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DELAY-RANGE-DEPENDENT ROBUST STABILITY CRITERIA FOR UNCERTAIN NEUTRAL-TYPE LUR'E SYSTEMS WITH SECTOR-BOUNDED NONLINEARITIES

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Abstract. *This paper is concerned with the robust stability of uncertain neutral-type Lur'e system with interval time-varying delays and sector-bounded nonlinearities. By use of augmented Lyapunov-Krasovskii functional (LKF) and delay-partitioning techniques, delay-dependent robust stability criteria are proposed in terms of linear matrix inequalities (LMIs) without using the general free-weighting matrix method. The criteria are less conservative than some previous ones. Numerical examples are presented to show the effectiveness and merits of the proposed approach.*

Keywords: Robust stability, Neutral-type Lur'e system, Sector bounded nonlinearity, Wirtinger-type inequality, Interval time-varying delay

1. **Introduction.** The stability analysis of dynamic systems with time-delays has been one of the important issues since time-delay frequently appears in many physical and industrial systems such as biological systems, chemical systems, electronic systems, and network control systems [1]. It is well known that time-delay is often attributed as the major source of poor performance and instability. Therefore, the stability analysis of delayed linear systems has been widely investigated by many researchers [2,3], and many effective methods have been developed to derive less conservative delay-dependent stability criteria [4,5]. However, most systems are nonlinear in practical engineering, so it is necessary and significant to consider the stability problems of delayed nonlinear systems. And it is well known that many certain nonlinear systems such as Chua's circuit and Lorenz systems can be modeled as Lur'e systems [6], which consist of a feedback connection of a linear dynamical system and a nonlinearity satisfying the sector bounded condition. The stability for this type of nonlinear systems was first introduced by Lur'e [7], and has been considered extensively in [8]. On the other hand, many practical systems such as the water pipes, population ecology, and chemical reactors are modeled in the neutral time-delay system which contains delays in both its states and its derivatives of the states. Delay-dependent stability criteria for nominal and uncertain neutral-type Lur'e systems with constant time delays and sector bounded nonlinearities were presented in [9]. However, the delay-range of the time-varying delay associated with the state vector is assumed to vary from zero to an upper bound. In actual practice, however, the delay-range may have non-zero lower bound, and such systems are referred to as interval time-varying delay systems. For Lur'e systems with interval time-varying delay, a new robust stability criterion is presented in [10]. In checking the conservatism of stability criteria, an important index is to get maximum delay bounds guaranteeing the asymptotic stability of time-delay systems. Therefore, how to choose an LKF, augmented vectors and

derive the derivation of a stability condition from the time-derivative of such a functional plays key roles in enhancing the feasible regions of stability criteria. In this regard, new cross-term bounding technique [11], parameterized neutral model transformation method [12], free weighting matrices technique [13] had contributed to enhancing the feasible regions of stability criteria for systems with time-delays. Recently, in order to reduce the conservatism of stability criteria for time-delay systems, some remarkable results [14,15] have been presented. Another popular technique in reducing the conservatism of stability criteria is delay-partitioning one. Since Gu [16] firstly proposed this method, it is well recognized that delay-partitioning approach can increase the feasible region of stability criteria owing to the fact that this method can obtain more tighter upper bounds obtained by calculation of the time-derivative of LKF, which leads to less conservative results. However, when the number of delay-partitioning increases, the matrix formulation becomes more complex and the computational burden and time-consumption grow bigger. Therefore, it is strongly needed that some new methods should be studied in applying delay-partitioning approach.

Motivated by this discussed above, in this paper, we contribute to the robust stability for a class of uncertain neutral-type Lur'e systems with interval time-varying delays and sector bounded nonlinearities. As a tradeoff between time-consumption and improvement of the feasible region, a new augmented LKF, which fractionizes the delay interval into two subsections, is constructed. By utilizing Wirtinger-type inequality [17] and reciprocally convex approach [18], new delay-dependent robust sufficient stability criteria are derived in terms of LMIs which will be formulated as convex optimization algorithms which are amenable to computer solution [19]. Finally, three numerical examples are included to show that the proposed approach improves existing methods and gives better results for stability than those reported earlier.

2. **Problem Statement and Preliminaries.**

Consider the following uncertain mixed neutral and Lur'e system with interval time-varying delays and sector bounded nonlinearities:

$$\begin{aligned}
&\dot{x}(t) - C\dot{x}(t-\tau(t)) = (A+\Delta A)x(t) + (A_1+\Delta A_1)\,x(t-h(t)) + (B+\Delta B)f(\sigma(t)),\\
&\sigma(t) = H^T x(t), \quad \forall t \geq 0, \qquad\qquad (1)\\
&x(s) = \phi(s), \quad \dot{x}(s) = \dot{\phi}(s), \quad s \in [-\max(h,\tau), 0],
\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $\sigma(t) \in \mathbb{R}^m$ denote the state and output vectors of the system, respectively. $\phi(s) \in \mathbb{R}^n$ is a continuous initial function specified on $[-\max(h,\tau), 0]$ with known positive scalars h, τ. $f(\sigma(t)) \in \mathbb{R}^m$ is the nonlinear function in the feedback path, which is given by:

$$[f(\sigma(t))] = [f_1(\sigma_1(t)) \quad f_2(\sigma_2(t)) \quad \cdots \quad f_m(\sigma_m(t))]^T, \tag{2}$$

where $\sigma_i(t)$ is the i-th component of the output vector $\sigma(t)$, and each term $f_i(\sigma_i(t))$ $(i = 1, 2, \ldots, m)$ satisfies the finite sector condition:

$$f_i(\sigma_i(t)) \in K_{[0,k_i]} = \left\{ f_i(\sigma_i(t))\,|f_i(0) = 0, 0 < \sigma_i(t) f_i(\sigma_i(t)) \leq k_i \sigma_i^2(t), \sigma_i(t) \neq 0 \right\}, \tag{3}$$

with known positive scalars k_i $(i = 1, 2, \ldots, m)$, or the infinite sector condition:

$$f_i(\sigma_i(t)) \in K_{[0,\infty]} = \{f_i(\sigma_i(t))|f_i(0) = 0, \sigma_i(t) f_i(\sigma_i(t)) > 0, \sigma_i(t) \neq 0\}. \tag{4}$$

A, A_1, B, C and H are real constant matrices with appropriate dimensions, and $\Delta A(t)$, $\Delta A_1(t)$ and $\Delta B(t)$ denote real-valued matrix functions representing parameter uncertainties, which are assumed to satisfy:

$$[\Delta A(t) \quad \Delta A_1(t) \quad \Delta B(t)] = DF(t)[E_1 \quad E_2 \quad E_3], \tag{5}$$

where D, E_a, E_{a1}, and E_b are known constant matrices with appropriate dimensions, and $F(t)$ is an unknown matrix with Lebesgue-measurable elements and satisfies:

$$F^T(t)F(t) \leq I, \quad \forall t \geq 0. \tag{6}$$

The time-varying delays $h(t)$ and $\tau(t)$ are continuous-time functions and are assumed to satisfy the following conditions:

$$0 < h_l \leq h(t) \leq h_u, \quad \dot{h}(t) \leq h_d, \quad 0 \leq \tau(t) \leq \tau, \quad \dot{\tau}(t) \leq \tau_d < 1, \quad \forall t > 0. \tag{7}$$

The purpose of this paper is to investigate robustly asymptotic stability criteria for system (1) satisfying conditions (3) (or (4)), (5)-(7). In order to realize the purpose, the following lemmas are required.

Lemma 2.1. [18] *For given scalar α in the interval $(0,1)$, $n \times n$-matrix $R > 0$, two matrices W_1 and W_2 in $\mathbb{R}^{n\times m}$, for all vector ξ in $\mathbb{R}^m$, the function $\Theta(\alpha, R)$ is given by:*

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi.$$

Then, if there exists a matrix X in $\mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, the following inequality holds:

$$\min_{\alpha\in(0,1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

Lemma 2.2. [17] *For a given matrix $R > 0$, the following inequality holds for all continuously differentiable function ω in $[a,b] \to \mathbb{R}^n$:*

$$\int_a^b \dot{\omega}^T(s) R \dot{\omega}(s) ds \geq \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) + \frac{3}{b-a}\Omega^T R \Omega,$$

where $\Omega = \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s)ds$.

Lemma 2.3. [20] *Given matrices Γ, Ξ and $\Omega = \Omega^T$, the following inequality:*

$$\Omega + \Gamma F(\sigma)\Xi + \Xi^T F^T(\sigma)\Gamma^T < 0$$

holds for any $F(\sigma)$ satisfying $F^T(\sigma)F(\sigma) \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that:

$$\Omega + \varepsilon^{-1}\Gamma^T\Gamma + \varepsilon\Xi^T\Xi < 0.$$

3. Main Results.

3.1. **Finite sector condition.** In this section, we will give sufficient conditions under which system (1) is robustly asymptotically stable. Firstly, the following theorem offers a stability criterion for system (1) with a nonlinear function $f(\sigma(t))$ satisfying the finite sector condition (3) and time-varying delays $h(t)$ and $\tau(t)$ satisfying (7). For convenience, we define e_i $(i = 1, 2, \ldots, 11)$ as block entry matrices. For example, $e_3 = [0 \;\; 0 \;\; I \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$. The other notations for some vectors and matrices are defined as:

$$\zeta^T(t) = \left[x^T(t), \int_{t-h_l}^t x^T(s)ds, \int_{t-h_c}^{t-h_L} x^T(s)ds, \int_{t-h_u}^{t-h_c} x^T(s)ds\right],$$

$$\xi_1^T(t) = \Big[x^T(t), x^T(t-h_l), x^T(t-h_c), x^T(t-h(t)), x^T(t-h_u), \int_{t-h_l}^t x^T(s)ds,$$

$$\frac{1}{h(t)-h_l}\int_{t-h(t)}^{t-h_l} x^T(s)ds, \frac{1}{h_c-h(t)}\int_{t-h_c}^{t-h(t)} x^T(s)ds, \int_{t-h_u}^{t-h_c} x^T(s)ds, f^T(\sigma(t)),$$

$$\dot{x}^T(t-\tau(t))\Big],$$

$$\xi_2^T(t)=\Big[x^T(t), x^T(t-h_l), x^T(t-h_c), x^T(t-h(t)), x^T(t-h_u), \int_{t-h_l}^{t} x^T(s)ds,$$

$$\int_{t-h_c}^{t-h_l} x^T(s)ds, \tfrac{1}{h(t)-h_c}\int_{t-h(t)}^{t-h_c} x^T(s)ds, \tfrac{1}{h_u-h(t)}\int_{t-h_u}^{t-h(t)} x^T(s)ds, f^T(\sigma(t)), \dot{x}^T(t-\tau(t))\Big],$$

$$\Omega_0 = x(t) + x(t-h_l) - \tfrac{2}{h_l}\int_{t-h_l}^{t} x(s)ds,$$

$$\Omega_1 = x(t-h_c) + x(t-h_u) - \tfrac{2}{\tilde{h}}\int_{t-h_u}^{t-h_c} x(s)ds,$$

$$\Omega_2 = x(t-h_l) + x(t-h_c) - \tfrac{2}{\tilde{h}}\int_{t-h_c}^{t-h_l} x(s)ds,$$

$$\Pi_1^1 = \left[e_2^T - e_4^T \;\; e_2^T + e_4^T - 2e_7^T \;\; e_4^T - e_3^T \;\; e_4^T + e_3^T - 2e_8^T\right],$$

$$\Pi_1^2 = \left[e_1^T - e_2^T \;\; e_1^T + e_2^T - \tfrac{2}{h_l}e_6^T \;\; e_3^T - e_5^T \;\; e_3^T + e_5^T - \tfrac{2}{\tilde{h}}e_9^T\right],$$

$$\Pi_2^1 = \left[e_3^T - e_4^T \;\; e_3^T + e_4^T - 2e_8^T \;\; e_4^T - e_5^T \;\; e_4^T + e_5^T - 2e_9^T\right],$$

$$\Pi_2^2 = \left[e_1^T - e_2^T \;\; e_1^T + e_2^T - \tfrac{2}{h_l}e_6^T \;\; e_2^T - e_3^T \;\; e_2^T + e_3^T - \tfrac{2}{\tilde{h}}e_7^T\right],$$

$$\Gamma_1 = \left[e_1^T \;\; e_6^T \;\; (h(t)-h_l)e_7^T + (h_c-h(t))e_8^T \;\; e_9^T\right],$$

$$\Gamma_2 = \left[e_1^T \;\; e_6^T \;\; e_7^T \;\; (h(t)-h_c)e_8^T + (h_u-h(t))e_9^T\right],$$

$$\Psi = \left[\eta^T \;\; e_1^T - e_2^T \;\; e_2^T - e_3^T \;\; e_3^T - e_5^T\right], \quad \Theta = e_2Q_{12}e_3 - e_3Q_{12}e_5 + e_1^T HKSe_{10},$$

$$\tilde{D} = \left[D^T \;\; 0 \;\; 0 \;\; 0\right], \; \eta = [A \;\; 0 \;\; 0 \;\; A_1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; B \;\; C], \; E = [E_1 \;\; 0 \;\; 0 \;\; E_2 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; E_3 \;\; 0],$$

$$\Phi = Q_3 + h_l^2R_0 + \tilde{h}^2(R_1+R_2) + \tfrac{h_l^4}{4}Z_0 + h_{s1}^2Z_1 + h_{s2}^2Z_2, \quad \Delta = Q_1 + Q_2 + h_l^2W_0 + \tilde{h}^2(W_1+W_2),$$

$$G = diag\left\{\Delta, Q_{11}-Q_1, Q_{22}-Q_{11}, (h_d-1)Q_2, -Q_{22}, -W_0, 0, 0, 0, -S, (\tau_d-1)Q_3\right\}$$

$$- \left(h_le_1^T - e_6^T\right) Z_0 \left(h_le_1 - e_6\right).$$

Now, we have the following theorem.

Theorem 3.1. *System (1) satisfying conditions (3) and (7) is robustly asymptotically stable for given scalar values of* h_l*,* h_u*,* h_d*,* $\tau_d < 1$*,* $k_l > 0$ $(l = 1, 2, \ldots, m)$ *and there exist positive-definite matrices* $P = [P_{ij}]_{4\times4}$*,* Q_1*,* Q_2*,* Q_3*,* R_i*,* W_i*,* Z_i*,* $(i = 0, 1, 2)$*,* $\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}$*, and positive definite diagonal matrices* $S = diag\{s_1, s_2, \ldots, s_m\}$*,* $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$*, and scalars* $\varepsilon_r > 0$ $(r = 1, 2)$*, and any matrices* X*,* Y *such that the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Sigma}_1 + Sym\left\{\Theta + \tilde{\Gamma}_1P\Psi^T + e_{10}^T\Lambda H^T\eta\right\} & \eta^T\Phi & \tilde{\Gamma}_1P\tilde{D}^T + e_{10}\Lambda H^TD & \varepsilon_1E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \quad (8)$$

$$\begin{bmatrix} \hat{\Sigma}_1 + Sym\left\{\Theta + \hat{\Gamma}_1P\Psi^T + e_{10}^T\Lambda H^T\eta\right\} & \eta^T\Phi & \hat{\Gamma}_1P\tilde{D}^T + e_{10}\Lambda H^TD & \varepsilon_1E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \quad (9)$$

$$\begin{bmatrix} \tilde{\Sigma}_2 + Sym\left\{\Theta + \tilde{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \tilde{\Gamma}_2 P\tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_2 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad (10)$$

$$\begin{bmatrix} \hat{\Sigma}_2 + Sym\left\{\Theta + \hat{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \hat{\Gamma}_2 P\tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_2 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad (11)$$

where

$$\tilde{\Sigma}_1 = G - \Pi_1^1 \begin{bmatrix} \tilde{R}_1 & X \\ * & \tilde{R}_1 \end{bmatrix} \Pi_1^{1T} - \Pi_1^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} \Pi_1^{2T} - \tilde{h}^2 \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}$$
$$- \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 \\ * & Z_2 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix} - \tilde{h}^2 e_8^T W_1 e_8 - e_9^T W_2 e_9,$$

$$\hat{\Sigma}_1 = G - \Pi_1^1 \begin{bmatrix} \tilde{R}_1 & X \\ * & \tilde{R}_1 \end{bmatrix} \Pi_1^{1T} - \Pi_1^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} \Pi_1^{2T} - \tilde{h}^2 \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}$$
$$- \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 \\ * & Z_2 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix} - \tilde{h}^2 e_7^T W_1 e_7 - e_9^T W_2 e_9,$$

$$\tilde{\Sigma}_2 = G - \Pi_2^1 \begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix} \Pi_2^{1T} - \Pi_2^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_1 \end{bmatrix} \Pi_2^{2T} - \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix},$$
$$-\tilde{h}^2 \begin{bmatrix} e_1 \\ e_9 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 \\ * & Z_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_9 \end{bmatrix} - e_7^T W_1 e_7 - \tilde{h}^2 e_9^T W_2 e_9,$$

$$\hat{\Sigma}_2 = G - \Pi_2^1 \begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix} \Pi_2^{1T} - \Pi_2^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_1 \end{bmatrix} \Pi_2^{2T} - \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix},$$
$$-\tilde{h}^2 \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 \\ * & Z_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} - e_7^T W_1 e_7 - \tilde{h}^2 e_8^T W_2 e_8,$$

$$\tilde{\Gamma}_1 = [e_1^T \quad e_6^T \quad \tilde{h}e_8^T \quad e_9^T], \quad \hat{\Gamma}_1 = [e_1^T \quad e_6^T \quad \tilde{h}e_7^T \quad e_9^T],$$
$$\tilde{\Gamma}_2 = [e_1^T \quad e_6^T \quad e_7^T \quad \tilde{h}e_9^T], \quad \hat{\Gamma}_2 = [e_1^T \quad e_6^T \quad e_7^T \quad \tilde{h}e_8^T],$$
$$\tilde{R}_0 = \begin{bmatrix} R_0 & 0 \\ 0 & 3R_0 \end{bmatrix}, \; \tilde{R}_1 = \begin{bmatrix} R_1 & 0 \\ 0 & 3R_1 \end{bmatrix}, \; \tilde{R}_2 = \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix}.$$

Proof: Choose a Lyapunov functional candidate as

$$V = \sum_{i=1}^{5} V_i, \quad (12)$$

where

$$V_1 = \zeta^T(t) P \zeta(t),$$
$$V_2 = 2\sum_{i=1}^{m} \lambda_i \int_0^{\sigma_i(t)} f_i(\sigma) d\sigma,$$
$$V_3 = \int_{t-h_c}^{t-h_l} \begin{bmatrix} x(s) \\ x(s-\tilde{h}) \end{bmatrix}^T \bar{Q} \begin{bmatrix} x(s) \\ x(s-\tilde{h}) \end{bmatrix} ds + \int_{t-h_l}^{t} x^T(s) Q_1 x(s) ds$$

$$+\int_{t-h(t)}^{t} x^T(s)Q_2x(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^T(s)Q_3\dot{x}(s)ds,$$

$$V_4 = h_l\int_{-h_l}^{0}\int_{t+\theta}^{t} x^T(s)W_0x(s)dsd\theta + \tilde{h}\int_{-h_c}^{-h_l}\int_{t+\theta}^{t} x^T(s)W_1x(s)dsd\theta$$
$$+\tilde{h}\int_{-h_u}^{-h_c}\int_{t+\theta}^{t} x^T(s)W_2x(s)dsd\theta + h_l\int_{-h_l}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_0\dot{x}(s)dsd\theta$$
$$+\tilde{h}\int_{-h_c}^{-h_l}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \tilde{h}\int_{-h_u}^{-h_c}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta,$$

$$V_5 = \frac{h_l^2}{2}\int_{-h_l}^{0}\int_{\theta}^{0}\int_{t+\beta}^{t} \dot{x}^T(s)Z_0\dot{x}(s)dsd\beta d\theta + h_{s1}\int_{-h_c}^{-h_l}\int_{\theta}^{0}\int_{t+\beta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\beta d\theta$$
$$+ h_{s2}\int_{-h_u}^{-h_c}\int_{\theta}^{0}\int_{t+\beta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\beta d\theta.$$

$h_c = \dfrac{h_l + h_u}{2}$, $\tilde{h} = \dfrac{h_u - h_l}{2}$, $h_{s1} = \dfrac{h_c^2 - h_l^2}{2}$, $h_{s2} = \dfrac{h_u^2 - h_c^2}{2}$.

Here, we will consider the time-derivative of V for two cases, $h_l \le h(t) \le h_c$ and $h_c \le h(t) \le h_u$.

Case I: $h_l \le h(t) \le h_c$. From V_1 and V_2, we have their time-derivative as:

$$\dot{V}_1 = 2\zeta^T(t)P\dot{\zeta}(t) = 2\xi_1^T(t)\Gamma_1 P\left(\Psi^T + \tilde{D}^T F(t)E\right)\xi_1(t), \tag{13}$$

$$\dot{V}_2 = 2\sum_{i=1}^{m}\lambda_i f_i(\sigma_i(t))\, h_i^T\dot{x}(t) = 2\xi_1^T(t)e_{10}^T\Lambda H^T(\eta + DF(t)E)\,\xi_1(t). \tag{14}$$

Also, we obtain $\dot{V}_3$ as follows:

$$\dot{V}_3 \le \xi_1^T(t)\left\{\begin{bmatrix} e_2 \\ e_3 \end{bmatrix}^T \bar{Q}\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} - \begin{bmatrix} e_3 \\ e_5 \end{bmatrix}^T \bar{Q}\begin{bmatrix} e_3 \\ e_5 \end{bmatrix} + e_1^T(Q_1+Q_2)e_1 - e_2^TQ_1e_2 \right.$$
$$\left. -(1-h_d)e_4^TQ_2e_4 - (1-\tau_d)e_{11}^TQ_3e_{11}\right\}\xi_1(t) + \dot{x}^T(t)Q_3\dot{x}(t). \tag{15}$$

By calculation of $\dot{V}_4$, we get

$$\dot{V}_4 = h_l^2x^T(t)W_0x(t) + \tilde{h}^2x^T(t)(W_1+W_2)x(t) + h_l^2\dot{x}^T(t)R_0\dot{x}(t) + \tilde{h}^2\dot{x}^T(t)(R_1+R_2)\dot{x}(t)$$
$$- h_l\int_{t-h_l}^{t} x^T(s)W_0x(s)ds - \tilde{h}\int_{t-h_c}^{t-h_l} x^T(s)W_1x(s)ds - \tilde{h}\int_{t-h_u}^{t-h_c} x^T(s)W_2x(s)ds$$
$$- h_l\int_{t-h_l}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds - \tilde{h}\int_{t-h_c}^{t-h_l} \dot{x}^T(s)R_1\dot{x}(s)ds - \tilde{h}\int_{t-h_u}^{t-h_c} \dot{x}^T(s)R_2\dot{x}(s)ds. \tag{16}$$

When $h_l \le h(t) \le h_c$, we have

$$-h_l\int_{t-h_l}^{t} x^T(s)W_0x(s)ds \le -\xi_1^T(t)e_6^TW_0e_6\xi_1(t), \tag{17}$$

$$-\tilde{h}\int_{t-h_c}^{t-h_l} x^T(s)W_1x(s)ds \le -\xi_1^T(t)\begin{bmatrix} (h(t)-h_l)e_7 \\ (h_c-h(t))e_8 \end{bmatrix}^T\begin{bmatrix} W_1 & W_1 \\ * & W_1 \end{bmatrix}\begin{bmatrix} (h(t)-h_l)e_7 \\ (h_c-h(t))e_8 \end{bmatrix}\xi_1(t), \tag{18}$$

$$-\tilde{h}\int_{t-h_u}^{t-h_c} x^T(s)W_2x(s)ds \le -\xi_1^T(t)e_9^TW_2e_9\xi_1(t). \tag{19}$$

Using Lemma 2.1 and Lemma 2.2, we have

$$-h_l\int_{t-h_l}^{t}\dot{x}^T(s)R_0\dot{x}(s)ds \leq -\left(x(t)-x(t-h_l)\right)^T R_0(x(t)-x(t-h_l)) - 3\Omega_0^T R_0\Omega_0, \quad (20)$$

$$\begin{aligned}-\tilde{h}\int_{t-h_c}^{t-h_l}\dot{x}^T(s)R_1\dot{x}(s)ds =& -\tilde{h}\int_{t-h(t)}^{t-h_l}\dot{x}^T(s)R_1\dot{x}(s)ds - \tilde{h}\int_{t-h_c}^{t-h(t)}\dot{x}^T(s)R_1\dot{x}(s)ds \\ \leq& -\xi_1^T(t)\Pi_1^1\begin{bmatrix}\tilde{R}_1 & X\\ * & \tilde{R}_1\end{bmatrix}\Pi_1^{1T}\xi_1(t), \end{aligned}\quad (21)$$

$$\begin{aligned}-\tilde{h}\int_{t-h_u}^{t-h_c}\dot{x}^T(s)R_2\dot{x}(s)ds \leq& -\left(x(t-h_c)-x(t-h_u)\right)^T R_2(x(t-h_c)-x(t-h_u)) \\ &- 3\Omega_1^T R_2\Omega_1, \end{aligned}\quad (22)$$

where X is any matrix satisfying $\begin{bmatrix}\tilde{R}_1 & X\\ * & \tilde{R}_1\end{bmatrix}\geq 0$.

From Equations (16)-(22), an estimation of $\dot{V}_4$ can be

$$\begin{aligned}\dot{V}_4 \leq& \xi_1^T(t)\left(e_1^T\left[h_l^2W_0+\tilde{h}^2(W_1+W_2)\right]e_1 - e_6^TW_0e_6 - e_9^TW_2e_9 - \Pi_1^1\begin{bmatrix}\tilde{R}_1 & X\\ * & \tilde{R}_1\end{bmatrix}\Pi_1^{1T}\right.\\ &\left.-\Pi_1^2\begin{bmatrix}\tilde{R}_0 & 0\\ 0 & \tilde{R}_2\end{bmatrix}\Pi_1^{2T} - \begin{bmatrix}(h(t)-h_l)e_7\\ (h_c-h(t))e_8\end{bmatrix}^T\begin{bmatrix}W_1 & W_1\\ * & W_1\end{bmatrix}\begin{bmatrix}(h(t)-h_l)e_7\\ (h_c-h(t))e_8\end{bmatrix}\right)\xi_1(t)\\ &+\dot{x}^T(t)\left[h_l^2R_0+\tilde{h}^2(R_1+R_2)\right]\dot{x}(t). \end{aligned}\quad (23)$$

By calculation of $\dot{V}_5$, we get

$$\begin{aligned}\dot{V}_5 \leq& \dot{x}^T(t)\left(\frac{h_l^4}{4}Z_0+h_{s1}^2Z_1+h_{s2}^2Z_2\right)\dot{x}(t) - \xi_1^T(t)\left\{\begin{bmatrix}h_le_1\\ e_6\end{bmatrix}^T\begin{bmatrix}Z_0 & -Z_0\\ * & Z_0\end{bmatrix}\begin{bmatrix}h_le_1\\ e_6\end{bmatrix}\right.\\ &+\begin{bmatrix}\tilde{h}e_1\\ (h(t)-h_l)e_7\\ (h_c-h(t))e_8\end{bmatrix}^T\begin{bmatrix}Z_1 & -Z_1 & -Z_1\\ * & Z_1 & Z_1\\ * & * & Z_1\end{bmatrix}\begin{bmatrix}\tilde{h}e_1^T\\ (h(t)-h_l)e_7^T\\ (h_c-h(t))e_8^T\end{bmatrix}\\ &\left.+\begin{bmatrix}\tilde{h}e_1\\ e_9\end{bmatrix}^T\begin{bmatrix}Z_2 & -Z_2\\ * & Z_2\end{bmatrix}\begin{bmatrix}\tilde{h}e_1\\ e_9\end{bmatrix}\right\}\xi_1(t). \end{aligned}\quad (24)$$

If the nonlinear function $f(\sigma(t))$ in the feedback path satisfying the finite sector conditions (3), for any $s_i \geq 0$, $i=1,2,\ldots,m$, it follows from (3) that

$$s_if_i(\sigma_i(t))\left[k_ih_i^Tx(t)-f_i(\sigma_i(t))\right]\geq 0, \quad i=1,2,\ldots,m,$$

which is equivalent to

$$2\left[x^T(t)HKSf(\sigma(t))-f^T(\sigma(t))Sf(\sigma(t))\right]\geq 0, \quad (25)$$

where $S=diag\{s_1,s_2,\ldots,s_m\}$, $K=diag\{k_1,k_2,\ldots,k_m\}$.

Then, it follows from inequalities (13)-(15), (23)-(25) that

$$\dot{V}\leq\sum_{i=1}^{5}\dot{V}_i+2\left[x^T(t)HKSf(\sigma(t))-f^T(\sigma(t))Sf(\sigma(t))\right]\leq\xi_1^T(t)\Upsilon_1\xi_1(t), \quad (26)$$

where

$$\Upsilon_1=\Sigma_1+Sym\left\{\Theta+\Gamma_1P\left(\Psi^T+\tilde{D}^TF(t)E\right)+e_{10}^T\Lambda H^T(\eta+DF(t)E)\right\}$$

$$+ (\eta + DF(t)E)^T \Phi(\eta + DF(t)E),$$

$$\Sigma_1 = G - \Pi_1^1 \begin{bmatrix} \tilde{R}_1 & X \\ * & \tilde{R}_1 \end{bmatrix} \Pi_1^{1T} - \Pi_1^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} \Pi_1^{2T} - \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 \\ * & Z_2 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_9 \end{bmatrix}$$
$$- \begin{bmatrix} (h(t)-h_l)e_7 \\ (h_c - h(t))e_8 \end{bmatrix}^T \begin{bmatrix} W_1 & W_1 \\ * & W_1 \end{bmatrix} \begin{bmatrix} (h(t)-h_l)e_7 \\ (h_c - h(t))e_8 \end{bmatrix} - e_9^T W_2 e_9$$
$$- \begin{bmatrix} \tilde{h}e_1 \\ (h(t)-h_l)e_7 \\ (h_c - h(t))e_8 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 & -Z_1 \\ * & Z_1 & Z_1 \\ * & * & Z_1 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ (h(t)-h_l)e_7 \\ (h_c - h(t))e_8 \end{bmatrix}.$$

If $\Upsilon_1 < 0$, then $\dot{V} < 0$, and system (1) is robustly stable. By Schur complement $\Upsilon_1 < 0$ is equivalent to

$$\begin{bmatrix} \Sigma_1 + Sym\left\{\Theta + \Gamma_1 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} + J_1 F(t) E + E^T F^T(t) J_1^T < 0, \quad (27)$$

where $J_1 = \left[\tilde{D} P \Gamma_1^T + D^T H \Lambda e_{10} \quad D^T \Phi\right]^T$.

It follows from Lemma 2.3 that Equation (27) holds if and only if there exists positive scalar $\varepsilon_1 > 0$ such that the following matrix inequalities hold:

$$\begin{bmatrix} \Sigma_1 + Sym\left\{\Theta + \Gamma_1 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} + \varepsilon_1^{-1} J_1 J_1^T + \varepsilon_1 E^T E < 0. \quad (28)$$

By Schur complement Equation (28) is equivalent to

$$\begin{bmatrix} \Sigma_1 + Sym\left\{\Theta + \Gamma_1 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi & \Gamma_1 P \tilde{D}^T + e_{10} \Lambda H^T D & \varepsilon_1 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0. \quad (29)$$

The left side of the above inequality is affine and consequently convex, with respect to $h(t) \in [h_l, h_c]$, so Equation (29) is equivalent to Equations (8) and (9). Therefore, if Equations (8) and (9) hold, then system (1) is robustly stable for $h_l \le h(t) \le h_c$.

Case II: $h_c \le h(t) \le h_u$.

For this case, the time-derivatives of V_1 and V_2 are:

$$\dot{V}_1 = 2\zeta^T(t) P \dot{\zeta}(t) = 2\xi_2^T(t) \Gamma_2 P \left(\Psi^T + \hat{D}^T F(t) E\right) \xi_2(t), \quad (30)$$

$$\dot{V}_2 = 2\sum_{i=1}^{m} \lambda_i f_i(\sigma_i(t)) h_i^T \dot{x}(t) = 2\xi_2^T(t) e_{10}^T \Lambda H^T (\eta + DF(t)E) \xi_2(t). \quad (31)$$

We can now obtain the calculation result of $\dot{V}_3$ as follows:

$$\dot{V}_3 \le \xi_2^T(t) \left\{ \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}^T \bar{Q} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} - \begin{bmatrix} e_3 \\ e_5 \end{bmatrix}^T \bar{Q} \begin{bmatrix} e_3 \\ e_5 \end{bmatrix} + e_1^T (Q_1 + Q_2) e_1 - e_2^T Q_1 e_2 \right.$$
$$\left. - (1 - h_d) e_4^T Q_2 e_4 - (1 - \tau_d) e_{11}^T Q_3 e_{11} \right\} \xi_2(t) + \dot{x}^T(t) Q_3 \dot{x}(t). \quad (32)$$

By calculation of $\dot{V}_4$, we get

$$\dot{V}_4 = h_l^2 x^T(t) W_0 x(t) + \tilde{h}^2 x^T(t) (W_1 + W_2) x(t) + h_l^2 \dot{x}^T(t) R_0 \dot{x}(t) + \tilde{h}^2 \dot{x}^T(t) (R_1 + R_2) \dot{x}(t)$$
$$- h_l \int_{t-h_l}^{t} x^T(s) W_0 x(s) ds - \tilde{h} \int_{t-h_c}^{t-h_l} x^T(s) W_1 x(s) ds - \tilde{h} \int_{t-h_u}^{t-h_c} x^T(s) W_2 x(s) ds$$

$$- h_l\int_{t-h_l}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds - \tilde{h}\int_{t-h_c}^{t-h_l} \dot{x}^T(s)R_1\dot{x}(s)ds - \tilde{h}\int_{t-h_u}^{t-h_c} \dot{x}^T(s)R_2\dot{x}(s)ds. \quad (33)$$

When $h_c \leq h(t) \leq h_u$, we have

$$-h_l\int_{t-h_l}^{t} x^T(s)W_0x(s)ds \leq -\xi_2^T(t)e_6^TW_0e_6\xi_2(t), \quad (34)$$

$$-\tilde{h}\int_{t-h_c}^{t-h_l} x^T(s)W_1x(s)ds \leq -\xi_2^T(t)e_7^TW_1e_7\xi_2(t), \quad (35)$$

$$-\tilde{h}\int_{t-h_u}^{t-h_c} x^T(s)W_2x(s)ds \leq -\xi_2^T(t)\begin{bmatrix} (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix}^T \begin{bmatrix} W_2 & W_2 \\ * & W_2 \end{bmatrix} \begin{bmatrix} (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix}\xi_2(t). \quad (36)$$

Using Lemma 2.1 and Lemma 2.2, we have

$$-h_l\int_{t-h_l}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds \leq -(x(t)-x(t-h_l))^TR_0(x(t)-x(t-h_l)) - 3\Omega_0^TR_0\Omega_0, \quad (37)$$

$$\begin{aligned} -\tilde{h}\int_{t-h_c}^{t-h_l} \dot{x}^T(s)R_1\dot{x}(s)ds \leq &-(x(t-h_l)-x(t-h_c))^TR_1(x(t-h_l)-x(t-h_c)) \\ &- 3\Omega_2^TR_1\Omega_2, \end{aligned} \quad (38)$$

$$\begin{aligned} -\tilde{h}\int_{t-h_u}^{t-h_c} \dot{x}^T(s)R_2\dot{x}(s)ds =& -\tilde{h}\int_{t-h(t)}^{t-h_c}\dot{x}^T(s)R_2\dot{x}(s)ds - \tilde{h}\int_{t-h_u}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)ds \\ \leq& -\xi_2^T(t)\Pi_2^1\begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix}\Pi_2^{1T}\xi_2(t), \end{aligned} \quad (39)$$

where Y is any matrix satisfying $\begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix} \geq 0$.

From Equations (33)-(39), an estimation of V_4 can be

$$\begin{aligned} \dot{V}_4 \leq \xi_2^T(t)\Bigg(& e_1^T\left[h_l^2W_0 + \tilde{h}^2(W_1+W_2)\right]e_1 - e_6^TW_0e_6 - e_7^TW_1e_7 - \Pi_2^2\begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_1 \end{bmatrix}\Pi_2^{2T} \\ & - \Pi_2^1\begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix}\Pi_2^{1T} - \begin{bmatrix} (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix}^T\begin{bmatrix} W_2 & W_2 \\ * & W_2 \end{bmatrix}\begin{bmatrix} (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix}\Bigg)\xi_2(t) \\ & + \dot{x}^T(t)\left[h_l^2R_0 + \tilde{h}^2(R_1+R_2)\right]\dot{x}(t). \end{aligned} \quad (40)$$

By calculation of $\dot{V}_5$, we get

$$\begin{aligned} \dot{V}_5 \leq\ & \dot{x}^T(t)\left(\frac{h_l^4}{4}Z_0 + h_{s1}^2Z_1 + h_{s2}^2Z_2\right)\dot{x}(t) - \xi_2^T(t)\left(\begin{bmatrix} h_le_1 \\ e_6 \end{bmatrix}^T\begin{bmatrix} Z_0 & -Z_0 \\ * & Z_0 \end{bmatrix}\begin{bmatrix} h_le_1 \\ e_6 \end{bmatrix}\right. \\ & + \begin{bmatrix} \tilde{h}e_1 \\ (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix}^T\begin{bmatrix} Z_2 & -Z_2 & -Z_2 \\ * & Z_2 & Z_2 \\ * & * & Z_2 \end{bmatrix}\begin{bmatrix} \tilde{h}e_1 \\ (h(t)-h_c)e_8 \\ (h_u-h(t))e_9 \end{bmatrix} \\ & \left. + \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix}^T\begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix}\begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix}\right)\xi_2(t). \end{aligned} \quad (41)$$

If the nonlinear function $f(\sigma(t))$ in the feedback path satisfies the finite sector conditions (3), for any $s_i \geq 0$, $i = 1, 2, \ldots, m$, it follows from (3) that

$$s_i f_i(\sigma_i(t))\left[k_i h_i^T x(t) - f_i(\sigma_i(t))\right] \geq 0, \quad i = 1, 2, \ldots, m,$$

which is equivalent to

$$2\left[x^T(t)HKSf(\sigma(t)) - f^T(\sigma(t))Sf(\sigma(t))\right] \geq 0, \tag{42}$$

where $S = diag\{s_1, s_2, \ldots, s_m\}$, $K = diag\{k_1, k_2, \ldots, k_m\}$.

Then, it follows from inequalities (30)-(32), (40)-(42) that

$$\dot{V} \leq \sum_{i=1}^{5} \dot{V}_i + 2\left[x^T(t)HKSf(\sigma(t)) - f^T(\sigma(t))Sf(\sigma(t))\right] \leq \xi_2^T(t)\Upsilon_2\xi_2(t), \tag{43}$$

where

$$\begin{aligned}
\Upsilon_2 &= \Sigma_2 + Sym\left\{\Theta + \Gamma_1 P\left(\Psi^T + \tilde{D}^T F(t)E\right) + e_{10}^T \Lambda H^T(\eta + DF(t)E)\right\} \\
&\quad + (\eta + DF(t)E)^T \Phi(\eta + DF(t)E), \\
\Sigma_2 &= G - \Pi_2^1 \begin{bmatrix} \tilde{R}_2 & Y \\ * & \tilde{R}_2 \end{bmatrix} \Pi_2^{1T} - \Pi_2^2 \begin{bmatrix} \tilde{R}_0 & 0 \\ 0 & \tilde{R}_1 \end{bmatrix} \Pi_2^{2T} - \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix}^T \begin{bmatrix} Z_1 & -Z_1 \\ * & Z_1 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ e_7 \end{bmatrix} \\
&\quad - \begin{bmatrix} (h(t) - h_c)e_8 \\ (h_u - h(t))e_9 \end{bmatrix}^T \begin{bmatrix} W_2 & W_2 \\ * & W_2 \end{bmatrix} \begin{bmatrix} (h(t) - h_c)e_8 \\ (h_u - h(t))e_9 \end{bmatrix} - e_7^T W_1 e_7 \\
&\quad - \begin{bmatrix} \tilde{h}e_1 \\ (h(t) - h_c)e_8 \\ (h_u - h(t))e_9 \end{bmatrix}^T \begin{bmatrix} Z_2 & -Z_2 & -Z_2 \\ * & Z_2 & Z_2 \\ * & * & Z_2 \end{bmatrix} \begin{bmatrix} \tilde{h}e_1 \\ (h(t) - h_c)e_8 \\ (h_u - h(t))e_9 \end{bmatrix}.
\end{aligned}$$

If $\Upsilon_2 < 0$, then $\dot{V} < 0$, and system (1) is robustly stability. By Schur complement $\Upsilon_2 < 0$ is equivalent to

$$\begin{bmatrix} \Sigma_2 + Sym\left\{\Theta + \Gamma_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} + J_2 F(t)E + E^T F^T(t) J_2^T < 0, \tag{44}$$

where $J_2 = \left[\tilde{D}P\Gamma_2^T + D^T H\Lambda e_{10} \quad D^T\Phi\right]^T$.

It follows from Lemma 2.3 that Equation (44) holds if and only if there exists positive scalar $\varepsilon_2 > 0$ such that the following matrix inequalities hold:

$$\begin{bmatrix} \Sigma_2 + Sym\left\{\Theta + \Gamma_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} + \varepsilon_2^{-1} J_2 J_2^T + \varepsilon_2 E^T E < 0, \tag{45}$$

By Schur complement Equation (45) is equivalent to

$$\begin{bmatrix} \Sigma_2 + Sym\left\{\Theta + \Gamma_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi & \Gamma_2 P\tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_2 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0. \tag{46}$$

The left side of the above inequality is affine and consequently convex, with respect to $h(t) \in [h_c, h_u]$, so Equation (46) is equivalent to Equations (10) and (11). Therefore, if Equations (10) and (11) hold, then system (1) is robustly stable for $h_c \leq h(t) \leq h_u$.

The proof of Theorem 3.1 is completed.

Remark 3.1. *By iteratively solving the LMIs given in Theorem 3.1 with respect to h_u for fixed h_l, τ_d, h_d the maximum upper bound of the time delay h_u can be found for guaranteeing asymptotic stability of system (1).*

Remark 3.2. *Recently, the reciprocally convex optimization technique and Wirtinger inequality to reduce the conservatism of stability criteria for systems with time-varying delays were proposed in [17]. Motivated by this work, the proposed methods of [17] were applied to the delay-partitioning method as shown in Equations (20)-(22) and (37)-(39).*

Remark 3.3. *The robust stability criteria investigated in Theorem 3.1 remove free-weighting matrices, which will make the computation quite complex. The stability criteria involve much less decision variable than those in [21]. So, Theorem 3.1 may be more useful than the theorem in [21].*

Remark 3.4. *If the nonlinear function $f(\sigma(t))$ in the feedback path satisfies the infinite sector conditions (4), for any $s_i \geq 0$, $i = 1, 2, \ldots, m$, it follows from (4) that*

$$s_i f_i(\sigma_i(t)) h_i^T x(t) \geq 0, \qquad i = 1, 2, \ldots, m,$$

which is equivalent to

$$2x^T(t)HSf(\sigma(t)) \geq 0, \tag{47}$$

where $S = diag\{s_1, s_2, \ldots, s_m\}$.

3.2. **Infinite sector condition.** Nextly, when the nonlinear function satisfies (4), the following corollary can be obtained.

Corollary 3.1. *System (1) satisfying conditions (4) and (7) is robustly asymptotically stable for given scalar values of h_l, h_u, h_d and $\tau_d < 1$, and there exist positive-definite matrices $P = [P_{ij}]_{4\times 4}$, Q_1, Q_2, Q_3, R_i, W_i, Z_i, $(i = 0, 1, 2)$, $\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}$, and positive definite diagonal matrices $S = diag\{s_1, s_2, \ldots, s_m\}$, $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, and scalars $\varepsilon_r > 0$ $(r = 1, 2)$, and any matrices X, Y such that the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Sigma}_1 + Sym\left\{\tilde{\Theta} + \tilde{\Gamma}_1 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \tilde{\Gamma}_1 P \tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_1 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \tag{48}$$

$$\begin{bmatrix} \hat{\Sigma}_1 + Sym\left\{\tilde{\Theta} + \hat{\Gamma}_1 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \hat{\Gamma}_1 P \tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_1 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \tag{49}$$

$$\begin{bmatrix} \tilde{\Sigma}_2 + Sym\left\{\tilde{\Theta} + \tilde{\Gamma}_2 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \tilde{\Gamma}_2 P \tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_2 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{50}$$

$$\begin{bmatrix} \hat{\Sigma}_2 + Sym\left\{\tilde{\Theta} + \hat{\Gamma}_2 P \Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T\Phi & \hat{\Gamma}_2 P \tilde{D}^T + e_{10}\Lambda H^T D & \varepsilon_2 E^T \\ * & -\Phi & \Phi D & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{51}$$

where $\tilde{\Theta} = e_2 Q_{12} e_3 - e_3 Q_{12} e_5 + e_1^T HSe_{10}$, $\tilde{\Sigma}_1$, $\hat{\Sigma}_1$, $\tilde{\Sigma}_2$, $\hat{\Sigma}_2$, $\tilde{\Gamma}_1$, $\hat{\Gamma}_1$, $\tilde{\Gamma}_2$ and $\hat{\Gamma}_2$ are defined in Theorem 3.1.

Proof: The proof is similar to the proof of Theorem 3.1.

Next, we extend the obtained stability conditions to the nominal form of system (1) without uncertainties:

$$\begin{aligned} &\dot{x}(t) - C\dot{x}(t-\tau(t)) = Ax(t) + A_1(t))x(t-h(t)) + Bf(\sigma(t)), \\ &\sigma(t) = H^T x(t), \quad \forall t \ge 0, \\ &x(s) = \phi(s), \quad \dot{x}(s) = \dot{\phi}(s), \quad s \in [-\max(h,\tau), 0]. \end{aligned} \tag{52}$$

We have the following theorem.

Theorem 3.2. *System (52) satisfying conditions (3) and (7) is asymptotically stable for given scalar values of* h_l, h_u, h_d, $\tau_d < 1$ *and* $k_l > 0$ $(l = 1, 2, \dots, m)$, *and there exist positive-definite matrices* $P = [P_{ij}]_{4\times 4}$, Q_1, Q_2, Q_3, R_i, W_i, Z_i, $(i = 0, 1, 2)$, $\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}$, *and positive definite diagonal matrices* $S = diag\{s_1, s_2, \dots, s_m\}$, $\Lambda = diag\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, *and any matrices* X, Y *such that the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Sigma}_1 + Sym\left\{\Theta + \tilde{\Gamma}_1 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{53}$$

$$\begin{bmatrix} \hat{\Sigma}_1 + Sym\left\{\Theta + \hat{\Gamma}_1 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{54}$$

$$\begin{bmatrix} \tilde{\Sigma}_2 + Sym\left\{\Theta + \tilde{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{55}$$

$$\begin{bmatrix} \hat{\Sigma}_2 + Sym\left\{\Theta + \hat{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{56}$$

where $\tilde{\Sigma}_1$, $\hat{\Sigma}_1$, $\tilde{\Sigma}_2$, $\hat{\Sigma}_2$, $\tilde{\Gamma}_1$, $\hat{\Gamma}_1$, $\tilde{\Gamma}_2$ *and* $\hat{\Gamma}_2$ *are defined in Theorem 3.1.*

Next, when the nonlinear function satisfies (4), the following corollary is obtained.

Corollary 3.2. *System (52) satisfying conditions (4) and (7) is asymptotically stable for given scalar values of* h_l, h_u, h_d *and* $\tau_d < 1$, *and there exist positive-definite matrices* $P = [P_{ij}]_{4\times 4}$, Q_1, Q_2, Q_3, R_i, W_i, Z_i, $(i = 0, 1, 2)$, $\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}$, *and positive definite diagonal matrices* $S = diag\{s_1, s_2, \dots, s_m\}$, $\Lambda = diag\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, *and any matrices* X, Y *such that the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Sigma}_1 + Sym\left\{\tilde{\Theta} + \tilde{\Gamma}_1 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{57}$$

$$\begin{bmatrix} \hat{\Sigma}_1 + Sym\left\{\tilde{\Theta} + \hat{\Gamma}_1 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{58}$$

$$\begin{bmatrix} \tilde{\Sigma}_2 + Sym\left\{\tilde{\Theta} + \tilde{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{59}$$

$$\begin{bmatrix} \hat{\Sigma}_2 + Sym\left\{\tilde{\Theta} + \hat{\Gamma}_2 P\Psi^T + e_{10}^T \Lambda H^T \eta\right\} & \eta^T \Phi \\ * & -\Phi \end{bmatrix} < 0, \tag{60}$$

where $\tilde{\Theta} = e_2 Q_{12} e_3 - e_3 Q_{12} e_5 + e_1^T H S e_{10}$, $\tilde{\Sigma}_1$, $\hat{\Sigma}_1$, $\tilde{\Sigma}_2$, $\hat{\Sigma}_2$, $\tilde{\Gamma}_1$, $\hat{\Gamma}_1$, $\tilde{\Gamma}_2$ *and* $\hat{\Gamma}_2$ *are defined in Theorem 3.1.*

4. **Numerical Examples.** In this section, we give three examples to show that the results proposed in this paper are less conservative than the existing ones.

Example 4.1. *Consider the nominal neutral-type Lur'e systems (52) with*

$$A = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix},\ B = \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix},\ C = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},\ H = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$

*The maximum admissible upper bounds (**MAUBs**) of h with different h_d and τ_d for the nominal system (52) with values of $K_{[0,\infty]}$ by using Corollary 4.1 along with the existing results are listed in Table 1. From Table 1, it is found that the **MAUBs** obtained by Corollary 4.1 are better than the previous results.*

TABLE 1. The **MAUBs** for different h_d and τ_d (Example 4.1)

τ_d	Methods\\h_d	0.2	0.4	0.6	0.8
	[21]	2.9997	2.1012	1.7307	1.3359
$\tau_d = 0.1$	[22]	3.0717	2.1171	1.7301	1.3367
	Corollary 4.1	3.1495	2.1637	1.7502	1.7153
	[21]	2.4563	1.8009	1.4823	1.1157
$\tau_d = 0.5$	[22]	2.5131	1.8146	1.4951	1.1168
	Corollary 4.1	2.5706	1.8512	1.5423	1.5261
	[21]	0.1130	0.1105	0.1105	0.1105
$\tau_d = 0.9$	[22]	0.1227	0.1197	0.1197	0.1197
	Corollary 4.1	0.2968	0.2814	0.2734	0.2714

Example 4.2. *Consider the Chua's circuit example and the system equation is given by*

$$\begin{aligned} \dot{x}(t) &= \alpha(y(t) - h(x(t))), \\ \dot{y}(t) &= x(t) - y(t) + z(t), \\ \dot{z}(t) &= -\beta y(t), \end{aligned}$$

with nonlinear characteristic

$$h(x(t)) = m_1 x(t) + \frac{1}{2}(m_0 - m_1)[|x(t) + c| - |x(t) - c|],$$

with parameters $m_0 = -\frac{1}{7}$, $m_1 = \frac{2}{7}$, $\alpha = 9$, $\beta = 14.28$ *and* $c = 1$. *The system can be represented in normal Lur'e system framework (52) with*

$$A = \begin{bmatrix} -\alpha m_1 - 1 & \alpha & 0 \\ 1 & -2 & 1 \\ 0 & -\beta & -1 \end{bmatrix},\ A_1 = \begin{bmatrix} -6.0029 & 0 & 0 \\ -1.3367 & 0 & 0 \\ 2.1264 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -\alpha(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix},\ H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C = 0.$$

*The feedback nonlinear function belongs to the sector $[0, K]$ with $K = 1$. The dynamic phenomena of the system are shown in Figure 1. The **MAUBs** of h for different h_d by*

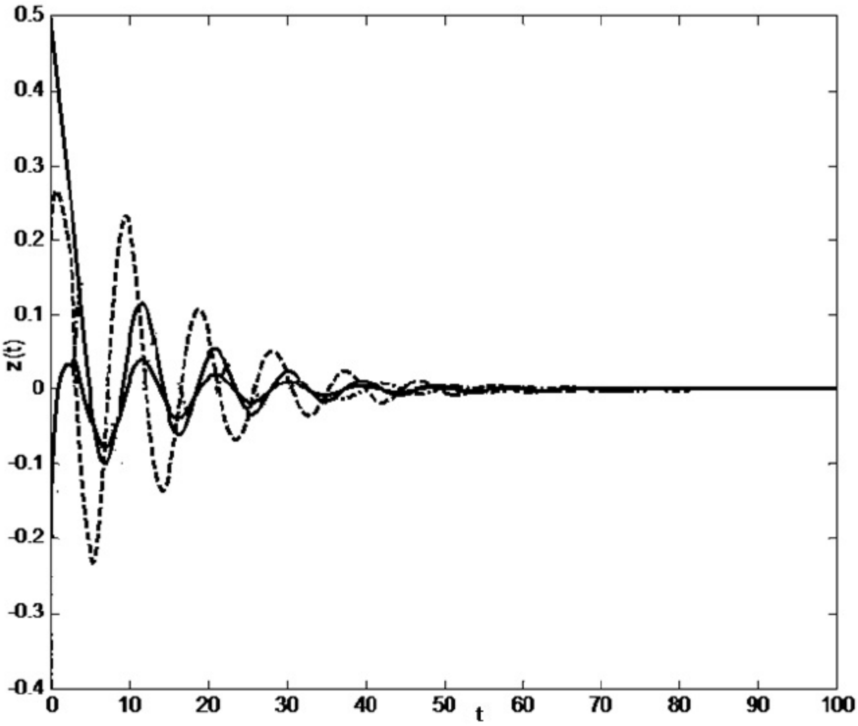

FIGURE 1. The dynamic phenomena of Example 4.2

TABLE 2. The **MAUBs** for different h_d (Example 4.2)

Methods\ h_d	0	0.3	0.6	0.9	≥ 1
[21]	0.1745	0.1698	0.1698	0.1698	0.1698
[22]	0.1747	0.1710	0.1703	0.1703	0.1703
Theorem 3.2	0.1770	0.1722	0.1718	0.1718	0.1718

using Theorem 3.2 against the existing results are listed in Table 2. From Table 2 it is clear that the proposed stability criterion is less conservative than the previous results.

Example 4.3. *Consider the uncertain system (1) with following parameters:*

$$h(x(t)) = m_1 x(t) + \frac{1}{2}(m_0 - m_1)[|x(t) + c| - |x(t) - c|],$$

with parameters $m_0 = -\frac{1}{7}$, $m_1 = \frac{2}{7}$, $\alpha = 9$, $\beta = 14.28$ *and* $c = 1$. *The system can be represented in normal Lur'e system framework (52) with*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \ H = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \ C = 0,$$

$$D = E_a = E_{a1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ E_b = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \ F(t) = \begin{bmatrix} \sin(\omega(t)) & 0 \\ 0 & \sin(\omega(t)) \end{bmatrix}.$$

The dynamic phenomena of the system are shown in Figure 2. The ***MAUBs*** *of* h *with* $h_d = 0$ *and* $\tau_d < 1$ *for different values of* $K_{[0,k_i]}$ *are listed in Table 3 by using Theorem 3.1 and Corollary 3.1 along with the existing criteria in [21,22]. We can found that the proposed stability criterion is less conservative than the ones in [21,22].*

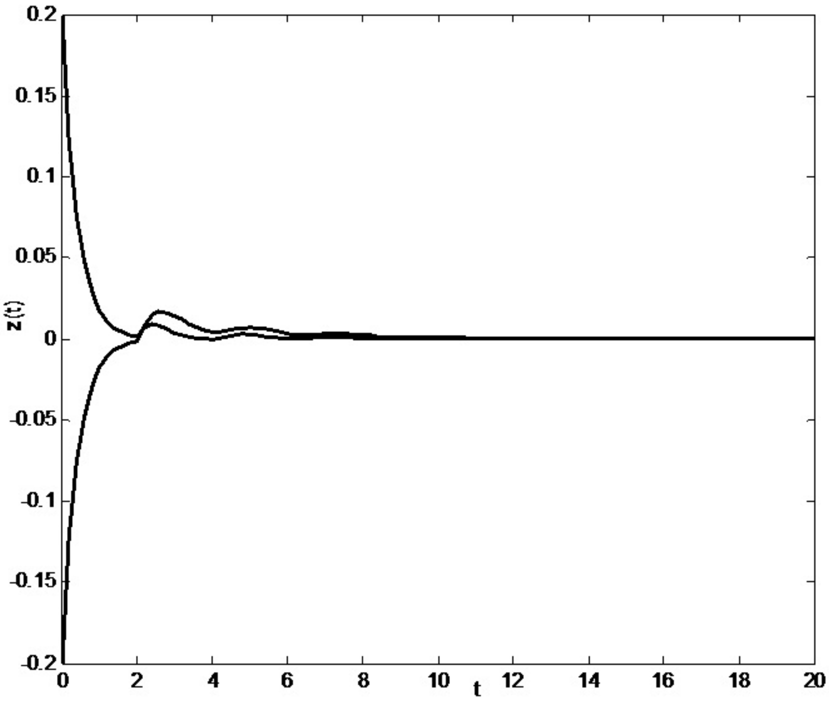

FIGURE 2. The dynamic phenomena of Example 4.3

TABLE 3. The **MAUBs** h for different K (Example 4.3)

K	Methods	h
$f(\cdot) \in K_{[0,0.5]}$	[21]	5.1666
	[22]	5.3486
	Theorem 3.1	5.9840
$f(\cdot) \in K_{[0,100]}$	[21]	4.9731
	[22]	5.1828
	Theorem 3.1	5.9377
$f(\cdot) \in K_{[0,\infty]}$	[21]	4.9431
	[22]	5.1654
	Corollary 3.1	5.9195

5. **Conclusion.** In this paper, some new stability criteria are proposed for a class of uncertain neutral-type Lur'e systems with interval time-varying delays and sector-bounded nonlinearity. The delay-dependent stability criteria are derived in the form of LMIs without using the general free-weighting matrix method. By utilizing reciprocal convex optimization approach and Wirtinger-type inequality which encompasses the Jensen inequality, the proposed criteria are less conservative than some existing results. In the future work, we will research variable segments instead of the equal partition. Finally, some standard numerical examples are used to illustrate the effectiveness of the proposed approaches and improve the existing methods.

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