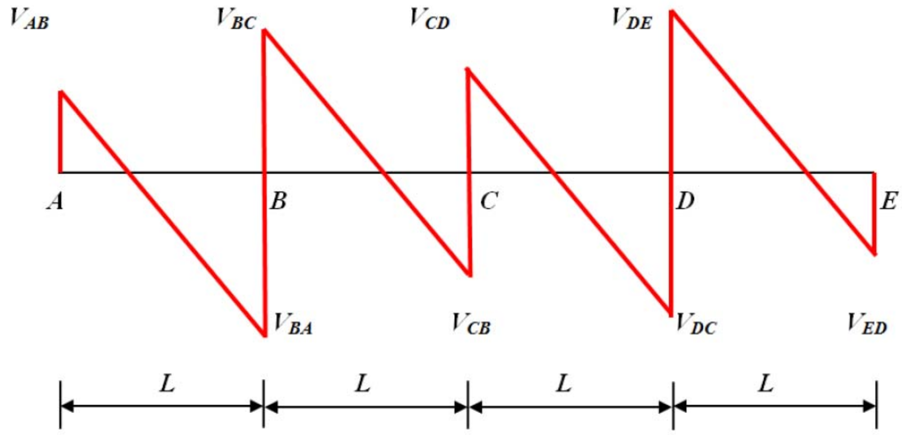


FIGURE 6. Diagram of typical shear forces

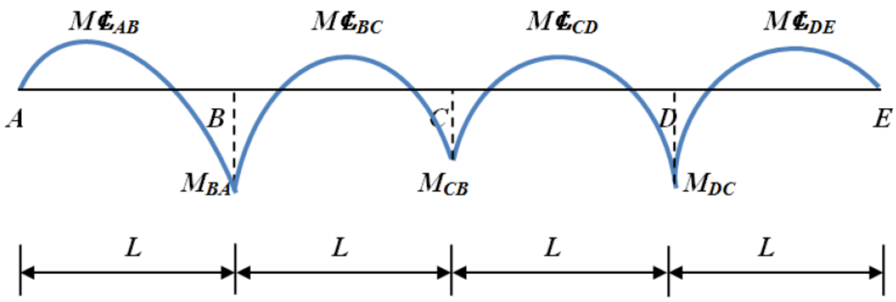


FIGURE 7. Diagram of typical bending moments

4. **Results.** Figure 6 shows diagram of typical shear forces and Figure 7 presents diagram of typical bending moments.

The differences of the Models 2 and 3 with respect to the Model 1 are presented in Tables 10 and 11, because the Model 1 is model proposed in this paper.

Table 10 presents the shear forces in the ends of the members for the three models; also the differences are greater in short beams for all cases. The greatest difference is presented in $L = 5.00$ m of a 6% in the support "A" of member "AB", and in the support "E" of member "ED", being the Model 1 greater with respect to the Model 3. The smallest

TABLE 10. The shear forces in kN

Shear forces	Case 1 $L = 10.00$ m					Case 2 $L = 7.50$ m					Case 3 $L = 5.00$ m				
	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$
V_{AB}	+137	+135	+135	0.99	0.99	+104	+102	+101	0.98	0.97	+71	+68	+67	0.96	0.94
V_{BA}	−207	−208	−208	1.00	1.00	−154	−156	−156	1.01	1.01	−100	−104	−104	1.04	1.04
V_{BC}	+183	+183	+184	1.00	1.01	+136	+137	+138	1.01	1.01	+90	+91	+92	1.01	1.02
V_{CB}	−161	−160	−159	0.99	0.99	−121	−120	−120	0.99	0.99	−82	−81	−80	0.99	0.98
V_{CD}	+161	+160	+159	0.99	0.99	+121	+120	+120	0.99	0.99	+82	+81	+80	0.99	0.98
V_{DC}	−183	−183	−184	1.00	1.01	−136	−137	−138	1.01	1.01	−90	−91	−92	1.01	1.02
V_{DE}	+207	+208	+208	1.00	1.00	+154	+156	+156	1.01	1.01	+100	+104	+104	1.04	1.04
V_{ED}	−137	−135	−135	0.99	0.99	−104	−102	−101	0.98	0.97	−71	−68	−67	0.96	0.94

V_{ij} = Shear forces of the beam "ij" in end "i"

V_{ji} = Shear forces of the beam "ij" in end "j"

Table 11. The final moments in kN-m

Final Moments	Case 1 $L = 10.00$ m					Case 2 $L = 7.50$ m					Case 3 $L = 5.00$ m				
	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$	M 1	M 2	M 3	$\frac{\text{M 2}}{\text{M 1}}$	$\frac{\text{M 3}}{\text{M 1}}$
M_{AB}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M℄_{AB}$	+273	+266	+265	0.97	0.97	+157	+150	+149	0.96	0.95	+74	+67	+66	0.91	0.89
M_{BA}	−348	−364	−368	1.05	1.06	−188	−204	−207	1.09	1.10	−73	−89	−92	1.22	1.26
M_{BC}	+348	−364	−368	1.05	1.06	+188	+204	+207	1.09	1.10	+73	−89	+92	1.22	1.26
$M℄_{BC}$	+138	+125	+125	0.91	0.91	+83	+70	+70	0.84	0.84	+45	+31	+31	0.69	0.69
M_{CB}	−237	−248	−245	1.05	1.03	−130	−141	−138	1.08	1.06	−53	−64	−61	1.21	1.15
M_{CD}	+237	−248	−245	1.05	1.03	+130	+141	+138	1.08	1.06	+53	−64	+61	1.21	1.15
$M℄_{CD}$	+138	+125	+125	0.91	0.91	+83	+70	+70	0.84	0.84	+45	+31	+31	0.69	0.69
M_{DC}	−348	−364	−368	1.05	1.06	−188	−204	−207	1.09	1.10	−73	−89	−92	1.22	1.26
M_{DE}	+348	−364	−368	1.05	1.06	+188	+204	+207	1.09	1.10	+73	−89	+92	1.22	1.26
$M℄_{DE}$	+273	+266	+265	0.97	0.97	+157	+150	+149	0.96	0.95	+74	+67	+66	0.91	0.89
M_{ED}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

M_{ij} = Negative moment of the beam "ij" in end "i"
$M℄_{ij}$ = Positive moment of the beam "ij"
M_{ji} = Negative moment of the beam "ji" in end "j"

difference is shown: in the support "C" of members "CB" and "CD", being the Model 1 greater with respect to the Model 2; in the support "B" of member "BC" and also in the support "D" of member "DC", being the Model 2 greater with respect to the Model 1, this difference is the 1%.

Table 11 shows the negative moments and positive for the three models; also the differences in short members are considerable for all cases. The greatest difference is of a 31% for member "BC" and "CD" in the center of the beam length, being the Model 1 greater with respect to the Models 2 and 3. The smallest difference is for the member "AB" and "DE" in the center of the beam length, being the Model 1 greater with respect to the Model 2 of a 9%.

5. **Conclusions.** With regard to Table 10, which shows the shear forces at the ends of the beams between the three models, the differences are larger between the Model 1 with respect to Models 2 and 3, when the length between supports is reduced.

Finally, in Table 11 illustrating the final moments, both negative and positive, also there are differences bigger in the Model 1 with respect to Models 2 and 3, when the length between supports is reduced, and not all are the side of safety with respect to Model 3 (moment-distribution classical method).

The shear forces and moments acting on the beams, these elements which are those governing the design of a structure were studied for three different models. The results showed differences between the three models, when members tend to be shorter, the differences are increased of conservative side as the unsafe side in proposed model with respect to the other two models.

On the other hand, when we design a structural member with materials of structural steel or reinforced concrete which are obtained its dimensions with the maximum moments, and for this problem in case of $L = 3.00$ m will be: for the Model 1 is 74 kN-m, for the Model 2 is 89 kN-m and for the Model 3 is 92 kN-m. Thus, using Model 1 (proposed model), we will have a saving of a 20.27% with respect to Model 2 and of a 24.32% in comparison to Model 3 (classical model, which is the moment-distribution method).

Therefore, for the normal practice of using the Model 2 and Model 3, this is not a recommended solution, when lengths are short between supports.

Then, the Model 1 (proposed model) passes to be the more appropriate model for structural analysis of continuous beams and also is adjusted to the real conditions, since the shear forces and moments are presented in any analysis of continuous beams and therefore the bending deformations and shear are produced.

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