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AN OPTIMAL PRODUCTION CAPACITY CONTROL INCLUDING OUTSIDE SUPPLIERS

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Abstract. *This study is part of an ongoing report on an analysis of production processes using a lead-time function. We present a strategy for determining the optimal production capacity using a quadratic form evaluation function in the production process. A mathematical model of production process is introduced by a stochastic differential equations with a lognormal type. In general, a production capacity is proportional to the rate of return. To determine the optimal production capacity, we calculated the optimal solution by introducing the Hamilton-Jacobi-Bellman equation. We determine the optimum parameters of the quadratic form evaluation function on the basis of the optimal capacity solution. We reported that an optimal production capacity is highly dependent on a volatility in workers. Further, we present the actual throughput data for a production flow process with high productivity (using a synchronous method) and in the absence of a production flow process (using an asynchronous method). The production efficiency of the synchronous process becomes clear from the actual data. For further verification, we confirmed the benefit of using the synchronization process to attempt to perform dynamic simulation.*
Keywords: Hamilton-Jacobi-Bellman equation, Lead-time function, Log-normal distribution, Financial theory

1. **Introduction.** Based on mathematical and physical understandings of production engineering, we are conducting research aiming at establishing an academic area called mathematical production engineering. As our business size is a small-to-medium-sized enterprise, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, when considering human intervention from outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of such intervention.

With respect to mathematical modeling of deterministic systems, a physical model of the production process was constructed using a one-dimensional diffusion equation in 2012 [1]. However, the many concerns that occured in the supply chain are major problems facing production efficiency and business profitability. A stochastic bi-linear partial differential equation with time delay was derived for outlet processes. The supply chain was modeled by considering a time delay system [3]. With respect to the analysis of production processes in stochastic system based on financial engineering, we have proposed that a production throughput rate was able to be estimated by utilizing Kalman filter theory based on the stochastic differential equation [2]. We have also proposed a stochastic differential equation (SDE) for the mathematical model describing production processes from the input of materials to the end. We utilized a risk-neutral principal in stochastic calculus based on the SDE [4].

With respect to a bottleneck in production processes, there is the famous theory of constraints (TOC) that describes the importance of avoiding bottlenecks in production processes [6]. Small fluctuations in an upstream subsystem appear as large fluctuations in the downstream (the so-called bullwhip effect) [12]. The bullwhip effect generates a large gap between the demand forecasts of the market and suppliers. Large fluctuations can be suppressed by the following mechanisms.

(1) Reducing the lead time, improving the throughput, and synchronizing the production process by the TOC.
(2) Sharing the demand information and performing mathematical evaluations.
(3) Analyzing the reduction and fluctuating demands of the subsystem (using nonlinear vibration theory).
(4) Basing the inventory management approach on stochastic demand.

When using manufacturing equipment, delays in one production step are propagated to the next. Hence, the use of manufacturing equipment itself may lead to delays. The improvement of production processes was presented that the "Synchronization with pre-process" method was the most desirable in practice using the actual data in production flow process based on the cash flow model by using the SDE of log-normal type [7]. In essence, we have proposed the best way, which is a synchronous method using the Vasicek model for mathematical finance [8]. Then, the supply chain theme, which was a time delay in the production processes, was proposed for the throughput improvement based on a stochastic differential equation of log-normal type [12].

Moreover, the analysis of the synchronized state indicated that this state was a much better method from the viewpoint of potential energy [12, 13]. We have also shown that the phase difference between stages in a process corresponded to the standard deviation of the working time [14]. When the phase difference was constant, the total throughput could be minimized. We showed that a synchronous process could be realized by the gradient system. The above problem is not limited to small- and medium-sized companies; in all cases, human interventions that directly affect the production process present a major challenge.

In general, we may reasonably consider that human interventions within and outside of the production system (internal and external forces, respectively) introduce uncertainties into the system's progress [4, 8]. The production system is formed by connecting both elements. When human intervention from outside companies involves an uncertainty, the noise element is frequently overlooked; instead, researchers have focus on efficient production or manufacturing the best system. Moreover, by including the noise element, we can recognize the unique advantage of the system.

With respect to an optimal control system, we reported an optimal control system for a semiconductor manufacturing equipment [9, 10]. This study, which determines an optimal production capacity based on SDE is not yet. In this study, to determine the optimal production capacity, we are seeking the optimum parameters to determine the optimal production capacity. A production capacity is modeled by a stochastic differential equation with a log-normal type, because a production capacity is proportional to a rate of return. Our historical data, which is a rate of return, shows a log-normal as to a probability distribution. To calculate an optimal capacity, we introduce a Hamilton-Jacobi-Bellman equation. As a result, we calculate optimum parameters of a production capacity.

From a theoretical result, an optimal production capacity is affected by a volatility. Therefore, using a production flow process, we compare three types of tests, which are an asynchronous method and two basic synchronous methods, to verify a volatility. Consequently, a synchronous method is the best way for a production process. In this study, we simulate a small-to-midsize firm without sufficient working capital to continue operations. Therefore, we need to raise working capital from financial institutions. Here, we call this cash flow. In essence, the rate of return (RoR) is at least proportional to the production lead time. In other words, if RoR forms a log-normal distribution, it is realistic to assume that the cash flow will also have the same log-normal distribution.

To evaluate the total production of a business, we utilize the actual throughput data of a firm with high productivity and implement a dynamic simulation for evaluation to confirm effectiveness of the synchronous and asynchronous processes. To the best of our knowledge, a determination of an optimal capacity has not been undertaken by previous studies.

2. Production Process Analysis for Lead-Time Function.

2.1. Production systems in the manufacturing equipment industry. In Figure 1, the production methods used in manufacturing equipment are briefly covered in this paper. More information is provided in our report [5]. This system is considered to be a "Make-to-order system with version control", which enables manufacturing after orders are received from clients, resulting in "volatility" according to its delivery date and lead time. In addition, there is volatility in the lead time, depending on the content of the make-to-order products (production equipment).

A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the production process.

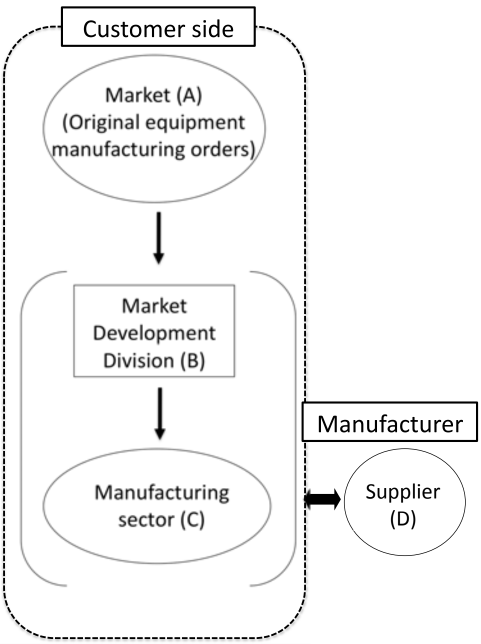

FIGURE 1. Business structure of company of research target

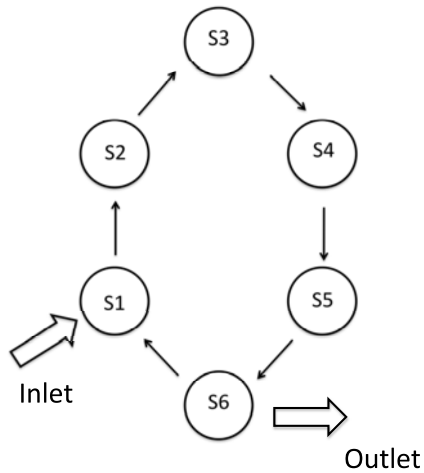

FIGURE 2. Production flow process

2.2. **Production flow process.** In Figure 2, the process consists of six stages. In each step S1-S6 of the manufacturing process, materials are being produced.

The direction of the arrows represents the direction of the production flow. Production materials are supplied through the inlet and the end-product is shipped from the outlet [7].

2.3. **Monthly rate of return.** From data for observed monthly rate of return (RoR), we calculate a probability density function (PDF, Figure 4) [5]. Results indicate that it conforms to a log-normal distribution (Figure 4, Theoretical). Our previous study provides further information.

$$f(x) = \frac{1}{\sqrt{2\pi}\,(x-\gamma_p)\,\sigma_p} \exp\left\{-\frac{1}{2}\left(\frac{(\ln x-\gamma_p)-\mu}{\sigma_p}\right)^2\right\} \tag{1}$$

Theoretical curve was calculated using EasyFit software (http://www.mathwave.com/), and as a result of Kolmogorov and Smirnov test, the observed values conformed to a log-normal type probability density function. Because, in the goodness-of-fit test of Kolmogorov-Smirnov, a null hypothesis that it is "log-normal" was not rejected with rejection rate 0.2, this data conforms to "log-normal" distribution. P-value was 0.588. The parameters of a theoretical curve were: $\mu = -0.134$ (average), $\sigma_p = 0.0873$ (standard deviation), $\gamma_p = -0.900$. The theoretical curve is given by Equation (1). For more information, please refer our previous study [5].

2.4. **Lead-time function.** Figure 5 shows that a throughput is proportional to a rate of return in production processes. Then, we introduce the lead-time function so that we can analyze a production process [15]. The lead time of production equipment is proportional to the RoR. Therefore, we determined that the lead time PDF was also the same PDF of RoR. Thus, the lead-time function $f(y)$ is assumed as a log-normal

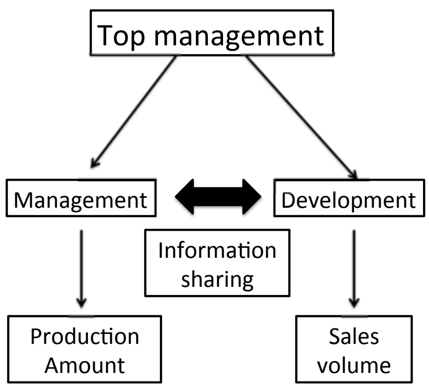

FIGURE 3. Information sharing between a management div. and a development div.

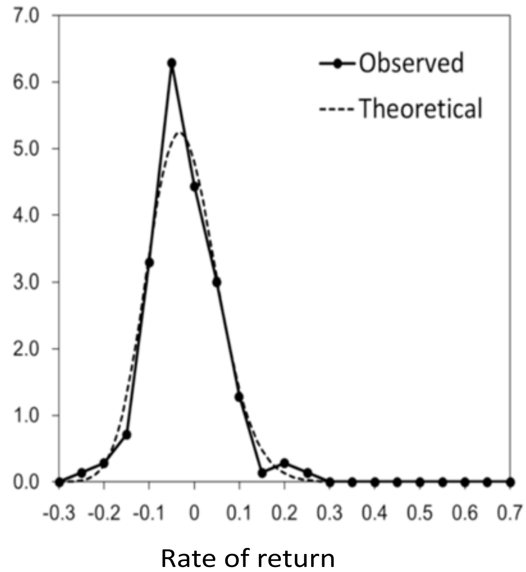

FIGURE 4. Probability density function of rate-of-return deviation: actual data (solid line) and data based on theoretical formula (dotted line)

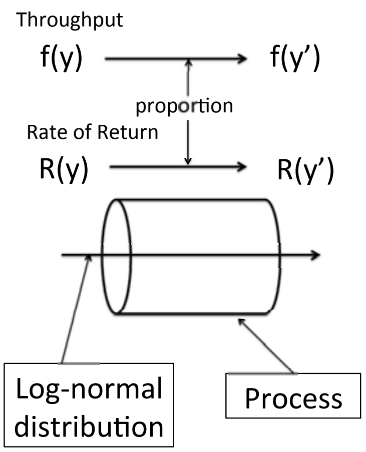

FIGURE 5. Throughput fluctuation in a process distribution amount

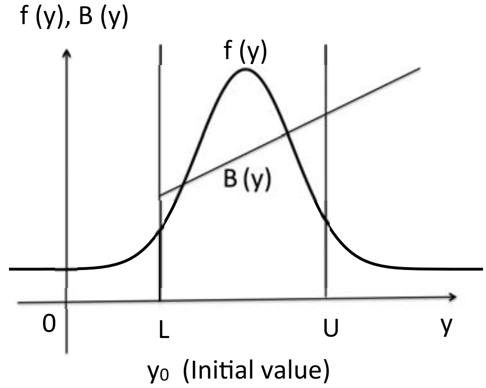

FIGURE 6. Lead-time function $f(y)$ and loss function $B(y)$

probability density function so that we can calculate the lead time using a continuous expected value calculation as shown in Figure 6.

Assumption 2.1. *Lead-time function of a probability density function with log-normal type.*

$$f(y) \equiv \frac{1}{\sqrt{2\pi}\sigma(y/y_0)} \exp\left\{-\frac{(\ln(y/y_0)-\mu)^2}{2\sigma^2}\right\} \tag{2}$$

where μ is an average value, σ is a volatility and y_0 is an initial lead time.

Now, let F be as a cash-in flow and let C_0 be as a fixed cost, and we calculate a continuous expected loss value F [15].

$$\begin{aligned} F &= \int_{-\infty}^{\infty} f(y)B(y)dy + C_0 \\ &= \int_{-\infty}^{L} B(y)f(y)dy + \int_{L}^{U} B(y)f(y)dy + \int_{U}^{\infty} B(y)f(y)dy + C_0 \end{aligned} \tag{3}$$

where,

$$B(y) = py + q, \quad p \geq 0 \tag{4}$$

where q is a constant parameter. L is a minimal lead time, $U = kL$ and k (> 1) is a constant parameter. p is as diminishing increasing function, for example, believes the following function. $p = \alpha\sqrt{y - kU}$, α is a constant value. U is a maximum lead time.

When $y < L$, production activities are not running. When $y > U$, the quantity ordered exceeds the physical limits of the production. Therefore, we must reduce the demand, and the problem becomes an analysis of $L \leq y \leq U$.

$$F = \int_{L}^{U} (py + q)f(y)dy + C_0 \tag{5}$$

For more information, please refer our previous study [15].

3. **Optimal Production Capacity Control.** We describe that this study obtains an optimal production capacity by determining the parameters of quadratic form evaluation function.

3.1. **Mathematical modeling of production capacity.** We describe the mathematical model for an optimal production capacity.

$$dX(t) = P(a)C(t)dt + \sigma C(t)dW(t) \tag{6}$$

where $P(a)$ is a production value, and this is the corresponding way of calling to finance theory. a is a trend coefficient for production value improvement. $C(t)$ denotes a production capacity. σ is a volatility and $W(t)$ is a standard Brownian motion.

With respect to a production value, a residual value is set as r and then we assume that the unit production value is one. Therefore, the required value of the management side is $1 - r$. Naturally, there is an upper limit to the production capacity which is described as follows:

$$r \cdot C(t) \leq \tilde{r}_t \cdot C(t) = E[X(t)] = P(a)C(t) \tag{7}$$

where $\tilde{r}_t$ is the upper limit of residual value and E[$\bullet$] is the expectation value.

With respect to optimal investment problems in the manufacturing industry, we have researched from a mathematical point of view in order to develop a strategy for the allocation balanced revenue by integrating both of management and production division. A corporate revenue is proportional to the production capacity and the order rates. The order rates can be regarded as a demand distribution by analyzing from the viewpoint of quantitation. Order rate was difficult to order at a constant rate throughout the year. In other words, the order rate varies every month. Considering our experience, the product value sometimes becomes to decrease in inverse proportion to the volume of orders. A corporate management is for an investment in production business about a product volume efficiency. Therefore, we need to quantify the product size. When investing, its limit exists in the company capability for business sales. Therefore, the corporate management must determine an investment to get the product value in consideration of the demand distribution.

Let an order rate be $\xi(t)$ and then, assume that the mass product effect is already known for business. A turnover $X(t)$ is proportional to an order rate $\xi(t)$ and a production capacity $C(t)$ and then, needless to say, its turnover has some relationship with an order rate and a production capacity. We consider as follows:

$$X(t) \approx \xi(\bullet)C(t) \tag{8}$$

The dynamic model of the production capacity represents Equation (6):

$$dC(t) = \frac{P(a)}{\xi(\bullet)}C(t)dt + \sigma_\xi C(t)dW(t) \tag{9}$$

where $\sigma_\xi = \sigma/\xi(\bullet)$ is a volatility.

With respect to $P(a)/\xi(\bullet)$, a product value decreases in inverse proportion to an order rate in some cases (mass production effect).

The mathematical model of a production capacity is as follows:

$$dC(t) = P^*(a)C(t)dt + \sigma_\xi C(t)dW(t) \tag{10}$$

where $P^*(a) = P(a)/\xi(\bullet)$.

Naturally, ensuring a turnover involves risks. For example, the external probability fluctuation factor, which is the demand fluctuation, material prices fluctuation, and logistics price itself fluctuation, etc, exists.

Therefore, the model with some fluctuation factors a, r and I is required to design the manufacturing business needed to design in conjunction with a turnover risk on the manufacturing business.

$$r = \{r(t) : r(t) \in [0, r_{\max}]\}, \quad a = \{[a(t) : a(t) \in [0, a_{\max}]\} \tag{11}$$

where $r_{\max}$ and $a_{\max}$ are a maximum value of r and a respectively.

Definition 3.1.

- *$u(r(t))$ represents an average value of the residual amount.*
- *$h(a(s))$ represents an average value of the product value at the time of trouble. For example, $h(a(s))$ denotes a quadratic function.*

These values $u(r(t))$ and $h(a(t))$ should be originally estimated by utilizing Kalman filter [4].

The sales involve risks, which are a change demand, material price fluctuations and logistics change, etc. Therefore, the value $u(r(t))$ is varied in conjunction with the risk of sales in production division. From view of the entire company, any unit residual value $r(t)$ is determined in after time t.

Definition 3.2. *$G(t)$ is the total expected payoff in the entire period.*

$$G(t) = \int_0^t e^{-rs}\{u(r(s)) - h(a(s))\}C(s)ds + e^{-rt}C(t) \tag{12}$$

Moreover, an adaptive process $Y(t)$, $\{Y(t) : 0 \le t < \infty\}$ exists:

$$G(t) = G(0) + \int_0^t e^{-rs}\sigma Y(s)C(s)dW(s) \tag{13}$$

where $0 \le t < \infty$. Differentiate Equation (12) and Equation (13) at t, and first, from Equation (12),

$$\begin{aligned} dG(t) &= e^{-rt}[\{u(r(t)) - h(a(t))\}C(t)] + d\left[e^{-rt}C(t)\right] \\ &= e^{-rt}[\{u(r(t)) - h(a(t))\}C(t) - rC(t)]dt + e^{-rt}dC(t) \end{aligned} \tag{14}$$

From Equation (18),

$$dG(t) = e^{-rs}\sigma G(t)C(t)dW(t) \tag{15}$$

From Equation (14) to Equation (15),

$$dC(t) = [rC(t) - \{u(r(t)) - h(a(t))\}C(t)]dt + \sigma Y(t)C(t)dW(t) \tag{16}$$

3.2. **Formulation of Hamilton-Jacobi-Belman equation.** Hamilton-Jacobi-Belman equation is described as follows [11]:

$$H(t) = \left[P(C(t)) - \theta J(G(t)) + \frac{\partial J}{\partial C}\{r \cdot C - \{u(r(t)) - h(a(t))\}C(t)\} + \frac{1}{2}\tilde{\sigma}^2 C \cdot \frac{\partial^2 J}{\partial C^2}\right] \tag{17}$$

where $P(C(t))$ is defined as follows. J is a cost function.

Definition 3.3. *Utility function*

$$E\left[\int_0^\infty e^{-rt} \cdot P(C(t))dt\right] \tag{18}$$

From the above description, the optimal production capacity problem is formulated by allocating the profits which are gained from each corporate division, by executing strategy. In Equation (8), we need to clarify $C(t)$.

We define the utility like a second order function as follows.

Definition 3.4. *Second order utility function*

$$E\left[\int_0^\infty \left\{\xi(g(r,a)C(t)) - \eta(g(r,a)C(t))^2\right\} e^{-rt}\right] \tag{19}$$

From Equations (18) and (19), we obtain as follows:

$$P(C(t)) = \xi[g(r,a)C(t) - \eta(g(r,a)C(t))]^2 \tag{20}$$

Therefore, Equation (17) is rewritten as follows:

$$\begin{aligned} H(t) = \Bigg[&\left\{\xi \cdot g(r,a)C(t) - \eta(g(r,a)C(t))^2\right\} - \theta \cdot J(G(t)) \\ &+ \frac{\partial J}{\partial C}[r \cdot C(t) - \{u(r(t)) - h(a(t))\}\theta(t)] + \frac{1}{2}\tilde{\sigma}^2 C(t)^2 \cdot \frac{\partial^2 J}{\partial C^2}\Bigg] \end{aligned} \tag{21}$$

where θ is a parameter.

Thus, from an optimal control theory, we obtain as follows:

$$\max_{C(t)} H(t) = \frac{\partial H(t)}{\partial C(t)} = 0 \tag{22}$$

From Equation (22), we obtain as follows:

$$\xi g(r,a) - 2\eta \cdot g(r,a)C(t) - \frac{\partial J}{\partial C}\{u(r(t)) - h(a(t))\} = 0 \tag{23}$$

From Equation (23), we obtain as follows:

$$C(t) = \frac{1}{2\eta g(r,a)}\left[\xi \cdot g(r,a) - \frac{\partial J}{\partial C(t)}\{u(r(t)) - h(a(t))\}\right] \tag{24}$$

where

$$\kappa(r,a) = u(r(t)) - h(a(t)) \tag{25}$$

Substituting Equations (24) and (25) to Equation (21) considering Equation (22), then we obtain as follows:

$$\begin{aligned} 0 = \Bigg[&\xi \cdot g(r,a) \cdot \left\{\frac{1}{2\eta g(r,a)}\left(\xi \cdot g(r,a) - \frac{\partial J}{\partial C(t)}\kappa(r,a)\right)\right\} \\ &- \eta g^2(r,a)\left\{\frac{1}{2\eta g(r,a)}\left(\xi \cdot g(r,a) - \frac{\partial J}{\partial C(t)}\kappa(r,a)\right)^2\right\} - \theta(t)J\Bigg] \\ &+ \frac{\partial J}{\partial C(t)}\Bigg[r \cdot C(t) - \kappa(r,a)\left\{\frac{1}{2\eta g(r,a)}\left(\xi \cdot g(r,a) - \frac{\partial J}{\partial C(t)}\right)\right. \\ &\left.+ \frac{1}{2}\tilde{\sigma}^2 C^2(t)\frac{\partial^2 J}{\partial C^2(t)}\right\}\Bigg] \end{aligned} \tag{26}$$

3.3. **Optimal control of a production capacity and a production revenue.** We define a quadratic form evaluation function as follows.

Definition 3.5. *Quadratic form function of* $J(G(t))$

$$J(G(t)) = a_J G^2(t) + bG(t) + c \tag{27}$$

where a_J, b *and* c *are parameters.*

Substituting Equation (27) to Equation (21), and using Equation (22), these parameters (a_J, b and c) are calculated from an identical equation on $G(t)$. We can obtain the optimal parameters a^*, b^* and c^*, which represent Equations (64), (67) and (68) respectively (see Appendix A).

From Equation (26), the optimal solution of production capacity $C^*(t)$ is as follows:

$$C^*(t) = \frac{1}{2\eta g}[\xi g - \{(2a^* G(t) + b^*)\}\{u(r(t)) - h(a(t))\}] \tag{28}$$

Equation (28) is rewritten as follows:

$$C^*(t) = \frac{1}{2\eta g}[\xi g - b^*\{u(r(t)) - h(a(t))\} - 2a^*\{u(r(t)) - h(a(t))\}G(t)] \tag{29}$$

Here, we replace Equation (29) as follows:

$$C^*(t) = w - \rho G(t) \tag{30}$$

where ρ represents an optimal feedback coefficient and $\rho = ((-2a)/(2\eta g))\kappa$.

We substitute Equation (30) to Equation (29), and then we obtain as follows:

$$\begin{aligned} dG(t) &= [rG(t) - \kappa\{w - \rho G(t)\}]dt\hat{\sigma}G(t)dW(t) \\ &= [(r + \kappa\rho)G(t) - \kappa w]dt + \hat{\sigma}G(t)dW(t) \end{aligned} \tag{31}$$

By calculating an expectation of Equation (31), we obtain as follows:

$$\frac{E[dG(t)]}{dt} = E[(r + \kappa\rho)G(t) - \kappa w] + E[\hat{\sigma}G(t)dW(t)] \tag{32}$$

The solution of Equation (32) is as follows:

$$E[G(t)] = \left(G(0) - \frac{\kappa w}{r + \kappa\rho}\right) e^{(r+\kappa\rho)t} + \frac{\kappa w}{r + \kappa\rho} \tag{33}$$

Therefore, we obtain as follows:

$$E[C^*(t)] = \rho\left[\left(G(0) - \frac{\kappa w}{r + \kappa\rho}\right) e^{(r+\kappa\rho)t} + \frac{\kappa w}{r + \kappa\rho}\right] + w \tag{34}$$

By providing the appropriate parameters, it was revealed that a stationary equilibrium value existed between a production continued evaluation value and a production capacity.

From the above relation, g, κ are as follows:

$$g \equiv g(r, a), \quad \kappa \equiv \kappa(r, a) \tag{35}$$

From the above results, Equation (29) represents an optimal production capacity.

However, as an state variable $G(t)$ is difficult to measure, we consider an assumption of the revenue model for production.

Assumption 3.1. *Revenue model in production division*

$$dX(t) = \alpha(a(t))X(t)dt + \sigma dW(t) \tag{36}$$

where $\alpha(a(t))$ is an coefficient representing a demand trend.

With respect to $\alpha(a(t))$, it can be obtained by calculating an expected rate of return as representing a probability distribution of a demand trend.

From Equation (30), let $w = 0$ for simplicity, and we obtain as follows:

$$C^*(t) = -\rho G(t) \tag{37}$$

From Equation (29), we obtain as follows:

$$\xi g - b^*\{u(r(t)) - h(a(t))\} = 0 \tag{38}$$

Consequently, we obtain as follows:

$$\{u(r(t)) - h(a(t))\} = \frac{\xi g}{b^*} \tag{39}$$

An optimal solution $C^*(t)$ is as follows:

$$C^*(t) = -\frac{1}{2\eta g}\{a^*(u(r(t)) - h(a(t)))\} = -\rho G(t) \tag{40}$$

Therefore, let an optimal solution of $G(t)$ to $G^*(t)$, from Equation (31), and we obtain as follows:

$$dG^*(t) = (r + \kappa\rho)G^*(t)dt + \tilde{\sigma}G^*(t)dW(t) \tag{41}$$

where, let $\kappa \equiv u(r(t)) - h(a(t))$, and κ represents substantially a trend coefficient of revenue.

Note that, the original model is as follows:

$$dG(t) = [rG(t) - \{u(r(t)) - h(a(t))\}C(t)] + \tilde{\sigma}G(t)dW(t) \tag{42}$$

Now, we analyze an expectation trend. From Equations (33) and (34), we obtain as follows:

$$K \equiv r + \kappa\rho, \quad L \equiv \kappa\omega \tag{43}$$

Then, we obtain as follows:

$$E[G(t)] = \left[G(0) - \frac{L}{K}\right]e^{Kt} + \frac{L}{K} \tag{44}$$

$$E[C(t)] = \rho\left[\left(G(0) - \frac{L}{K}\right)e^{Kt} + \frac{L}{K}\right] + \omega = \rho\left(G(0) - \frac{L}{K}\right)e^{Kt} + \left(\frac{\rho L}{K} + \omega\right) \tag{45}$$

From Equation (29), $C^*(t)$ can be rewritten as follows:

$$C^*(t) = \frac{1}{2\eta g}(\xi g - \kappa b^* - 2a^*\kappa G(t)) = \frac{(\xi g - \kappa b^*)}{2\eta g} - \frac{2a^*\kappa}{2\eta g}G(t) \tag{46}$$

where, the parameters w and ρ are as follows:

$$\omega = \frac{(\xi g - \kappa b^*)}{2\eta g} \tag{47}$$

$$\rho = \frac{a^*\kappa}{\eta g} \tag{48}$$

where a^*, b^* are as follows:

$$a^* = \frac{\tilde{\sigma} - \theta + 2r}{1 - \frac{2\kappa}{\eta g}} \tag{49}$$

$$b^* = \frac{1}{[A] + \frac{\kappa^2}{\eta g}a^*} \times \left[\frac{\xi}{\eta} + \frac{\kappa\xi g a^*}{\eta g}\right] \tag{50}$$

where $[A]$ is as follows:

$$[A] =\equiv \frac{a^*\kappa}{\eta g} + r - \theta \tag{51}$$

Consequently, a value of K in Equation (43) can be considered as follows:

1) $K > 0$
2) $K = 0$

3) $K < 0$

where, from Equation (46)-(50), as $a^* = \tilde{\sigma}^2$, $\rho \approx \tilde{\sigma}^2$ and $w \approx \{a^*, b^*\}$, then, b^* is a linear fractional function of a^*.

Thus, $K(\equiv r + \kappa\rho)$ is an increasing function of $\tilde{\sigma}$. Moreover, as classified finely as follows:

(a) $\left(G(0) - \frac{L}{K}\right) < 0$, $\left(\frac{L}{K}\right) > 0$, $K < 0$, $L < 0$
(b) $\left(G(0) - \frac{L}{K}\right) > 0$, $\left(\frac{L}{K}\right) > 0$, $K > 0$, $L > 0$
(c) $\left(G(0) - \frac{L}{K}\right) > 0$, $\left(\frac{L}{K}\right) = 0$, $K < 0$, $L = 0$

Now, in the real production process, because $C^*(t) \to \infty$ cannot be there, it can be ignored in the case of Equation (b). Also, if it is $C^*(t) \to 0$, the case of Equation (c) can be ignored from that the production process is not working. Therefore, if you have any meaning, it becomes the case of Equation (a).

Consequently, a steady-state equilibrium value is as follows:

$$E[C^*] = \frac{L}{K} = \frac{L}{r + \kappa\rho} \approx \tilde{\sigma}^{-1} \tag{52}$$

where $E[C^*]$ represents an expectation of an optimal production capacity.

$$E[G^*] = \frac{\rho L}{K} + \omega = \frac{\rho L}{r + \kappa\rho} + \omega \approx \omega \tag{53}$$

where $E[G^*]$ represents an expectation of an optimal production revenue.

From the above description, K is an increasing function as to a volatility σ. Therefore, the smaller a volatility of a production system is, the larger individual revenue is.

4. **Verification of the Theoretical Optimal Control Capacity.** From the above results, K is restricted by a volatility, and affects a production throughput significantly. Therefore, we present an actual testing result, which is obtained by using a production flow process.

4.1. **Analysis of the test run results.** The production throughput is evaluated using a number of equipment pieces in comparison with the target number of equipment pieces (production ranking) and simulating asynchronous and synchronous production (see Appendix A). The asynchronous method is prone to worker fluctuations imposed by various delays, whereas worker fluctuations in the synchronous method are small. In terms of the production lead times results presented in the Appendix A, the productivity ranking tests indicate that test run 3 > test run 2 > test run 1, where test run 1 is asynchronous and test runs 2 and 3 are synchronous.

Here, the throughput values calculated from the throughput probability in test run 1-test run 3, are as follows:

- Test run 1: 4.4 (pieces of equipment)/6 (pieces of equipment) = 0.73
- Test run 2: 5.5 (pieces of equipment)/6 (pieces of equipment) = 0.92
- Test run 3: 5.7 (pieces of equipment)/6 (pieces of equipment) = 0.95

With respect to the actual data, please refer to our previous study [7, 15].

4.2. **Dynamic simulation of production processes.** We attempted to perform a dynamic simulation of the production process by utilizing the simulation system that NTT DATA Mathematical Systems Inc. (www.msi.co.jp) has developed. With respect to the meaning of the individual parts in Figure 7, we conducted a simulation of the following procedure. For more information, please refer to our previous study [15].

- When the simulation began, it generated one of the products on a “generate” part going to “finish”.
- In each process, including the six workers in parallel, the slowest worker waited till the work was completed.
- When the work of each process was completed, it moved to the next process.
- Simultaneously as each process was completed, it recorded the working time of each process.

With respect to Table 1 and Table 2,

- Process No. indicates each process (1-6).
- Average indicates the average time.
- STD indicates the standard deviation of process time (sec).
- Worker efficiency (WE) indicates the efficiency of six workers.

“record” calculates the worker’s operating time, which is obtained by multiplying the specified WE data for the log-normal distributed random numbers in Table 1.

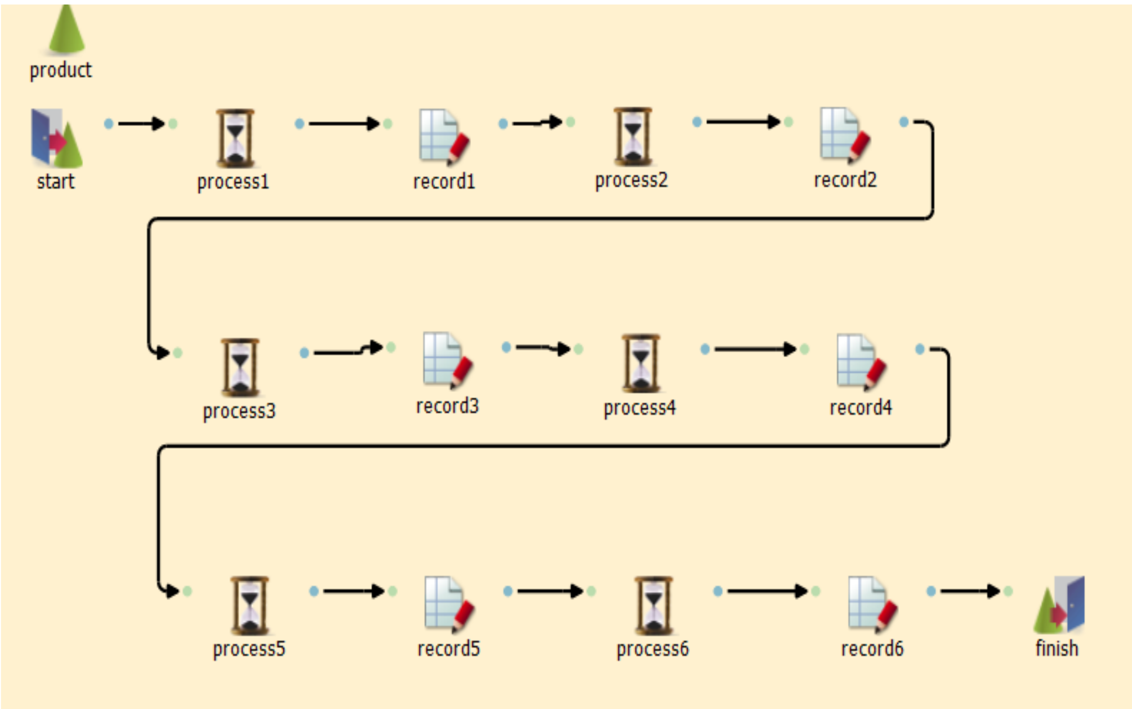

Figure 7. Simulation model of production flow system

Table 1. Working data for six production asynchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	22	25	22	25	21
STD	2.1	2.5	1.6	1.9	2.0	1.9
W.E 1	0.83	1.0	0.66	0.76	0.88	0.91
W.E 2	1.27	1.26	1.21	1.31	1.17	1.20
W.E 3	0.96	1.11	1.01	1.12	0.88	0.89
W.E 4	0.92	0.96	1.06	0.98	0.91	0.9
W.E 5	1.2	1.03	1.07	0.89	1.03	1.1
W.E 6	1.09	1.1	1.2	0.98	1.13	0.89

TABLE 2. Working data for six production synchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	20	20	20	20	20
STD	1.1	1.5	1.2	1.4	1.0	1.4
W.E 1	1.0	1.0	1.0	1.0	1.0	1.0
W.E 2	1.0	1.0	1.2	1.3	1.1	1.2
W.E 3	1.7	1.1	1.0	1.1	1.0	1.0
W.E 4	1.0	1.0	1.0	1.0	1.0	1.0
W.E 5	1.0	1.0	1.0	1.0	1.0	1.0
W.E 6	1.0	1.3	1.2	1.0	1.1	1.0

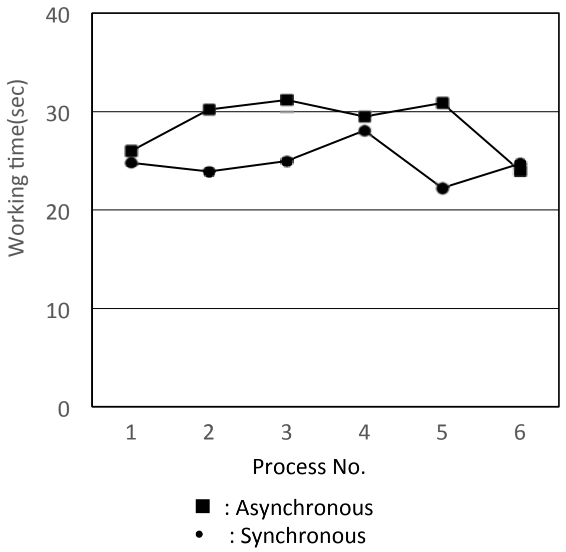

FIGURE 8. Working time for process number one to six

Figure 8 shows the operating time of process 1-6 (record1-record6). As the working time of the synchronous process is volatiler, the work efficiency became higher than the asynchronous process. In Figure 8, the total working time of asynchronous and synchronous processes are 1241.7 (sec) and 586.4 (sec) respectively. The synchronous process shows more better production efficiency than the asynchronous process.

5. **Conclusion.** We calculated the parameters of quadratic form evaluation function. Then, it was clarified that they can be affected by a volatility. To evaluate the validity of the parameters obtained on the basis of the production flow processes, we compared the method variation between an asynchronous method, which had a large volatility and a synchronous method, which had small volatility. Consequently, the product manufacturing throughput was clarified that the synchronization method was superior. The validity of the parameter values that determined the optimal capacity was able to guarantee. Next, we plan to evaluate production processes by utilizing the test run data on the basis of the Black-Scholes equation.

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Appendix A. **Derivation of the Parameters a and b.** Substitute Equation (27) to Equation (26). Then, we obtain as follows:

$$\begin{aligned}\xi \cdot g(r,a) - \frac{\partial J}{\partial G(t)}\kappa(r,c) &= \xi \cdot g(r,a) - (2aG(t)+b)\kappa(r,a) \\ &= (\xi \cdot g(r,a) - b\kappa(r,a)) - 2aG(t) = \pi(r,a) - 2aC(t) \end{aligned} \quad (54)$$

where

$$\pi(r,a) = \xi \cdot g(r,a) - b\kappa(r,a) \quad (55)$$

Then, substitute Equations (27) and (55) to Equation (26). We obtain as follows:

$$\begin{aligned}\text{First term} = &\left[\xi g(r,a) \times \frac{1}{2\eta g(r,a)}(\pi(r,a) - 2aG(t))\right] - \eta^2 g^2(r,a) \\ &\times \frac{1}{4\eta^2 g^2(r,a)}(\pi(r,a) - 2aG(t))^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{\xi}{2r\eta}(\pi(r,a) - 2aG(t)) - \frac{1}{4}(\pi(r,a) - 2aG(t))^2 \\
&= \frac{\xi}{2\eta}\pi(r,a) - \frac{a\xi}{\eta}G(t) - \frac{1}{4}\pi^2(r,a) + a\pi(r,a)G(t) - a^2G^2(t) \\
&= -a^2g^2(t) - \frac{a\xi}{\eta}g(t) + \left(\frac{\xi}{2\eta}\pi(r,a) - \frac{1}{4}\pi^2(r,a)\right) \qquad (56)
\end{aligned}
$$

$$\text{Second term} = -a\theta G^2(t) - b\theta G(t) - c\theta \qquad (57)$$

$$
\begin{aligned}
\text{Third term} &= (2aG(t) + b)\left\{rG(t) - \frac{\kappa(r,a)}{2\eta g(r,a)}(\pi(r,a) - 2aG(t))\right\} \\
&= 2arG^2(t) - \frac{a\kappa(r,a)\cdot G(t)}{\eta g(r,a)}(\pi(r,a) - 2aG(t)) + brG(t) \\
&\quad - \frac{b\kappa(r,a)}{2\eta g(r,a)}(\pi(r,a) - 2aG(t)) \\
&= 2arG^2(t) - \frac{a\kappa(r,a)\cdot\pi(r,a)}{\eta g(r,a)}G(t) + \frac{2a^2\kappa(r,a)}{\eta g(r,a)}G^2(t) + brG(t) \\
&\quad - \frac{b\kappa(r,a)\cdot\pi(r,a)}{2\eta g(r,a)} + \frac{2ab\pi(r,a)}{2\eta g(r,a)}G(t) \\
&= \left\{2ar + \frac{2a^2\kappa(r,a)}{\eta g(r,a)}\right\}G^2(t) + \left\{br - \frac{a\kappa(r,a)\cdot\pi(r,a)}{\eta g(r,a)} + \frac{2ab\pi(r,a)}{2\eta g(r,a)}\right\} \\
&\quad \times G(t) - \frac{b\kappa(r,a)\cdot\pi(r,a)}{2\eta g(r,a)} \qquad (58)
\end{aligned}
$$

$$\text{Fourth term} = \frac{1}{2}\tilde{\sigma}^2G^2(t)\times 2a = a\tilde{\sigma}^2G^2(t) \qquad (59)$$

Using Equations (56)-(59), the identical equation is as follows:

$$-a^2 - a\theta + 2ar + \frac{2a^2\kappa}{\eta g} + a\tilde{\sigma}^2 = 0 \qquad (60)$$

$$-\frac{a\xi}{\eta} - b\theta + br - \frac{a\kappa\pi}{\eta g} + \frac{2ab\kappa}{2etag} = 0 \qquad (61)$$

$$\frac{\xi\pi}{2\eta} - \frac{\pi^2}{4} - c\theta - \frac{b\kappa\pi}{2\eta g} = 0 \qquad (62)$$

Then, from Equation (60), we obtain as follows:

$$\left(\frac{2\kappa}{\eta g} - 1\right)a^2 + \left(\tilde{\sigma}^2 - \theta + 2r\right)a = a\left\{\left(\frac{2\kappa}{\eta g} - 1\right)a + \left(\tilde{\sigma}^2 - \theta + 2r\right)\right\} = 0 \qquad (63)$$

From $a \neq 0$, we obtain as follows:

$$a^* = \frac{\tilde{\sigma}^2 - \theta + 2r}{1 - \frac{2\kappa}{\eta g}} \qquad (64)$$

From Equation (62), we obtain as follows:

$$\frac{\kappa\pi}{2\eta g}b + \theta c = \frac{\xi\pi}{2\eta} - \frac{\pi^2}{4} \qquad (65)$$

Substituting Equation (64) to Equation (61), we obtain as follows:

$$\left(\frac{2a^*\kappa}{2\eta g} + r - \theta\right)b = \frac{\xi}{\eta}a^* + \frac{\kappa\pi}{\eta g}a^* = \left(\frac{\xi}{\eta} + \frac{\kappa\pi}{\eta g}\right)a^* \qquad (66)$$

b^* is as follows:

$$b^* = \frac{1}{\left(\frac{a^*\kappa}{\eta g} + r - \theta\right) + \frac{\kappa^2}{\eta g}a^*}\left[\frac{\xi}{\eta} + \frac{\kappa}{\eta g}\xi g\right] \tag{67}$$

Substituting Equation (67) to Equation (65), c^* is as follows:

$$c^* = \frac{1}{\theta}\left\{\frac{\xi\pi}{2\eta} - \frac{\pi^2}{4} - \frac{\kappa\pi}{2\eta g}b^*\right\} \tag{68}$$