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CONTROL THEORY BASED AQM FOR HIGH SPEED NETWORKS

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Abstract. *In this paper, we address the problem of synthesizing static antiwindup compensator for a linearized model of a high speed network TCP/IP router. Theorectically, the class of systems considered are input saturating linear systems with known and constant time delay in both the input and the states. The synthesis of these antiwindup compensators is carried out on an LMI-based framework whose development is based on the use of a Lyapunov-Krasovskii functional, a generalized sector condition, and a negligibly constrained usage of Finsler Lemma. In practical terms, high speed networks need routers with the ability to quickly return the packet queue size to the predetermined steady state level, otherwise, quality of service criteria might be compromised. In this sense, we perform the active queue management of a TCP/IP router on a high speed network in order to improve its speed on returning to steady state levels of queue and transmission rate. We compare our results by computing the minimization of the $\mathcal{L}_2$-gain upper bound of the disturbances for which the time function of the queue length and TCP congestion window size are assured to be bounded.*
Keywords: Antiwindup, Time delay, LMI, Continuous time

1. **Introduction.** Since early 60's researchers have been proposing techniques to compensate the effects of the saturation on control systems. An extensive review can be seen in [1]. In this sense, antiwindup compensation has been researched intensively but only recently over time delay systems. It is well known that time delays are present in many control applications and are also source of performance degradation and even instability (see [2]). When a time delay plant saturates, the effects can be magnified given the presence of time delays. This justifies the research of antiwindup techniques for such systems. They can be grouped according to where the plant is time delayed: input, states or output. In [3, 4], for instance, plants present input and/or output delays. In [5-7], plants have only state delay. As far as we know, in [8] that it started to consider plants delayed both in input and states. This combination of delay is likely to occur in specially distributed systems with a remote saturating input. We have been researching precisely this category, mostly motivated by its application on TCP/IP congested routers (see [9-11]). From a theorectical perspective, all the cited works on antiwindup synthesis for time delay systems are based on Lyapunov-Krasovskii functionals [14] and LMI conditions [15], for they are efficient ways to determine the stability of the closed loop system. Also, there are some peculiarities among these works. For instance, [8] does not consider any type of disturbances on the plant. This prevents any direct comparison with [9-11], where the antiwindup compensators are synthesised in order to minimize the $\mathcal{L}_2$-gain of the disturbance to the plant output. Nonetheless, our works have always been driven towards reducing the conservativeness of the conditions that ensure both the existence

of an antiwindup compensator as well as the closed loop stability of the whole system. Specifically in this, present work outperforms the previous ones.

In practical terms, our work proposed a novel technique for active queue management (AQM) of TCP/IP routers in high speed networks (HSN). It invites extension of the best practices in this matter, which nowadays consists of a Random Early Detect (RED) [18] technique consisting in making the packet discard probability a direct function of the queue size. Out of it, as a default policy, most of the routers use drop tail, a technique which has a similar performance to RED excepting that the router queue average size is higher. We are looking forward to implementing our results in real routers for getting a more indepth understanding of its benefits.

The paper is organized as follows. Section 2 states the problem investigated on a theoretical approach. Section 3 presents the main development resources. Section 4 presents the results in forms of theorem. Section 5 casts an optimization problem of interest for deriving the antiwindup compensator. Section 6 applies the result on the AQM model of a TCP/IP router on an HSN. Section 7 comprises the concluding remarks.

Notation: For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A^T denotes the transpose of A. $A_{(i)}$ denotes the i^{th} line of matrix A. $\star$ stands for symmetric blocks; I denotes an identity matrix of appropriate order. $\underline{\lambda}(P)$ and $\bar{\lambda}(P)$ denote the minimal and maximal eigenvalues of matrix P, respectively. $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \Re^n)$ is the Banach Space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the norm $\parallel \phi \parallel_c = \sup_{-\tau \le t \le 0} \parallel \phi(t) \parallel$. $\parallel \cdot \parallel$ refers to the Euclidean vector norm. $\mathcal{C}_\tau^v$ is the set defined by $\mathcal{C}_\tau^v = \{\phi \in \mathcal{C}_\tau; || \phi ||_c < v, v > 0\}$. For $v \in \Re^m$, $sat(v) : \Re^m \to \Re^m$ denotes the classical symmetric saturation function defined as $(sat(v))_{(i)} = sat(v_{(i)}) = sign(v_{(i)}) \min(u_{o(i)}, |v_{(i)}|)$, $\forall i = 1, \cdots, m$, where $u_{o(i)} > 0$ denotes the ith magnitude bound. $blockdiag(\cdots)$ is a block diagonal matrix whose diagonal blocks are the ordered arguments. $He\{A\} = A + A^T$.

2. Problem Statement.

Consider the following plant model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - \tau) + Bu(t - \tau) + B_\omega \omega(t) \\ y(t) &= C_y x(t) \\ z(t) &= C_z x(t) + D_z u(t) \end{aligned} \tag{1}$$

where vectors $x(t) \in \Re^n$, $u(t) \in \Re^m$, $\omega(t) \in \Re^q$, $y(t) \in \Re^p$, $z(t) \in \Re^l$ are the plant state, input, disturbance, measured output and regulated output, respectively. The time delay τ is assumed to be known and constant. Matrices A, A_d, B, B_ω, C_y, C_z, D_z are of proper dimensions.

The plant inputs are supposedly bounded.

$$-u_{o_{(i)}} \le u_{(i)} \le u_{o_{(i)}}, \quad u_{o_{(i)}} > 0, \quad i = 1, \ldots, m \tag{2}$$

The disturbance vector $\omega(t)$ is assumed to be limited in energy, that is, $\omega(t) \in \mathcal{L}_2$. Hence, for some scalar δ, $0 \le \frac{1}{\delta} < \infty$, one has

$$\|\omega(t)\|_2^2 = \int_0^\infty \omega(t)^T \omega(t) dt \le \frac{1}{\delta} \tag{3}$$

In order to control system (1), we assume that the following controller has been designed for stabilizing the system disregarding the control bounds given in (2)

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t - \tau) + B_c u_c(t) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) \end{aligned} \tag{4}$$

where $x_c(t) \in \Re^{n_c}$, $u_c(t) \in \Re^p$ and $y_c(t) \in \Re^m$. Matrices A_c, $A_{c,d}$, B_c, C_c, D_c are of proper dimensions. The nominal interconnection of the controller (4) with the plant (1) is given by $u_c(t) = y(t)$ and $u(t-\tau) = y_c(t-\tau)$. In consequence of the control bounds, the *de facto* control signal to be injected in the system is

$$u(t-\tau) = \text{sat}(y_c(t-\tau))$$

To mitigate the effects of saturation, we inject in the states of the controller an anti-windup signal

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t-\tau) + B_c u_c(t) + E_c \psi(y_c(t)) + F_c \psi(y_c(t-\tau)) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) \end{aligned} \tag{5}$$

where $\psi(\cdot) = sat(\cdot) - (\cdot)$

Comment 1: This paper regards for antiwindup synthesis, so we are not concerned with the computation of controller (4). We assume it has been previously computed and it would ensure the global asymptotic stability of system (1) if $u(t-\tau) = y_c(t-\tau)$.

3. **Preliminaries.**

Through the following matrices

$$\mathbf{A} = \begin{bmatrix} A & 0 \\ B_c C_y & A_c \end{bmatrix}, \quad \mathbf{A}_d = \begin{bmatrix} A_d + B D_c C_y & B C_c \\ 0 & A_{c,d} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathbf{B}_\omega = \begin{bmatrix} B_\omega \\ 0 \end{bmatrix}$$
$$\check{\mathbf{I}} = \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix}, \quad \mathbf{D}_z = D_z, \quad \mathbf{C}_z = \begin{bmatrix} C_z + D_z D_c C_y & D_z C_c \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} D_c C_y & C_c \end{bmatrix}$$

closed loop system (1), (5) is represented as follows

$$\begin{aligned} \dot{\xi}(t) &= \mathbf{A}\xi(t) + \mathbf{A}_d \xi(t-\tau) + \check{\mathbf{I}} E_c \psi(y_c(t)) + (\mathbf{B} + \check{\mathbf{I}} F_c)\psi(y_c(t-\tau)) + \mathbf{B}_\omega \omega(t) \\ z(t) &= \mathbf{C}_z \xi(t) + \mathbf{D}_z \psi(y_c(t)) \end{aligned} \tag{6}$$

where $\xi(t) = \begin{bmatrix} x(t)^T & x_c(t)^T \end{bmatrix}^T$ and $y_c(t) = \mathbf{K}\xi(t)$.

Initial condition of system (6) is denoted by function ϕ_ξ, defined in the interval $[-\tau, 0]$

$$\begin{aligned} \phi_\xi(\theta) &= \begin{bmatrix} x(\theta)^T & x_c(\theta)^T \end{bmatrix}^T \\ &= \begin{bmatrix} \phi_x(\theta)^T & \phi_{x_c}(\theta)^T \end{bmatrix}^T, \quad \forall \theta \in [-\tau, 0], \quad (t_0, \phi_\xi) \in \Re^+ \times \mathcal{C}_\tau^v \end{aligned}$$

Let G, $G_\tau \in \Re^{m \times (n+n_c)}$ and the sets

$$\begin{aligned} \mathcal{S}(u_o) &= \left\{ \xi(t) \in \Re^{n+n_c}; |(\mathbf{K}_{(i)} + G_{(i)})\xi(t)| \le u_{o_{(i)}}, i \in [1, m] \right\} \\ \mathcal{S}_\tau(u_o) &= \left\{ \xi(t-\tau) \in \Re^{n+n_c}; |(\mathbf{K}_{(i)} + G_{\tau_{(i)}})\xi(t-\tau)| \le u_{o_{(i)}}, i \in [1, m] \right\} \end{aligned}$$

over which the following lemma is stated.

Lemma 3.1. *Generalized Sector Condition* [12]*: If $\xi(t) \in \mathcal{S}(u_o)$ and $\xi(t-\tau) \in \mathcal{S}_\tau(u_o)$ then the relations*

$$\begin{aligned} &\psi(y_c(t))^T T \big(\psi(y_c(t)) - G\xi(t)\big) \le 0 \\ &\psi(y_c(t-\tau))^T T_\tau \big(\psi(y_c(t-\tau)) - G_\tau \xi(t-\tau)\big) \le 0 \end{aligned}$$

are true for any positive diagonal T, $T_\tau \in \Re^{m \times m}$.

Lemma 3.2. *Jensen Inequality* [13]*: For any scalar $\tau > 0$, positive definite matrix $Q \in \Re^{m \times m}$ and function $x : [0, \tau] \to \Re^m$ so that the integral is definite, and the following inequality holds:*

$$\tau \int_0^\tau x(\theta)^T Q x(\theta) d\theta \ge \left(\int_0^\tau x(\theta) d\theta \right)^T Q \left(\int_0^\tau x(\theta) d\theta \right)$$

Lemma 3.3. *Finsler Lemma* [15]*: If there exists a matrix* $\mathbf{M}_1 \in \Re^{m\times m}$*, a vector* $x(t) \in \Re^m$ *and a matrix* $\mathcal{B} \in \Re^{p\times m}$ *such that* $x(t)^T\mathbf{M}_1 x(t) < 0$*,* $\forall x(t) \neq 0 \mid \mathcal{B}x(t) = 0$ *is verified, then there exists a matrix* $\mathbf{F} \in \Re^{m\times p}$ *so that*

$$\mathbf{M}_1 + \mathbf{F}\mathcal{B} + \mathcal{B}^T\mathbf{F}^T < 0$$

In other words, both statements are equivalent.

Remark 3.1. *Note that in case* $G = G_\tau = -\mathbf{K}$*,* $S(u_o) \equiv \Re^{n+n_c}$ *and* $S_\tau(u_o) \equiv \Re^{n+n_c}$*.*

4. **Main Results.** Now we derive a synthesis framework for globally stabilizing static antiwindup compensator based on a Lyapunov-Krasovskii candidate

$$V(t) = \xi(t)^T P\xi(t) + \int_{t-\tau}^{t} \xi(\theta)^T R\xi(\theta)d\theta + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}(\beta)^T Q\dot{\xi}(\beta)d\beta d\theta \tag{7}$$

where $P = P^T > 0$, $R = R^T > 0$, $Q = Q^T > 0 \in \Re^{(n+n_c)\times(n+n_c)}$. The following theorem can be stated.

Theorem 4.1. *If there exists symmetric positive definite* P*,* R*,* $Q \in \Re^{(n+n_c)\times(n+n_c)}$*, matrices* F_{12}*,* F_{22}*,* F_{32}*,* $F_{42} \in \Re^{(n+n_c)\times(n+n_c)}$*,* F_{52}*,* F_{62}*,* G*,* $G_\tau \in \Re^{m\times(n+n_c)}$*,* $F_{72} \in \Re^{q\times(n+n_c)}$*, scalars* γ*,* α *and structured matrices* F_{11}*,* F_{21}*,* F_{31}*,* $F_{41} \in \Re^{(n+n_c)\times(n+n_c)}$*,* F_{51}*,* $F_{61} \in \Re^{m\times(n+n_c)}$*, where*

$$F_{11} = \begin{bmatrix} F_{11a} & 0 \\ F_{11b} & aI_{n_c} \end{bmatrix},\ F_{21} = \begin{bmatrix} F_{21a} & 0 \\ F_{21b} & bI_{n_c} \end{bmatrix},\ F_{31} = \begin{bmatrix} F_{31a} & 0 \\ F_{31b} & cI_{n_c} \end{bmatrix},\ F_{41} = \begin{bmatrix} F_{41a} & 0 \\ F_{41b} & dI_{n_c} \end{bmatrix}$$
$$F_{51} = \begin{bmatrix} F_{51a} & eI_{m\times n_c} \end{bmatrix},\ F_{61} = \begin{bmatrix} F_{61a} & fI_{m\times n_c} \end{bmatrix},\ F_{71} = \begin{bmatrix} F_{71a} & gI_{q\times n_c} \end{bmatrix}$$

and parameters a*,* b*,* c*,* d*,* e*,* f*,* g *are determined* a priori*, such that the following LMIs are verified*

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 \end{bmatrix} < 0 \tag{8}$$

$$\Sigma_1 = \begin{bmatrix} \tau Q - F_{11} - F_{11}^T & \star \\ P - F_{21} + (\mathbf{A}+\mathbf{A}_d)^T F_{11}^T - F_{12}^T & \begin{pmatrix} R + F_{21}(\mathbf{A}+\mathbf{A}_d) - F_{22} \\ +(\mathbf{A}+\mathbf{A}_d)^T F_{21}^T - F_{22}^T \end{pmatrix} \\ -F_{31} + F_{12}^T & F_{31}(\mathbf{A}+\mathbf{A}_d) - F_{32} + F_{22}^T \\ -F_{41} - \mathbf{A}_d^T F_{11}^T + F_{12}^T & F_{41}(\mathbf{A}+\mathbf{A}_d) - F_{42} - \mathbf{A}_d^T F_{21}^T + F_{22}^T \\ -F_{51} + E_c^T \check{\mathbf{I}}^T F_{11}^T & -T\mathbf{K} + F_{51}(\mathbf{A}+\mathbf{A}_d) - F_{52} - E_c^T \check{\mathbf{I}}^T F_{21}^T \\ -F_{61} + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{11}^T & F_{61}(\mathbf{A}+\mathbf{A}_d) - F_{62} + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{21}^T \\ -F_{71} + \mathbf{B}_\omega^T F_{11}^T & F_{71}(\mathbf{A}+\mathbf{A}_d) - F_{72} + \mathbf{B}_\omega^T F_{21}^T \\ 0 & \mathbf{C}_z \end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix} \star & \star \\ \star & \star \\ -R + F_{32} + F_{32}^T & \star \\ F_{42} - \mathbf{A}_d^T F_{31}^T + F_{32}^T & -\frac{1}{\tau}Q - F_{41}\mathbf{A}_d + F_{42} - \mathbf{A}_d^T F_{41}^T + F_{42}^T \\ F_{52} + E_c^T \check{\mathbf{I}}^T F_{31}^T & -F_{51}\mathbf{A}_d + F_{52} + E_c^T \check{\mathbf{I}}^T F_{41}^T \\ -T_\tau \mathbf{K} + F_{62} + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{31}^T & -F_{61}\mathbf{A}_d + F_{62} + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{41}^T \\ F_{72} + \mathbf{B}_\omega^T F_{31}^T & -F_{71}\mathbf{A}_d + F_{72} + \mathbf{B}_\omega^T F_{41}^T \\ 0 & 0 \end{bmatrix}$$

$$\Sigma_3 = \begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ -2T + F_{51}\check{\mathbf{I}}E_c + E_c^T\check{\mathbf{I}}^T F_{51}^T & \star \\ F_{61}\check{\mathbf{I}}E_c + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{51}^T & -2T_\tau + F_{61}(\mathbf{B} + \check{\mathbf{I}}F_c) + (\mathbf{B} + \check{\mathbf{I}}F_c)^T F_{61}^T \\ F_{71}\check{\mathbf{I}}E_c + \mathbf{B}_\omega^T F_{51}^T & F_{71}(\mathbf{B} + \check{\mathbf{I}}F_c) + \mathbf{B}_\omega^T F_{61}^T \\ \mathbf{D}_z & 0 \end{bmatrix}$$

$$\Sigma_4 = \begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ -\alpha I_q + F_{71}\mathbf{B}_\omega + \mathbf{B}_\omega^T F_{71}^T & \star \\ 0 & -\gamma I_l \end{bmatrix}$$

then a static antiwindup compensation, E_c, $F_c \in \Re^{n_c \times m}$ as defined in (5) ensures that

1. *when $\omega(t) \neq 0$, the trajectories of the closed-loop system remains limited for all $\phi_\xi(\theta) \in \mathcal{C}_\tau^v$ at any initial conditions;*
2. $\|z(t)\|_2^2 \leq \gamma V(0) + \gamma\alpha\|\omega(t)\|_2^2$;
3. *if $\omega(t) = 0$, $\forall t \geq t_1 \geq 0$, $\xi(t)$ converges asymptotically to the origins.*

Proof: Assuring that closed loop origin of system (6) is globally asymptotically stable, through a Lyapunov-Krasovskii functional, implies in ensuring that $\forall\ \xi(t) \neq 0$, $V(t) > 0$, $\dot{V}(t) < 0$. By construction, we ensure $V(t) > 0$, $\forall\ \xi(t) \neq 0$. It remains to ensure $\dot{V}(t) < 0$, $\forall\ \xi(t) \neq 0$. Therefore, we define

$$\mathcal{J}(t) = \dot{V}(t) - \alpha\omega(t)^T\omega(t) + \frac{1}{\gamma}z(t)^T z(t)$$

Then, if $\mathcal{J}(t) < 0$, it follows that

$$\int_0^T \mathcal{J}(t)dt = V(T) - V(0) - \alpha\int_0^T \omega(t)^T\omega(t)dt + \frac{1}{\gamma}\int_0^T z(t)^T z(t)dt < 0 \tag{9}$$

Thus, the above relation implies that $V(T) \leq V(0) + \alpha\|\omega(t)\|_2^2 \leq \beta + (\alpha/\delta) \leq \mu^{-1}$. Hence, from (7) the satisfaction of (9) implies that $\xi(T)^T P\xi(T) \leq V(T) \leq \mu^{-1}$, that is, for all $T > 0$ the trajectories of the system do not leave the set $\varepsilon(P, \mu^{-1}) = \{\xi \in \Re^{n+n_c},\ \xi(t)^T P\xi(t) \leq \mu^{-1}\}$ for all $\omega(t)$ satisfying (3). Moreover, for $T \to +\infty$ (9) yields $\|z(t)\|_2^2 < \gamma\alpha\,\|\omega(t)\|_2^2 + \gamma V(0)$, thus, finite.

Now, from Lemma 3.1, provided that $\xi(t) \in \mathcal{S}(u_o)$ and $\xi(t-\tau) \in \mathcal{S}_\tau(u_o)$ an upper bound for $\mathcal{J}(t)$ becomes

$$\begin{aligned} \mathcal{J}(t) \leq\ & \dot{\xi}(t)^T P\xi(t) + \xi(t)^T P\dot{\xi}(t) + \xi(t)^T R\xi(t) - \xi(t-\tau)^T R\xi(t-\tau) + \tau\dot{\xi}(t)^T Q\dot{\xi}(t) \\ & - \int_{t-\tau}^t \dot{\xi}(\theta)^T Q\dot{\xi}(\theta)d\theta - \alpha\omega(t)^T\omega(t) + \frac{1}{\gamma}z(t)^T z(t) - \psi(y_c(t))^T T\mathbf{K}\xi(t) \\ & - \xi(t)^T\mathbf{K}^T T\psi(y_c(t)) - 2\psi(y_c(t))^T T\psi(y_c(t)) - \psi(y_c(t-\tau))^T T_\tau\mathbf{K}\xi(t-\tau) \\ & - \xi(t-\tau)^T\mathbf{K}^T T_\tau\psi(y_c(t-\tau)) - 2\psi(y_c(t-\tau))^T T_\tau\psi(y_c(t-\tau)) \end{aligned}$$

Applying Lemma 3.2 on the integral term on the right side of the above inequality and defining $\int_{t-\tau}^{t} \dot{\xi}(\theta)d\theta = \xi(t) - \xi(t-\tau)$, the following statement holds

$$-\int_{t-\tau}^{t} \dot{\xi}(\theta)^T Q \dot{\xi}(\theta) d\theta \leq -\left(\int_{t-\tau}^{t} \dot{\xi}(\theta)^T d\theta\right)^T \frac{1}{\tau} Q \left(\int_{t-\tau}^{t} \dot{\xi}(\theta) d\theta\right)$$

Thus, one has

$$\left(\int_{t-\tau}^{t} \dot{\xi}(\theta)^T d\theta\right)^T \frac{1}{\tau} Q \left(\int_{t-\tau}^{t} \dot{\xi}(\theta) d\theta\right) = \big(\xi(t) - \xi(t-\tau)\big)^T \frac{1}{\tau} Q \big(\xi(t) - \xi(t-\tau)\big)$$

and $\mathcal{J}(t)$ becomes bounded by

$$\begin{aligned}\mathcal{J}(t) \leq\ & \dot{\xi}(t)^T P \xi(t) + \xi(t)^T P \dot{\xi}(t) + \xi(t)^T R \xi(t) - \xi(t-\tau)^T R \xi(t-\tau) + \tau \dot{\xi}(t)^T Q \dot{\xi}(t) \\ & - (\xi(t) - \xi(t-\tau))^T \frac{1}{\tau} Q (\xi(t) - \xi(t-\tau)) - \alpha \omega(t)^T \omega(t) + \frac{1}{\gamma} z(t)^T z(t) \\ & - \psi(y_c(t))^T T \mathbf{K} \xi(t) - \xi(t)^T \mathbf{K}^T T \psi(y_c(t)) - 2\psi(y_c(t-\tau))^T T_\tau \psi(y_c(t-\tau)) \\ & - \psi(y_c(t-\tau))^T T_\tau \mathbf{K} \xi(t-\tau) - \xi(t-\tau)^T \mathbf{K}^T T_\tau \psi(y_c(t-\tau)) - 2\psi(y_c(t))^T T \psi(y_c(t))\end{aligned}$$

Define $\eta(t) \in \Re^{4\times(n+n_c)+2\times m+q}$ such that

$$\eta(t) = \begin{bmatrix} \dot{\xi}(t)^T & \xi(t)^T & \xi(t-\tau)^T & \big(\xi(t) - \xi(t-\tau)\big)^T & \psi(y_c(t))^T & \psi(y_c(t-\tau))^T & \omega(t)^T \end{bmatrix}^T$$

Since we want to ensure $\mathcal{J}(t) < 0$, it suffices ensuring its upper bound as negative definite. Once we can matricially represent it as $\eta(t)^T \mathbf{M}_1 \eta(t) < 0$, we now look forward to ensuring $\mathbf{M}_1 < 0$. $\mathbf{M}_1$ goes bellow.

$$\begin{bmatrix} \frac{\tau}{2}Q & 0 & 0 & 0 & 0 & 0 & 0 \\ P & \frac{1}{2}R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\tau}Q & 0 & 0 & 0 \\ 0 & -T\mathbf{K} & 0 & 0 & -T & 0 & 0 \\ 0 & 0 & -T_\tau \mathbf{K} & 0 & 0 & -T_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\alpha}{2} I_q \end{bmatrix} + (\star) + \frac{1}{\gamma} \mathcal{C}_z^T \mathcal{C}_z$$

where $\mathcal{C}_z = \begin{bmatrix} 0 & \mathbf{C}_z & 0 & 0 & \mathbf{D}_z & 0 & 0 \end{bmatrix}$.

Clearly, we cannot achieve $\mathbf{M}_1 < 0$, $\forall \eta(t)$. Thus, we apply Lemma 3.3, what shall ensure $\mathbf{M}_1 < 0$ only for the *de facto* trajectories of the closed loop system (6). We then define matrix $\mathcal{B}$ as

$$\mathcal{B} = \begin{bmatrix} -I & \mathbf{A} + \mathbf{A}_d & 0 & -\mathbf{A}_d & \check{\mathbf{I}} E_c & \mathbf{B} + \check{\mathbf{I}} F_c & \mathbf{B}_\omega \\ 0 & -I & I & I & 0 & 0 & 0 \end{bmatrix}$$

and from Lemma 3.3 we now look forward to ensuring $\mathbf{M}_1 + \mathbf{F}\mathcal{B} + \mathcal{B}^T \mathbf{F}^T < 0$.

Remark 4.1. *The latter condition has not yet been cast in the related literature, and the minimal constraining on the elements of* $\mathbf{F}$ *leads to potentially less conservative results.*

We choose $\mathbf{F}$ to be

$$\begin{bmatrix} F_{i1} & F_{i2} \\ \vdots & \vdots \end{bmatrix}, \quad i = 1, \ldots, 7$$

with $\mathbf{F}\mathcal{B}$ being

$$\begin{bmatrix} -F_{11} & F_{11}(\mathbf{A}+\mathbf{A}_d)-F_{12} & F_{12} & -F_{11}\mathbf{A}_d+F_{12} & F_{11}\check{\mathbf{I}}E_c & F_{11}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{11}\mathbf{B}_w \\ -F_{21} & F_{21}(\mathbf{A}+\mathbf{A}_d)-F_{22} & F_{22} & -F_{21}\mathbf{A}_d+F_{22} & F_{21}\check{\mathbf{I}}E_c & F_{21}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{21}\mathbf{B}_w \\ -F_{31} & F_{31}(\mathbf{A}+\mathbf{A}_d)-F_{32} & F_{32} & -F_{31}\mathbf{A}_d+F_{32} & F_{31}\check{\mathbf{I}}E_c & F_{31}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{31}\mathbf{B}_w \\ -F_{41} & F_{41}(\mathbf{A}+\mathbf{A}_d)-F_{42} & F_{42} & -F_{41}\mathbf{A}_d+F_{42} & F_{41}\check{\mathbf{I}}E_c & F_{41}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{41}\mathbf{B}_w \\ -F_{51} & F_{51}(\mathbf{A}+\mathbf{A}_d)-F_{52} & F_{52} & -F_{51}\mathbf{A}_d+F_{52} & F_{51}\check{\mathbf{I}}E_c & F_{51}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{51}\mathbf{B}_w \\ -F_{61} & F_{61}(\mathbf{A}+\mathbf{A}_d)-F_{62} & F_{62} & -F_{61}\mathbf{A}_d+F_{62} & F_{61}\check{\mathbf{I}}E_c & F_{61}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{61}\mathbf{B}_w \\ -F_{71} & F_{71}(\mathbf{A}+\mathbf{A}_d)-F_{72} & F_{72} & -F_{71}\mathbf{A}_d+F_{72} & F_{71}\check{\mathbf{I}}E_c & F_{71}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{71}\mathbf{B}_w \end{bmatrix}$$

Let $\mathbf{M}_2 = \mathbf{M}_1 + \mathbf{F}\mathcal{B} + \mathcal{B}^T\mathbf{F}^T$, with $\mathbf{M}_2$ given by

$$\left[\begin{matrix} \frac{\tau}{2}Q - F_{11} & F_{11}(\mathbf{A}+\mathbf{A}_d)-F_{12} & F_{12} & -F_{11}\mathbf{A}_d+F_{12} \\ P-F_{21} & \frac{1}{2}R - F_{22} + F_{21}(\mathbf{A}+\mathbf{A}_d) & F_{22} & -F_{21}\mathbf{A}_d+F_{22} \\ -F_{31} & F_{31}(\mathbf{A}+\mathbf{A}_d)-F_{32} & -\frac{1}{2}R+F_{32} & -F_{31}\mathbf{A}_d+F_{32} \\ -F_{41} & F_{41}(\mathbf{A}+\mathbf{A}_d)-F_{42} & F_{42} & -\frac{1}{2\tau}Q+F_{42}-F_{41}\mathbf{A}_d \\ -F_{51} & -T\mathbf{K}-F_{52}+F_{51}(\mathbf{A}+\mathbf{A}_d) & F_{52} & -F_{51}\mathbf{A}_d+F_{52} \\ -F_{61} & F_{61}(\mathbf{A}+\mathbf{A}_d)-F_{62} & -T_\tau\mathbf{K}+F_{62} & -F_{61}\mathbf{A}_d+F_{62} \\ -F_{71} & F_{71}(\mathbf{A}+\mathbf{A}_d)-F_{72} & F_{72} & -F_{71}\mathbf{A}_d+F_{72} \end{matrix}\right. \cdots$$

$$\cdots \left.\begin{matrix} F_{11}\check{\mathbf{I}}E_c & F_{11}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{11}\mathbf{B}_w \\ F_{21}\check{\mathbf{I}}E_c & F_{21}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{21}\mathbf{B}_w \\ F_{31}\check{\mathbf{I}}E_c & F_{31}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{31}\mathbf{B}_w \\ F_{41}\check{\mathbf{I}}E_c & F_{41}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{41}\mathbf{B}_w \\ -T+F_{51}\check{\mathbf{I}}E_c & F_{51}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{51}\mathbf{B}_w \\ F_{61}\check{\mathbf{I}}E_c & -T_\tau+F_{61}(\mathbf{B}+\check{\mathbf{I}}F_c) & F_{61}\mathbf{B}_w \\ F_{71}\check{\mathbf{I}}E_c & F_{71}(\mathbf{B}+\check{\mathbf{I}}F_c) & -\frac{\alpha}{2}+F_{71}\mathbf{B}_w \end{matrix}\right] + (\star) + \frac{1}{\gamma}\mathcal{C}_z^T\mathcal{C}_z$$

Now, we apply Schur complement, spliting the term $\frac{1}{\gamma}\mathcal{C}_z^T\mathcal{C}_z$. This leads us to LMI (8). This concludes the proof of theorem.

5. **Optimization: Minimization of Disturbance $\mathcal{L}_2$-gain.** In this section we minimize the $\mathcal{L}_2$-gain upper bound of $\omega(t)$ to $z(t)$. The initial conditions are null ($\phi_\xi(\theta) = 0$, $\forall t \in [-\tau, 0]$). Hence, for a non-null positive bound on the $\mathcal{L}_2$-norm of the admissible disturbances $\frac{1}{\delta}$, the idea is to minimize the upper bound for the $\mathcal{L}_2$-gain of $\omega(t)$ on $z(t)$. With $\beta = 0$, this can be obtained as follows

$$\underline{\gamma} = \min_{(8)} \gamma, \qquad \min_{\gamma < 1.20\cdot\underline{\gamma},(8)} \alpha \tag{10}$$

6. **Application on the AQM of a TCP/IP Router on an HSN.** In this section we illustrate our methodology. We assume a TCP/IP router, whose model is given by (1), has an AQM policy given by a PI controller, (4), borrowed from [16, 17]. To this PI we add our antiwindup compensator, resulting in (5). The state variables are the TCP congestion window and the router queue size. The input is the packet discard probability, and the disturbance is User Datagram Protocol (UDP) traffic. Setup has an instantaneous number of connections given by $N = 6000$, a maximum round trip time $R = 0.2$, a link bandwidth capacity of $C = 375000$, a steady state discard probability of $p_0 = 0.013$, and a queue size level of $q_0 = 1750$. For this setup, we assume $u_o = 0.987$. Bellow it is given the plant model explicitly.

$$A = \begin{bmatrix} -0.4000 & -6.6667\times 10^{-5} \\ 3.0000\times 10^4 & -5 \end{bmatrix}, \; B = \begin{bmatrix} -390.6250 \\ 0 \end{bmatrix}, \; B_\omega = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -0.4000 & -0.0001 \\ 0 & 0 \end{bmatrix}, \; C_y = C_z = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; D_z = 0$$
$$A_c = 0, \; A_{c,d} = 0, \; B_c = 1, \; C_c = 2.7654 \times 10^{-7}, \; D_c = 3.4567 \times 10^{-7}$$

For a specific time delay $\tau = 0.1$, corresponding static antiwindup compensator from (10) is

$$E_c = 5.1373 \times 10^5, \quad F_c = 2.4318 \times 10^6$$

The combination of parameters $a, \ldots, g$ was found by grid search, the obtained parameters were $a = 0.50$, $b = 0.50$, $c = 0.50$, $d = 0.50$, $e = -1.00$, $f = -1.00$, $g = 0$, resulting in $\mathcal{L}_2$-gain of $\sqrt{\gamma\alpha} = 0.2365$, with $\gamma = 0.2421$, $\alpha = 0.2311$. The dynamic antiwindup from [11] was unable to solve the plant proposed, so we could not compare the compensators.

In order to better visualize the results, we have performed a simulation where we apply a step function with amplitude of 1×10^8, being applied to the plant in the interval $[5, 15]$. This shall be interpreted as a burst traffic of UDP packets 100Mbps, lasting 10 seconds. The response of the system is depicted in Figure 1 and is compared amongst three different controllers: RED from [16], PI controller proposed in [17] and the same controller along with the antiwindup compensator. The top graphic depicts the queue size variation over the steady state level. The bottom graphic depicts the packet discard probability variation over the steady state level.

Note that while the UDP traffic burst is happenning the queue size grows at uncontrollable rates. The difference between the techniques manifests once the bursts ends, at 15s. Then our method is more successful in reseting the queue level to its initial level, as well as the discard probability. We can clearly see that the queue size tracks the steady state

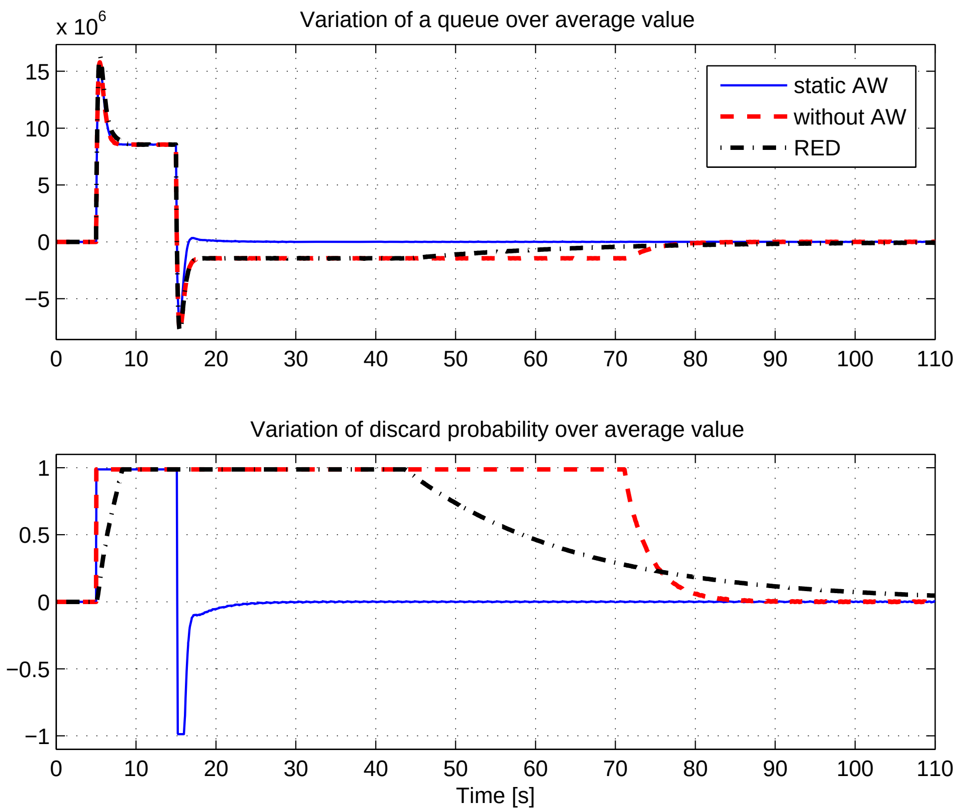

FIGURE 1. TCP/IP router queue size X discard probability

level much faster when our compensator is used. This encourages us to move forward on our research as to make this available for router vendors as a firmware patch.

7. **Conclusions.** In this work we have presented a methodology for synthesizing static antiwindup compensators for systems subject to time delays and input saturation. Conditions in an LMI form have been proposed in order to compute an antiwindup compensator, ensuring that the trajectories are bounded for $\mathcal{L}_2$-norm bounded disturbances, while ensuring the internal asymptotic stability of the closed loop system. We have applied the results on a TCP/IP router model with AQM given by a PI controller and compared with RED technique.

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