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# NOVEL GAUSSIAN APPROXIMATE FILTER METHOD FOR STOCHASTIC NON-LINEAR SYSTEM

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Abstract. *In the traditional process of nonlinear non-Gaussian filtering system model, there are many disadvantages on filtering results such as inaccuracy initial value selection and low convergence, due to complex system model or the influence of diversity interference. To improve the filter accuracy and solve the above problems, we put forward a new Gaussian approximate filter method, also give the general solution and special solution on the new method. In this paper, we demonstrate that existing Gaussian approximate filter methods are based on our new scheme. The new Gaussian approximate filter method adopts measurement point correction status quadrature points to better utilize the one-step predicted density, non-Gaussian information and high-order moment information of the posterior density, which can directly update quadrature points without repeatedly producing quadrature points. In addition, our new method not only is suitable for deterministic system model but for stochastic system model. At the end of this paper, we apply our method into single variable non-stationary growth model and vertical free-fall model to verify the performance of new method. What is more, we make comparison with the existing Gaussian approximate filter methods, and the results show that our method is more effective and superior.*

**Keywords:** Gaussian approximate filter method, General solution, Special solution, Quadrature points, Measurement point correction status

## 1. Introduction.

Nonlinear filtering [1-3] has been widely used in target tracking [4], signal processing [5], communication and automatic control [6]. Based on the random state space model, main task of nonlinear filtering is to calculate the posterior probability density function. We generally use Bayesian estimation theory to deal with nonlinear filtering problems by recursively computing the posterior probability density function of state. Bayesian estimation theory provides an optimal solution for dynamic state estimation problems [7]. However, Bayesian estimation contains multi-dimensional nonlinear integral that is difficult to solve, so it is unable to obtain the analytical solution of the posterior probability density function for nonlinear stochastic dynamic system. Therefore, there is no optimal nonlinear filter. To complete the state estimation of nonlinear stochastic dynamic system, we must use the approximate method to obtain suboptimal nonlinear filter. These approximate methods can be divided into two categories [8,9]: global method and local method. Under the Gauss hypothesis, Gaussian approximation filters mainly calculate the Gaussian weighted integral. Unscented Kalman filter (UKF) [10-12] is a typical Gaussian approximation filter, which uses three order unscented transformation to calculate Gaussian weighted integral. In order to improve the accuracy of UKF, Wang and Rui [13] proposed improved UKF. Firstly, the fading factor was introduced into the filter process based on strong tracking filter to avoid the filter divergence, and then wavelet transform was used to estimate the statistical characteristics of measurement noise to improve unscented Kalman filter tracking ability. Liu and Yin [14] used a minimum skewness monomorphic sampling strategy to reduce the amount of calculation of unscented Kalman filter and improve the accuracy of unscented Kalman filter. Yu et al. [15] put forward an improved dual unscented Kalman filter (IDUKF) with random control inputs and sequential dual estimation structure, in which the parameter was linearly observed and uncorrelated with the state. In [16], based on the square-root unscented KF (SRUKF), traditional Maybeck's estimator was modified and extended to nonlinear systems. The square root of the process noise covariance matrix $Q$ or that of the measurement noise covariance matrix $R$ was estimated straightforwardly. Novel fault detection method for nonlinear systems was proposed using the residuals generated by the second-order divided difference filter (DD2 filter) and the local approach. DD2 filter was based on the polynomial approximation of the nonlinear transformations obtained with a particular multi-dimensional extension of Stirling's interpolation formula [17]. In order to improve the stability of UKF in high dimensional state estimation, improved high-degree cubature Kalman filter (IHDCKF) was proposed [18].

Updated integral points are obtained by linear transformation of state prediction integral points. It ignores the correction action of measurement integral points for state integral points. In order to solve this problem, this paper puts forward an updating point method. We take correction function of measurement integral points into consideration for state integral points, which is more effective to obtain non-Gaussian information and high order moment information of state step prediction probability density function and state posterior probability density function. In addition, the proposed integral points updating method not only is suitable for determining system, but also for random system model. Secondly, we deduce general solution and special solution for proposed integral points updating method. Finally, we apply this method into Gaussian approximation filters and get a new Gaussian approximation filtering method. The experiments of our new method on single variable non-stationary growth model and vertical free-fall model demonstrate the effectiveness and superiority compared with other Gaussian approximation filtering methods. The structure of this paper is as follows. In Section 2, we introduce nonlinear Gaussian approximation filter including existing improved quadrature Kalman Filter. Then we detailed explain the proposed new Gaussian approximation filter method in Section 3. Next, experiments are shown in Section 4. Finally, there is a conclusion in Section 5.

## 2. Nonlinear Gaussian Approximation Filter.

**2.1. Gaussian approximation filter framework.** Considering the following discrete nonlinear control system [19-21] in state apace:

$$x_k = f_{k-1}(x_{k-1}) + w_{k-1}, \tag{1}$$

$$z_k = h_k(x_k) + v_k \tag{2}$$

where (1) and (2) are system equation and measurement equation respectively. $k$ is discrete-time series. $x_k \in R^n$ is state vector. $z_k \in R^m$ is system measurement vector. $f_{k-1}(\cdot)$ and $h_k(\cdot)$ are unrelated zero mean Gaussian white noise, which meets $E[w_k w_l^T] = Q_k \delta_{kl}$ and $E[v_k v_l^T] = R_k \delta_{kl}$ respectively. $\delta_{kl}$ is Kronecker-$\delta$ function. Initial state $x_0$ is Gaussian random vector whose mean value is $\widehat{x}_{0|0}$ and covariance matrix is $P_{0|0}$. Initial state $x_0$, system noisy $w_k$ and measurement noisy $v_k$ are independent. Nonlinear filtering obtains the minimum variance estimation $E[x_k|Z_k]$ according to current moment noise measurement and previous noise measurement. $Z_k = z_j, 1 \leq j \leq k$.

Under Gaussian approximation filter, state and measurement joint step prediction probability density function can be approximated as Gaussian, namely:

$$p(x_k, z_k|Z_{k-1}) = N\left(\begin{bmatrix} x_k \\ z_k \end{bmatrix}; \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{z}_{k|k-1} \end{bmatrix}, \begin{bmatrix} P_{k|k-1} & P_{\mathbf{x}z,k|k-1} \\ (P_{\mathbf{x}z,k|k-1})^T & P_{\mathbf{z}z,k|k-1} \end{bmatrix}\right) \tag{3}$$

where state step prediction $\hat{x}_{k|k-1}$ and its corresponding covariance matrix $P_{k|k-1}$ are the first and second order moment of $p_{x_k|Z_{k-1}}$ respectively. Measurement step prediction $\hat{z}_{k|k-1}$ and its corresponding covariance matrix $P_{\mathbf{z}z,k|k-1}$ are the first and second order moment of $p_{z_k|Z_{k-1}}$ respectively. $P_{\mathbf{x}z,k|k-1}$ is cross-covariance matrix of state and measurement. The following is the calculation for above matrixes.

$$\hat{x}_{k|k-1} = \int_{R^n} f_{k-1}(x_{k-1})N(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})dx_{k-1}. \tag{4}$$

$$\begin{aligned} P_{k|k-1} = &\int_{R^n} f_{k-1}(x_{k-1})f_{k-1}^T(x_{k-1})N\left(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}\right)dx_{k-1} \\ &- \hat{x}_{k|k-1}\hat{x}_{k|k-1}^T + Q_{k-1}. \end{aligned} \tag{5}$$

$$\hat{z}_{k|k-1} = \int_{R^n} h_k(x_k)N\left(x_k; \hat{x}_{k|k-1}, P_{k|k-1}\right)dx_k. \tag{6}$$

$$P_{\mathbf{z}z,k|k-1} = \int_{R^n} h_k(x_k)h_k^T(x_k)N\left(x_k; \hat{x}_{k|k-1}, P_{k|k-1}\right)dx_{k-1} - \hat{z}_{k|k-1}\hat{z}_{k|k-1}^T + R_k. \tag{7}$$

$$P_{\mathbf{x}z,k|k-1} = \int_{R^n} h_k^T(x_k)N\left(x_k; \hat{x}_{k|k-1}, P_{k|k-1}\right)dx_k - \hat{x}_{k|k-1}\hat{z}_{k|k-1}^T. \tag{8}$$

According to Formula (3), state posterior probability density function can be updated as Gaussian:

$$p(x_k|Z_k) = \frac{p(x_k, z_k|Z_{k-1})}{p(z_k|Z_{k-1})} = N(x_k; \hat{x}_{k|k}, P_{k|k}) \tag{9}$$

where state estimation $\hat{x}_{k|k}$ and corresponding estimation error covariance matrix is:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + W_k(z_k - \hat{z}_{k|k-1}), \tag{10}$$

$$P_{k|k} = P_{k|k-1} + W_k P_{\mathbf{z}z,k|k-1} W_k^T, \tag{11}$$

$$W_k = P_{\mathbf{x}z,k|k-1} P_{\mathbf{z}z,k|k-1}^{-1} \tag{12}$$

where $W_k$ is Kalman filter gain. As we all know, the basic framework of nonlinear filter is Kalman filter. So Gaussian approximation filter framework adopts that of Kalman filter.

Gaussian approximation filter is composed of time update and measurement update, and Formulas (4) and (5) belong to time update. (6), (7), (8) and (10), (11), (12) are measurement update. We mainly calculate the Gaussian weighted integral in (4)-(8). So Gaussian weighted integral can be written as the following unified form,

$$I[g] = \int_{R^n} g(x)N(x; \mu, \Sigma)dx. \tag{13}$$

In general, we adopt the following formula to solve (13).

$$\int_{R^n} g(x)N(x; \mu, \Sigma)dx \approx \sum_{i=1}^{N} \omega_i g(x_i) \tag{14}$$

where $x_i$ and $\omega_i$ are integral points and corresponding of Gaussian density $N(x; \mu, \Sigma)$. $N$ is the number of integral points. We use (14) to calculate Gaussian weighted integral in (4)-(8). Different $x_i$ and $\omega_i$ will result in different Gaussian approximation filters. Before updating time and measurement, existing Gaussian approximation methods need to produce integral points based on the Gaussian hypothesis. After updating, it will give up these integral points. However, it will repeat the above processes in the next time. In addition, it only retains low moment information of $p(x_k|Z_k)$ and $p(x_k|Z_{k-1})$, which limits estimation accuracy of Gaussian approximation filtering and loses high moment information.

2.2. **Existing improved quadrature Kalman filter.** Existing improved quadrature Kalman filter [14,22] does not need Gaussian hypothesis to repeat generate integral points. And we define the following variate.

$$\tilde{\xi}_k^- = \left[\xi_k^{(1)-} - \hat{x}_{k|k-1}, \cdots, \xi_k^{(N)-} - \hat{x}_{k|k-1}\right], \tag{15}$$

$$\tilde{\xi}_k^+ = \left[\xi_k^{(1)+} - \hat{x}_{k|k}, \cdots, \xi_k^{(N)+} - \hat{x}_{k|k}\right] \tag{16}$$

where $\xi_k^{(i)-}$ is integral points of state step prediction. $\tilde{\xi}_k^-$ is state prediction integral points error matrix. $\xi_k^{(i)+}$ is updated integral points at $k$ time. $\tilde{\xi}_k^+$ is state updating integral points error matrix. So the processes of existing improved quadrature Kalman filter can be described as:

1) Initialization. According to (14), we can get integral points $\xi_0^{(i)+}$ of density function $N(x_0; \hat{x}_{0|0}, P_{0|0})$ and corresponding weight $\omega_i$.
2) Time updating.
   - Spread updated integral points of previous time $\xi_{k-1}^{(i)+}$.
   $$\xi_k^{(i)-} = f_{k-1}\left(\xi_{k-1}^{(i)+}\right). \tag{17}$$
   - Compute state step prediction $\hat{x}_{k|k-1}$ and corresponding error covariance matrix $P_{k|k-1}$.
   $$\hat{x}_{k|k-1} = \sum_{i=1}^{N} \omega_i \xi_k^{(i)-}. \tag{18}$$
   $$P_{k|k-1} = \sum_{i=1}^{N} \omega_i \xi_k^{(i)-} \left(\omega_i \xi_k^{(i)-}\right)^T - \hat{x}_{k|k-1}\hat{x}_{k|k-1}^T. \tag{19}$$
   - Compute state prediction integral points error matrix $\tilde{\xi}_k^-$ by Formula (15).
3) Measurement updating.
   - Calculate measurement step prediction $\hat{z}_{k|k-1}$, corresponding covariance matrix $P_{\mathbf{z}z,k|k-1}$ and cross-covariance matrix $P_{\mathbf{x}z,k|k-1}$.
   $$\hat{z}_{k|k-1} = \sum_{i=1}^{N} \omega_i h_k \xi_k^{(i)-}. \tag{20}$$
   $$P_{\mathbf{z}z,k|k-1} = \sum_{i=1}^{N} \omega_i h_k \xi_k^{(i)-} h_k^T \xi_k^{(i)-} - \hat{z}_{k|k-1}\hat{z}_{k|k-1}^T + R_k. \tag{21}$$
   $$P_{\mathbf{x}z,k|k-1} = \sum_{i=1}^{N} \omega_i h_k \xi_k^{(i)-} h_k^T \xi_k^{(i)-} - \hat{x}_{k|k-1}\hat{z}_{k|k-1}^T. \tag{22}$$
   - Compute kalman filter gain $W_k$ through (12) and updating state estimation $\hat{x}_{k|k}$, corresponding estimation error covariance matrix $P_{k|k}$ by (10) and (11).

- Compute state updating integral points error matrix $\tilde{\xi}_k^+$.

$$\tilde{\xi}_k^+ = S_{k|k} S_{k|k-1}^{-1} \tilde{\xi}_k^- . \tag{23}$$

where $S_{k|k}$ and $S_{k|k-1}$ are root matrix of $P_{k|k}$ and $P_{k|k-1}$ respectively.

- Calculate updated integral points $\xi_k^{(i)+}$.

$$\left[\xi_k^{(1)+}, \cdots, \xi_k^{(N)+}\right] = \tilde{\xi}_k^+ + \left[\hat{x}_{k|k}, \cdots, \hat{x}_{k|k}\right]. \tag{24}$$

From (23), we can know that state updating integral points error matrix $\tilde{\xi}_k^+$ is obtained by linear transformation of state prediction integral points error matrix $\tilde{\xi}_k^-$, which ignores correction function of measurement integral points for state integral points and limits estimation accuracy of existing improved the quadrature Kalman filter. If $w_{k-1} = 0$ in (3), then the quadrature Kalman filter cannot be applied into random system model. Therefore, we propose an improved Gaussian approximation filter method using measurement integral points to correct state integral points. It has higher accuracy than other methods. In addition, it can be applied into determining system and random system.

## 3. **The Improved Gaussian Approximation Filter Method.**

3.1. **Updating integral points.** We firstly define the following variate before introducing new integral points.

$$\chi_k^{(i)} = f_{k-1}\left(X_{k-1}^{(i)+}\right), \tag{25}$$

$$\tilde{\chi}_k = \left[\chi_k^{(1)} - \hat{x}_{k|k-1}, \cdots, \chi_k^{(N)} - \hat{x}_{k|k-1}\right], \tag{26}$$

$$\tilde{X}_k^- = \left[X_k^{(1)-} - \hat{x}_{k|k-1}, \cdots, X_k^{(N)-} - \hat{x}_{k|k-1}\right], \tag{27}$$

$$Z_k^{(i)-} = h_k\left(X_k^{(i)-}\right), \tag{28}$$

$$\tilde{Z}_k^- = \left[Z_k^{(1)-} - \hat{z}_{k|k-1}, \cdots, Z_k^{(N)-} - \hat{z}_{k|k-1}\right], \tag{29}$$

$$\tilde{X}_k^+ = \left[X_k^{(1)+} - \hat{x}_{k|k}, \cdots, X_k^{(N)+} - \hat{x}_{k|k}\right], \tag{30}$$

$$\mathbf{w} = [w_1, \cdots, w_N]^T, \quad W = diag\mathbf{w} \tag{31}$$

where $X_{k-1}^{(i)+}$ is updated integral points at $k$ time. $\chi_k^{(i)}$ is integral points of $X_{k-1}^{(i)+}$ propagated by system function $f_{k-1}(\cdot)$. $\tilde{\chi}_k$ is error matrix of propagated integral points. $X_k^{(i)-}$ and $\tilde{X}_k^-$ are step prediction state integral points and state prediction integral points error matrix respectively. $Z_k^{(i)-}$ and $\tilde{Z}_k^-$ are step prediction measurement integral points and measurement prediction integral points error matrix respectively. $\mathbf{X}_k^{(i)+}$ is updated integral points at $k$ time. $\tilde{\mathbf{X}}_k^+$ is state updating integral points error matrix. $w$ is column vector composed of weight. $W$ is diagonal matrix composed of weight. So we can get the following formulas:

$$\tilde{\chi}_k w = 0; \quad \tilde{\chi}_k W \tilde{\chi}_k^T = P_{k|k-1} - Q_{k-1}; \quad \tilde{Z}_k^- w = 0. \tag{32}$$

$$\tilde{Z}_k^- W \left(\tilde{Z}_k^-\right)^T = P_{\mathbf{z}z,k|k-1} - R_k. \tag{33}$$

$$\tilde{X}_k^- W \left(\tilde{Z}_k^-\right)^T = P_{\mathbf{x}z,k|k-1}. \tag{34}$$

In this section, we adopt the following methods to update integral points.

$$\tilde{\mathbf{X}}_k^- = F\tilde{\chi}_k. \tag{35}$$

$$\tilde{\mathbf{X}}_k^+ = G\tilde{\mathbf{X}}_k^- - H\tilde{\mathbf{Z}}_k^-. \tag{36}$$

(35) and (36) satisfy the following constraint conditions.

$$\tilde{\mathbf{X}}_k^- w = 0; \quad \tilde{\mathbf{X}}_k^- W\left(\tilde{\mathbf{X}}_k^-\right)^T = P_{k|k-1}, \tag{37}$$

$$\tilde{\mathbf{X}}_k^+ w = 0; \quad \tilde{X}_k^+ W(\tilde{X}_k^+)^T = P_{k|k} \tag{38}$$

where $P_{k|k}$ is introduced in (11). The above constraint conditions show that updated integral points $\mathbf{X}_k^{(i)+}$ and $\mathbf{X}_k^{(i)-}$ can match mean value and covariance matrix of state posterior density $p(x_k|Z_k)$ and state step prediction density $p(x_k|Z_{k-1})$. Meanwhile, it keeps weight $w_i$ unchanged. After getting $\tilde{\mathbf{X}}_k^-$ and $\tilde{\mathbf{X}}_k^+$, we add state step prediction $\hat{x}_{k|k-1}$ and state estimation $\hat{x}_{k|k}$ into column of $\tilde{X}_k^-$ and $\tilde{X}_k^+$ to get state prediction integral points $X_k^{(i)-}$ and state updating integral points $X_k^{(i)+}$.

Equation (23) shows that other methods are only suitable for nonlinear system with determining system model (i.e., system noise covariance matrix $Qk-1=0$) to design the integral points updating method. And they ignore the effect correction function of measurement integral points on state integral points. From Equation (32) we can know that our method aims to the nonlinear system with random system model to design the integral points updating method. Therefore, we propose improved Gaussian approximation filter method, which not only can be applied into determining system, but random system. Equation (36) indicates that our new method utilizes the measurement integral points to correct state integral points.

Then, we will give the detailed calculation method on matrix $F$, $G$ and $H$.

**Theorem 3.1.** *If matrix $F$ meets following relationship:*

$$F = S_{k|k-1}M_1^T\left(S_{k|k-1}^1\right)^{-1} \tag{39}$$

*where $S_{k|k-1}$ is root matrix of $P_{k|k-1}$, $M_1$ is arbitrary orthogonal matrix, $S_{k|k-1}^1$ is root matrix of $P_{k|k-1} - Q_{k-1}$, namely,*

$$S_{k|k-1}S_{k|k-1}^T = P_{k|k-1}, \quad M_1M_1^T = I, \tag{40}$$

$$S_{k|k-1}^1\left(S_{k|k-1}^1\right)^T = P_{k|k-1} - Q_{k-1} \tag{41}$$

*where $I$ is unit matrix, therefore, constraint Equation (37) is true.*

**Proof:** From (32) and (35), we can get that:

$$\tilde{\mathbf{X}}_k^- w = F\tilde{\chi}_k w = F0 = 0. \tag{42}$$

So (37) is true for any matrix $F$. According to (32) and (35), we calculate matrix $\tilde{X}_k^- W(\tilde{X}_k^-)^T$,

$$\tilde{X}_k^- W\left(\tilde{X}_k^-\right)^T = F\tilde{\chi}_k W\left(F\tilde{\chi}_k\right)^T = F\tilde{\chi}_k W\tilde{\chi}_k^T F^T = F\left(P_{x|x-1} - Q_{k-1}\right)F^T. \tag{43}$$

We substitute (43) into (37), and get constraint equation,

$$F\left(P_{k|k-1} - Q_{k-1}\right)F^T = P_{k|k-1}. \tag{44}$$

According to (39)-(41), (44) can be calculated as:

$$\begin{aligned} &F\left(P_{k|k-1} - Q_{k-1}\right)F^T \\ &= S_{k|k-1}M_1^T\left(S_{k|k-1}^1\right)^{-1} \times \left(P_{k|k-1} - Q_{k-1}\right)\left(S_{k|k-1}M_1^T\left(S_{k|k-1}^1\right)^{-1}\right)^T \\ &= S_{k|k-1}S_{k|k-1}^T = P_{k|k-1}. \end{aligned} \tag{45}$$

If $F$ meets Equations (39)-(41), then (44) is true as well as (37). Next, we compute matrix $G$ and $H$. Firstly, we should find error matrix $\tilde{X}_k'^{+}$ satisfying the following constraint equation,

$$\tilde{X}_k'^{+} = A\tilde{X}_k^{-} - B\tilde{Z}_k^{-}. \tag{46}$$

$$\tilde{X}_k'^{+} w = 0. \tag{47}$$

$$\tilde{X}_k'^{+} W (\tilde{X}_k'^{+})^T = P_{k|k} - W_k R_k W_k^T. \tag{48}$$

Then we use the following linear transformation to transform $\tilde{X}_k'^{+}$ as $\tilde{\mathbf{X}}_k^{+}$.

$$\tilde{\mathbf{X}}_k^{+} = L\tilde{X}_k'^{+}. \tag{49}$$

Using (36), (46) and (49), we can get:

$$\tilde{X}_k^{+} = LA\tilde{X}_k^{-} - LB\tilde{Z}_k^{-} = G\tilde{X}_k^{-} - H\tilde{Z}_k^{-}. \tag{50}$$

So $G$ and $H$ can be expressed as:

$$G = LA, \quad H = LB. \tag{51}$$

**Theorem 3.2.** *If matrix $A$ and $B$ meet the following relation:*

$$A = BP_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} + EM_2^T S_{k|k-1}^{-1}, \tag{52}$$

$$EE^T = P_{k|k} - W_k R_k W_k^T - B\left[P_{\mathbf{x}z,k|k-1} - R_k - P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1}\right] B^T, \tag{53}$$

*where $W_k$ is Kalman filter gain, $M_2$ is arbitrary orthogonal matrix,*

$$M_2 M_2^T = I, \tag{54}$$

*then, constraint Equations (47) and (48) are true.*

**Proof:** We use Formulas (32), (37) and (46) to get:

$$\tilde{\mathbf{X}}_k'^{+} w = \left(A\tilde{\mathbf{X}}_k^{-} - B\tilde{\mathbf{Z}}_k^{-}\right) w = A\tilde{\mathbf{X}}_k^{-} w - B\tilde{\mathbf{Z}}_k^{-} w = A0 - B0 = 0. \tag{55}$$

So constraint Equation (47) is true for any matrix $A$, $B$. By Equation (46), $\tilde{\mathbf{X}}_k'^{+} W \left(\tilde{\mathbf{X}}_k'^{+}\right)^T$ can be described as:

$$\begin{aligned}\tilde{\mathbf{X}}_k'^{+} W \left(\tilde{\mathbf{X}}_k'^{+}\right)^T &= A\left(\tilde{\mathbf{X}}_k^{-} W \left(\tilde{\mathbf{X}}_k^{-}\right)^T\right) A^T - B\left(\tilde{\mathbf{X}}_k^{-} W \left(\tilde{\mathbf{Z}}_k^{-}\right)^T\right) A^T \\ &\quad - A\left(\tilde{\mathbf{X}}_k^{-} W \left(\tilde{\mathbf{Z}}_k^{-}\right)^T\right) B^T + B\left(\tilde{\mathbf{Z}}_k^{-} W \left(\tilde{\mathbf{Z}}_k^{-}\right)^T\right) B^T.\end{aligned} \tag{56}$$

Then, we put (32) (34) (37) into (56), and get:

$$\tilde{\mathbf{X}}_k'^{+} W \left(\tilde{\mathbf{X}}_k'^{+}\right)^T = AP_{k|k-1}A^T - BP_{\mathbf{x}z,k|k-1}^T A^T - AP_{\mathbf{x}z,k|k-1}B^T + B(P_{\mathbf{z}z,k|k-1})B^T. \tag{57}$$

We substitute (57) into (47) and get a constraint equation which is equivalent to (48).

$$AP_{k|k-1}A^T - BP_{\mathbf{x}z,k|k-1}^T A^T - AP_{\mathbf{x}z,k|k-1}B^T + B(P_{\mathbf{z}z,k|k-1})B^T = P_{k|k} - W_k R_k W_k^T. \tag{58}$$

Equation (58) can be divided by matrix decomposition.

$$\begin{aligned}&AS_{k|k-1}(AS_{k|k-1})^T - B\left(S_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1}\right)^T \times (AS_{k|k-1})^T \\ &- (AS_{k|k-1})S_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1} B^T + BP_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1} B^T \\ =\, &P_{k|k} - W_k R_k W_k^T - B\left(P_{\mathbf{z}z,k|k-1}\right) - R_k - P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1})B^T.\end{aligned} \tag{59}$$

After computing, (59) can be expressed as:

$$\begin{aligned} & AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^T AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^{T^T} \\ = & P_{k|k} - W_kR_kW_k^T - B\left(P_{\mathbf{z}z,k|k-1} - R_k - P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1}\right)B^T. \end{aligned} \tag{60}$$

According to (52)-(54), $AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^T$ and

$$AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^T AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^{T^T}$$

can be calculated as (61) and (62):

$$\begin{aligned} & AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^T \\ = & BP_{\mathbf{x}z,k|k-1}\left(S_{k|k-1}^T\right)^{-1} + EM_2^T - BP_{\mathbf{x}z,k|k-1}\left(S_{k|k-1}^T\right)^{-1} = EM_2^T. \end{aligned} \tag{61}$$

$$\begin{aligned} & AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^T AS_{k|k-1} - B\left(S_{k|k-1}^{-1}P_{\mathbf{x}z,k|k-1}\right)^{T^T} \\ = & P_{k|k} - W_kR_kW_k^T - B\left(P_{\mathbf{z}z,k|k-1} - R_k - P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} P_{\mathbf{x}z,k|k-1}\right)B^T. \end{aligned} \tag{62}$$

From (62) we can know that if $A$ and $B$ meet (52)-(54), so (58), (59) and (47), (48) are true.

**Theorem 3.3.** *If matrix $L$ satisfies the following relation*

$$L = S_{k|k}M_3^T D^{-1} \tag{63}$$

*where $S_{k|k}$ is root matrix of $P_{k|k}$, $M_3$ is arbitrary orthogonal matrix, $D$ is root matrix of $P_{k|k} - W_kR_kW_k^T$, so*

$$S_{k|k}S_{k|k}^T, \quad M_3M_3^T = I \tag{64}$$

$$DD^T = P_{k|k} - W_kR_kW_k^T. \tag{65}$$

*Constraint Equation (38) is true.*

**Proof:** $\tilde{X}_k^+ w$ can be expressed as follows by (47) and (49).

$$\tilde{X}_k^+ w = L\tilde{X}_k'^+ w = L0 = 0. \tag{66}$$

So constraint equation (38) is true for any matrix $L$. Using (48) and (49), we calculate $\tilde{X}_k^+ W\left(\tilde{X}_k^+\right)^T$,

$$\tilde{X}_k^+ W\left(\tilde{X}_k^+\right)^T = L\Big(P_{k|k} - W_kR_kW_k^T\Big)L^T. \tag{67}$$

We substitute (67) into (38) and get a constraint equation which is equivalent to (42).

$$L\Big(P_{k|k} - W_kR_kW_k^T\Big)L^T = P_{k|k}. \tag{68}$$

According to (63)-(65), $L\Big(P_{k|k} - W_kR_kW_k^T\Big)L^T$ can be calculated as:

$$L\Big(P_{k|k} - W_kR_kW_k^T\Big)L^T = S_{k|k}M_3^T D^{-1}DD^T\Big(D^T\Big)^{-1}M_3S_{k|k}^T = S_{k|k}S_{k|k}^T = P_{k|k}. \tag{69}$$

So if matrix $L$ meets Equations (63)-(65), then (68) and (38) are true. From Theorems 3.2 and 3.3, matrix $G$ and $F$ can be expressed as:

$$G = S_{k|k} M_3^T D^{-1} \Big( B P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} + E M_2^T S_{k|k-1}^{-1} \Big). \tag{70}$$

$$H = S_{k|k} M_3^T D^{-1} B. \tag{71}$$

(70) and (71) show that $G$ and $F$ are the function of unknown matrix $B$. To obtain the detailed value of matrix $G$ and $H$, we need to give the detailed $B$ value. Then, we will give two possible values of $B$ in the following. First, when $B = 0$, state updating integral points error matrix $\tilde{X}_k^+$ has only relation with state prediction integral points error matrix $\tilde{X}_k^-$. So Theorem 3.4 will give the solution of matrix $G$ and $H$.

**Theorem 3.4.** *If $B = 0$, and the following formula is true,*

$$P_{k|k} = W_k R_k W_k^T > 0, \tag{72}$$

*so $G$ and $H$ can be calculated by:*

$$G = S_{k|k} M_4^T S_{k|k-1}^{-1}, \quad H = 0 \tag{73}$$

*where $M_4$ is arbitrary orthogonal matrix.*

$$M_4 M_4^T = I. \tag{74}$$

**Proof:** It plugs $B = 0$ into Formula (71) and can get:

$$H = S_{k|k} M_3^T D^{-1} B = S_{k|k} M_3^T D^{-1} 0 = 0. \tag{75}$$

$B = 0$ is plugged into Formula (53) and it can get:

$$E E^T = P_{k|k} - W_k R_k W_k^T. \tag{76}$$

Utilizing (66) and (76), we get:

$$D D^T = E E^T = P_{k|k} - W_k R_k W_k^T. \tag{77}$$

Formula (72) and (77) have the orthogonal matrix $\overline{M}$:

$$E \overline{M} = D. \tag{78}$$

We plug (78) and $B = 0$ into (70) and obtain:

$$G = S_{k|k} (M_2 \overline{M} M_3)^T S_{k|k-1}^{-1} = S_{k|k} (M_4)^T S_{k|k-1}^{-1} \tag{79}$$

where

$$M_4 = M_2 \overline{M} M_3. \tag{80}$$

Because $M_2$, $\overline{M}$ and $M_3$ are orthogonal matrixes,

$$M_4 M_4^T = M_2 \overline{M} M_3 M_3^T \overline{M}^T M_2^T = I. \tag{81}$$

According to Formulas (79)-(81), we can know that $M_4$ is orthogonal matrix. And (73), (74) are true. $M_2$ and $M_3$ are any orthogonal matrixes, so $M_4$ is also orthogonal matrix.

**Theorem 3.5.** *If $B = W_k$, then matrix $G$ and $H$ can be calculated as:*

$$\begin{aligned} G = S_{k|k} M_3^T D^{-1} \Big( & W_k P_{\boldsymbol{x}z,k|k-1}^{-1} P_{k|k-1}^{-1} + S_{k|k-1} M_5 S_{k|k-1}^{-1} \\ & - W_k P_{\boldsymbol{x}z,k|k-1}^T S_{k|k-1}^{-T} M_5 S_{k|k-1}^{-1} \Big). \end{aligned} \tag{82}$$

$$H = S_{k|k} M_3^T D^{-1} W_k. \tag{83}$$

*$M_5$ is an orthogonal matrix.*

$$M_5 M_5^T = I. \tag{84}$$

**Proof:** We plug $B = W_k$ into (71) and get,

$$H = S_{k|k} M_3^T D^{-1} W_k. \tag{85}$$

$B = W_k$ is plugged into (53). It gets,

$$EE^T = P_{k|k} - W_k P_{\mathbf{x}z,k|k-1} W_k^T + W_k P_{\mathbf{x}z,k|k-1}^T P_{\mathbf{x}z,k|k-1}^{-1} W_k^T. \tag{86}$$

We use (12) and obtain,

$$W_k P_{\mathbf{x}z,k|k-1} W_k^T = W_k P_{\mathbf{x}z,k|k-1}^T = P_{\mathbf{x}z,k|k-1} W_k^T. \tag{87}$$

Then we plug (11) and (87) into (86),

$$EE^T = \left(S_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T}\right)\left(S_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T}\right)^T. \tag{88}$$

Using (11) and (87), the matrix $S_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T}$ can be expressed as:

$$S_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T} = \left(P_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1} W_k^T\right) S_{k|K-1}^{-T} = P_{k|k} S_{k|k-1}^{-T}. \tag{89}$$

We put (89) into (88) and get,

$$EE^T = P_{k|k} P_{k|k-1}^{-1} P_{k|k}. \tag{90}$$

$P_{k|k}$ and $P_{k|k-1}$ are state estimation error covariance matrix and step prediction error covariance matrix respectively. So they are positive definite matrix,

$$P_{k|k} > 0, \quad P_{k|k-1} > 0. \tag{91}$$

We plug (91) into (90) and get,

$$EE^T = P_{k|k} P_{k|k-1}^{-1} P_{k|k} > 0. \tag{92}$$

$\widehat{M}$ is an orthogonal matrix.

$$E = \left(S_{k|k-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T}\right) \widehat{M}. \tag{93}$$

We plug $B = W_k$ and (93) into (70), and get:

$$\begin{aligned} G = S_{k|k} M_3^T D^{-1} \Big( & W_k P_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} \\ & + S_{k|k-1} M_5 S_{k|k-1}^{-1} - W_k P_{\mathbf{x}z,k|k-1}^T S_{k|k-1}^{-T} M_5 S_{k|k-1}^{-1} \Big) \end{aligned} \tag{94}$$

where

$$M_5 = \widehat{M} M_2^T. \tag{95}$$

$\widehat{M}$ and $M_2$ are orthogonal matrixes. So we can get,

$$M_5 M_5^T = \widehat{M} M_2^T M_2 \widehat{M}^T = I. \tag{96}$$

According to Formula (94) and Formula (96), we know that $M_5$ is orthogonal matrix and (82) and (84) are true. $M_2$ is arbitrary orthogonal matrix, so $M_5$ is arbitrary orthogonal matrix too.

We can further simplify the specific solution in Theorem 3.5. Setting $M_5 = I$ and plugging it into (82), we can get,

$$G = S_{k|k} M_3^T D^{-1}. \tag{97}$$

Therefore, $G = S_{k|k}M_3^T D^{-1}$ and $H = S_{k|k}M_3^T|D^{-1}W_k$ are specific solutions of our method.

3.2. **Improved Gaussian approximation filter based on quadrature rule.** In this section, we give the detailed processes for proposed method, which contains time updating and measurement updating.

1) Initialization. According to (14) and related quadrature rule, it generates integral point $X_0^{(i)+}$ and corresponding weight value $\omega_i$ of posterior density function $N(x_0; \widehat{x}_{0|0}, P_{0|0})$.
2) Time updating.
   - We use (25) to spread the updated integral point $X_{k-1}^{(i)+}$ in last time and get the transmitted integral point $\chi_k^{(i)}$.
   - Compute state step prediction $\widehat{x}_{k|k-1}$.

$$\widehat{x}_{k|k-1} = \sum_{i=1}^{N} \omega_i \chi_k^{(i)}. \tag{98}$$

   - Use (26) to calculate error matrix $\widetilde{\chi}k$ of integral point.
   - Calculate state step prediction error covariance matrix $P_{k|k-1}$.

$$P_{k|k-1} = \widetilde{\chi}_k W \widetilde{\chi}_k^T + Q_{k-1}. \tag{99}$$

   - Use (39)-(41) and (35) to calculate matrix $F$ and state prediction integral point error matrix $\widetilde{X}_k^-$ respectively.
   - Compute step prediction state integral point $X_k^{(i)-}$.

$$\left[X_k^{(1)-}, \cdots, X_k^{(N)-}\right] = \widetilde{X}_k^- + \left[\widehat{x}_{k|k-1}, \cdots, \widehat{x}_{k|k-1}\right]. \tag{100}$$

3) Measurement updating.
   (a) Use (28) to calculate step prediction measurement integral point $Z_k^{(i)-}$.
   (b) Calculate measurement step prediction $\widehat{z}_{k|k-1}$.

$$\widehat{z}_{k|k-1} = \sum_{i=1}^{N} \omega_i Z_k^{(i)-}. \tag{101}$$

   (c) Use (29) to calculate measurement prediction integral point error matrix $\widetilde{Z}_k^-$.
   (d) Calculate measurement prediction error covariance matrix $P_{\mathbf{z}z,k|k-1}$ and cross-covariance matrix of state and measurement $P_{\mathbf{x}z,k|k-1}$.

$$P_{\mathbf{z}z,k|k-1} = \widetilde{Z}_k^- W \left(\widetilde{Z}_k^-\right)^T + R_k. \tag{102}$$

$$P_{\mathbf{x}z,k|k-1} = \widetilde{X}_k^- W \left(\widetilde{Z}_k^-\right)^T. \tag{103}$$

   (e) Use (12) to compute Kalman filter gain $W_k$, and use (10) and (11) to update state estimation $\widehat{x}_{k|k}$ and corresponding estimation error covariance matrix $P_{k|k}$.
   (f) Use (53), (54), (64), (65), (70), (71) to compute $G$ and $H$, and use (36) to calculate state updating integral point error matrix $\widetilde{X}_k^+$.
   (g) Calculate updated integral point $X_k^{(i)+}$.

$$\left[X_k^{(1)+}, \cdots, X_k^{(N)+}\right] = \widetilde{X}_k^+ + \left[\widehat{x}_{k|k}, \cdots, \widehat{x}_{k|k}\right]. \tag{104}$$

Our new method can directly update integral point without generating integral point repeatedly. Then we make a comparison to standard Gaussian approximation filter method with our new method.

First, standard Gaussian approximation filter method adopts Formulas (4)-(8) to calculate state and measurement step prediction average value $\widehat{x}_{k|k-1}$, $\widehat{z}_{k|k-1}$, and covariance matrix $P_{k|k-1}$, $P_{\mathbf{z}z,k|k-1}$, $P_{\mathbf{x}z,k|k-1}$. However, new method uses the following way.

$$\widehat{x}_{k|k-1} = \int_{R^n} f_{k-1}(x_{k-1})p(x_{k-1}|Z_{k-1})dx_{k-1}. \tag{105}$$

$$P_{k|k-1} = \int_{R^n} f_{k-1}(x_{k-1})f_{k-1}^T(x_{k-1}) \times p(x_{k-1}|Z_{k-1})dx_{k-1} - \widehat{x}_{k|k-1}\widehat{x}_{k|k-1}^T + Q_{k-1}. \tag{106}$$

$$\widehat{z}_{k|k-1} = \int_{R^n} h_k(x_k p(x_k|Z_{k-1})dx_k). \tag{107}$$

$$P_{\mathbf{z}z,k|k-1} = \int_{R^n} h_k(x_k)h_k^T(x_k)p(x_k|Z_{k-1})dx_k - \widehat{z}_{k|k-1}\widehat{z}_{k|k-1}^T + R_k. \tag{108}$$

$$P_{\mathbf{x}z,k|k-1} = \int_{R^n} x_k h_k^T(x_k)p(x_k|Z_{k-1})dx_k - \widehat{x}_{k|k-1}\widehat{z}_{k|k-1}^T. \tag{109}$$

And the state posterior probability density function $p(x_{k-1}|Z_{k-1})$, state step prediction probability density function $p(x_k|Z_{k-1})$ can be expressed as:

$$p\left(x_{k-1}|Z_{k-1}\right) = \sum_{i=1}^{N} \omega_i \delta\left(x_{k-1} - X_{k-1}^{(i)+}\right), \tag{110}$$

$$p\left(x_k|Z_{k-1}\right) = \sum_{i=1}^{N} \omega_i \delta\left(x_k - X_k^{(i)-}\right) \tag{111}$$

where $X_{k-1}^{(i)+}$ and $X_{k-1}^{(i)-}$ are directly updated by Theorems 3.1-3.3. Through comparing Formulas (4)-(8) and (105)-(111), when standard Gaussian approximation filter method calculates state and measurement step prediction average value, it needs to assume that state posterior probability density function and state step prediction probability density function are Gaussian condition. The new method does not require these assumptions.

Second, they adopt different methods to get integral points. Standard Gaussian approximation filter method uses numerical technique to produce integral points $\zeta_k^{(i)-}$ and $\zeta_k^{(i)+}$ under the condition of Gaussian assumption of state posterior probability density function and state step prediction probability density function. However, the new method adopts three-order sphere diameter volume rule to generate integral points $\zeta_k^{(i)-}$ and $\zeta_k^{(i)+}$.

$$\zeta_k^{(i)-} = \widehat{x}_{k|k-1} + S_{k|k-1}\lambda_i; \quad \zeta_k^{(i)+} = \widehat{x}_{k|k} + S_{k|k}\lambda_i \tag{112}$$

$$\lambda_i = \begin{cases} \sqrt{n}e_i & \text{when } i = 1, \cdots, n \\ -\sqrt{n}e_{i-n} & \text{when } i = n+1, \cdots, 2n \end{cases} \tag{113}$$

where $e_i$ denotes unit column vector of $i$-th element. The new method only uses state posterior probability density function and state step prediction probability density function to update integral points $X_k^{(i)-}$ and $X_k^{(i)+}$. Using (25)-(27) and (35), $X_k^{(i)-}$ can be shown as:

$$X_k^{(i)-} = \widehat{x}_{k|k-1} + F\left[f_{k-1}(X_{k-1}^{(i)+}) - \widehat{x}_{k|k-1}\right] \tag{114}$$

where $\widetilde{X}_k^{(i)-}$ and $\widetilde{\chi}_k^{(i)}$ denote $i$-th column of error matrix $\widetilde{X}_k^-$ and $\widetilde{\chi}_k$ respectively.

We apply (39) into (114), so $X_k^{(i)-}$ can be redefined as:

$$X_k^{(i)-} = \widehat{x}_{k|k-1} + S_{k|k-1}\eta_i \tag{115}$$

where

$$\eta_i = M_1^T \left(S_{k|k-1}^1\right)^{-1} \left[f_{k-1} \times \left(X_{k-1}^{(i)+}\right) - \widehat{x}_{k|k-1}\right]. \tag{116}$$

Using (27)-(29), (36), $X_k^{(i)-}$ can be expressed as:

$$X_k^{(i)+} = \widehat{x}_{k|k} + G\left[X_k^{(i)-} - \widehat{x}_{k|k-1}\right] - H\left[h_k(X_k^{(i)-}) - \widehat{z}_{k|k-1}\right] \tag{117}$$

where $\widetilde{X}_k^{(i)+}$ and $\widetilde{Z}_k^{(i)-}$ are the $i$-th column of error matrix $\widetilde{X}_k^+$ and $\widetilde{Z}_k^-$ respectively.

We plug (70) and (71) into (117), so $X_k^{(i)+}$ is:

$$X_k^{(i)+} = \widehat{x}_{k|k} + S_{k|k}\theta_i. \tag{118}$$

$\theta_i$ can be denoted by:

$$\begin{aligned}\theta_i = {} & M_3^T D^{-1}\left(BP_{\mathbf{x}z,k|k-1}^T P_{k|k-1}^{-1} + EM_2^T S_{k|k-1}^{-1}\right) \times \left[X_k^{(i)-} - \widehat{x}_{k|k-1}\right] \\ & - B\left[h_k(X_k^{(i)-}) - \widehat{z}_{k|k-1}\right].\end{aligned} \tag{119}$$

From (112), (113), (115), (118) we can know that, $\lambda_i$ is used for constructing integral point $\zeta_k^{(i)-}$ and $\zeta_k^{(i)+}$. $\eta_i$ and $\theta_i$ is used for updating integral point $X_k^{(i)-}$ and $X_k^{(i)+}$. According to (113), $\lambda_i$ is constrained on the axis, so $\zeta_k^{(i)-}$ and $\zeta_k^{(i)+}$ cannot capture high-order moment information of state posterior probability density function and state step prediction probability density function. From (118), (121), $\eta_i$ and $\theta_i$ are not constrained on the axis and they contain the nonlinear information of system model $f_{k-1}(\cdot)$ and measurement model $h_k(\cdot)$. Therefore, high-order moment and non-Gaussian information is kept in $X_k^{(i)-}$ and $X_k^{(i)+}$. Our new method can better approximate state posterior probability density function and state step prediction probability density function than standard Gaussian approximation filter.

Finally, we make a comparison to computation complexity of different methods. Computation complexity includes the number of floating point arithmetic of calculation matrix multiplication, Cholesky decomposition, inverse matrix, system function and measurement function. So the float point arithmetic total number of the different methods is:

$$\begin{aligned}f_1 = {} & O\left(12n^3 + m^3\right) + 9n^3 + 3n^2N + 2nN^2 + 2nNm + mN^2 + Nm^2 \\ & + 7mn^2 + 6nm^2 + Nn + Nm + nm + NM_f + NM_h,\end{aligned} \tag{120}$$

$$\begin{aligned}f_2 = {} & O\left(3n^3 + m^3\right) + n^3 + 2n^2N + nNm + Nm^2 + mn^2 + 2nm^2 \\ & + 3Nn + 2Nm + n^2 + m^2 + NM_f + NM_h,\end{aligned} \tag{121}$$

$$\begin{aligned}f_3 = {} & O\left(2n^3 + m^3\right) + n^2N + nNm + Nm^2 + mn^2 + 2nm^2 + 3Nn \\ & + 2Nm + 2nm + n^2 + m^2 + NM_f + NM_h\end{aligned} \tag{122}$$

where $f_1$, $f_2$ and $f_3$ are the float point arithmetic total number of our new method, existing improved Gaussian approximation filter and standard Gaussian approximation filter respectively. $O(\cdot)$ is the order number of floating point calculation. $N$ is number of integral point. $M_f$ and $M_h$ denote the floating point arithmetic number in system function and measurement function. From (121)-(123), we know that:

$$f_1 > f_2 > f_3. \tag{123}$$

Formula (123) shows that the new method has higher approximation accuracy and higher computational complexity than other Gaussian approximation filter methods.

4. **Experiments and Analysis.** In this section, we use single variable non-stationary growth model and vertical free-fall model to verify our method's effectiveness and superiority. We conduct the experiments on MATLAB platform. According to the real situation, parameters are selected as $B = W_k$, $M_3 = M_5 = I$. We use the same integration rule to compare new method and existing improved methods. Before experiments, we adopt unscented transformation to process the method (free parameter $\kappa = 3 - n$) and get the standard UKF, improved UKF ($B = 0$) and proposed UKF ($B = W_k$). In addition, to further demonstrate the superiority of the proposed method, we compare the standard Cubature Kalman filters (CKF), odd-order filter DDF (its interval length is $h = \sqrt{3}$), existing proposed UKF and our method.

4.1. **Single variable non-stationary growth model.** Single variable non-stationary growth model has been widely used as a standard question to verify the performance of nonlinear filter, and its state space model can be expressed as follows:

$$x_k = 0.5x_{k-1} + 25\frac{x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2k) + w_{k-1}, \tag{124}$$

$$z_k = \frac{x_k^2}{20} + v_k \tag{125}$$

where system noise vector $w_k$ and measurement noise vector $v_k$ are independent white Gaussian noise. Their zero mean variances are $Q_k = 1$ and $R_k = 3$. Initial state $x_0$ is Gaussian random variable with mean value 0.1 and variance 1. In each iteration, initial state estimation $\widehat{x}_{0|0}$ is randomly selected from Gaussian density function $N(x_0; 0.1, 1)$. All filters have the same initial conditions. Simulation time $T = 10000s$. Root mean square error (RMSE) is as the performance index:

$$RMSE_{x_k} = \sqrt{\frac{1}{T}\sum_{k=1}^{T}\left(x_k^l - \hat{x}_k^l\right)^2} \tag{126}$$

where $x_k^l$ and $\hat{x}_k^l$ are the real value and estimation value respectively at $l$-th iteration. Figure 1 is the RMSE comparison curve after 50 iterations. Table 1 is the average RMSE value. Table 2 is the single step running time. Note when $n + \kappa = h^2$ and $n = 1$, standard UKF is equal to 2-order UKF.

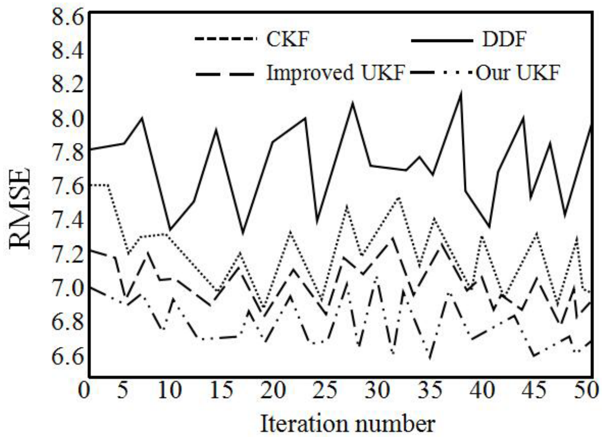


Figure 1. RMSE comparison curve

TABLE 1. Average RMSE with different methods

| Filter | CKF | DDF | Improved UKF | Our UKF |
|---|---|---|---|---|
| Average RMSE | 7.418 | 7.788 | 7.273 | 7.008 |

TABLE 2. Single step running time with different methods

| Filter | CKF | DDF | Improved UKF | Our UKF |
|---|---|---|---|---|
| Running time | $0.21 \times 10^{-3}$ | $0.22 \times 10^{-3}$ | $0.28 \times 10^{-3}$ | $0.32 \times 10^{-3}$ |

From Figure 1 and Table 1, the improved UKF and our method have smaller RMSE than standard UKF and DDF. When $B = W_k$, RMSE of our method is smaller than $B = 0$. From Table 2, this paper's method needs more single step running time. Therefore, the new method has higher estimation precision.

4.2. **Vertical free-fall model.** Vertical free-fall model has been widely used to verify the nonlinear filtering performance. In this subsection, we will use the measurement amplitude of radar to estimate the height, velocity and ballistic coefficient of the vertical fall. Vertical free-fall model is as shown in Figure 2. $x_1(t)$ and $x_2(t)$ denote the height of faller and falling velocity respectively. $r(t)$ is the distance between radar and faller. $M$ is horizontal distance. $Z$ is the height of radar.

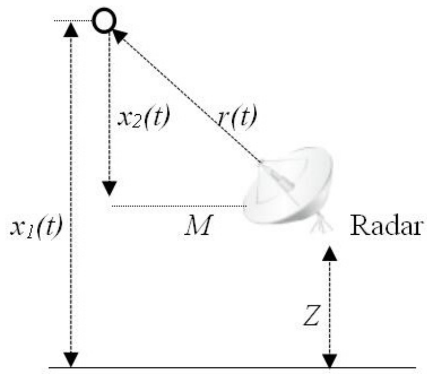


FIGURE 2. Vertical free-fall model

Continuous time dynamic equation of vertical free-fall is:

$$\hat{x}_1(t) = -x_2(t), \tag{127}$$

$$\hat{x}_2(t) = -e^{-\gamma x_1(t)} x_2^2(t) x_3(t), \tag{128}$$

$$\hat{x}_3(t) = 0 \tag{129}$$

where $x_3(t)$ denotes ballistic coefficient. $\gamma$ is known constant. So measurement function is:

$$y_k = \sqrt{M^2 + (x_{1,k} - Z)^2} + v_k \tag{130}$$

where $v_k \sim N(0, R_k)$. Radar measures the distance every 1 second. In order to eliminate the effect of system strongly nonlinear, we use 4-order Runge-Kutta to make integration for (132)-(134). Parameters are: $T = 60s$, $\gamma = 5 \times 10^{-5}$, $Z = 10^5 ft$, $R_k = 10^4 ft^2$. System true state initial value $x_0 = [3 \times 10^5 \quad 2 \times 10^4 \quad 10^{-3}]^T$. Initial filter value $\hat{x}_{0|0} =$

$[3.5 \times 10^5 \quad 2.5 \times 10^4 \quad 3 \times 10^{-5}]^T$. Covariance matrix $P_{0|0} = diag{\times}10^6, 4 \times 10^6, {\times}10^{-4}$. We use average absolute error (AAE) as the performance index. AAE is defined as:

$$AAE_k = \frac{1}{t}\sum_{n=1}^{t} |x_k^n - x_{k|k}^n| \tag{131}$$

where $t$ is iteration number. We also use standard DDF, standard UKF, existing improved UKF and our method to make a comparison. After 250 iterations, the results are as Figures 3-5. Table 3 is the mean of average absolute error (MEAE). Table 4 is the single step running time. Figure 3 shows that MEAE with DDF is the worst, followed by UKF and improved UKF. MEAE with our new method can reach convergence in a short time of nearly 15s. Hight MEAE with our method is 73.209 smaller than DDF(143.702), UKF(144.002) and improved UKF(92.059). The differences of speed MEAE are obvious that our method is the smallest in Figure 4. Although the curve of the four method is similar, our UKF is the optimal choice. Also ballistic coefficient MEAE with new method is just $1.885 \times 10^{-5}$, which is the best method shown in Figure 5.

From Figures 3-5, we can know that our new method has smaller average absolute error value. When $B = W_k$, the new method has the lowest average absolute error value compared with $B = 0$.

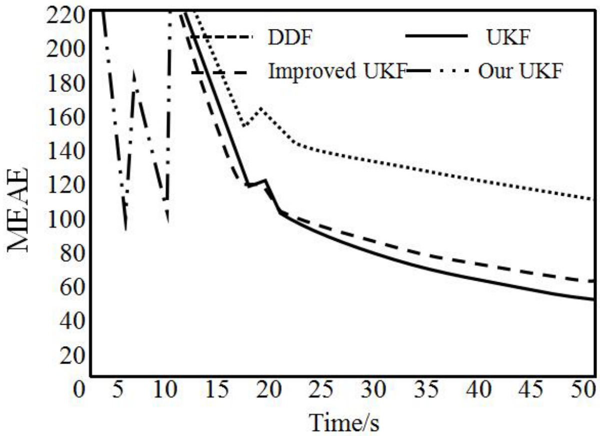


Figure 3. Hight MEAE comparison curve

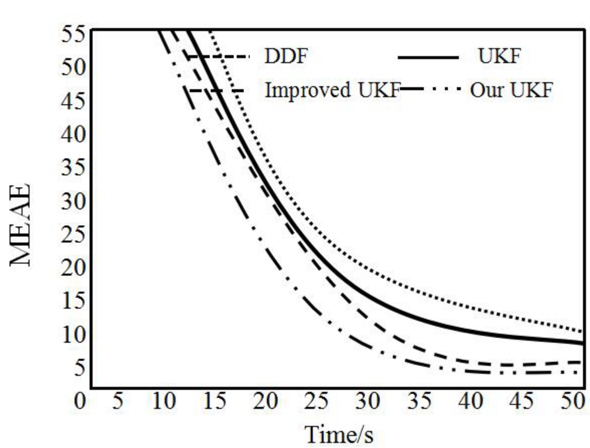


Figure 4. Speed RMSE comparison curve

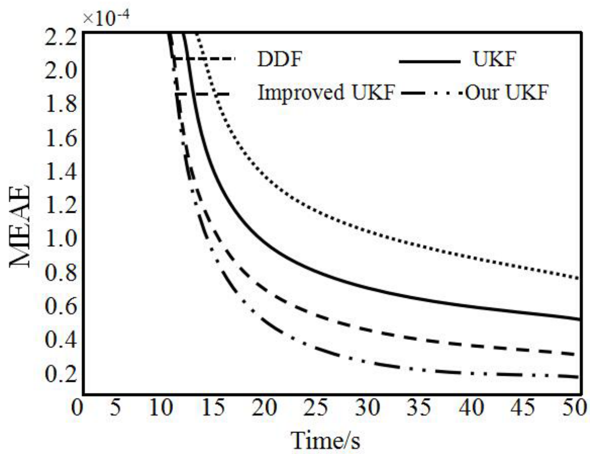


Figure 5. Ballistic coefficient RMSE comparison curve

TABLE 3. Average absolute error mean with different methods

| Filter | Hight MEAE | Speed MEAE | Ballistic coefficient MEAE |
|---|---|---|---|
| DDF | 143.702 | 9.635 | $5.174 \times 10^{-5}$ |
| UKF | 144.002 | 9.658 | $5.183 \times 10^{-5}$ |
| Improved UKF | 92.059 | 5.722 | $3.086 \times 10^{-5}$ |
| Paper's method | 73.209 | 3.498 | $1.885 \times 10^{-5}$ |

TABLE 4. Single step running time with different methods

| Filter | DDF | UKF | Improved UKF | Our UKF |
|---|---|---|---|---|
| Running time | $0.28 \times 10^{-3}$ | $0.25 \times 10^{-3}$ | $0.34 \times 10^{-3}$ | $0.49 \times 10^{-3}$ |

5. **Conclusions.** This paper proposes a new Gaussian approximation filtering method. The new method can better capture the non-Gaussian information and high-order moment of state step prediction probability density function and state posterior probability density function. In addition, the proposed method not only can be fit for determined model, but also fit for the random model. The simulation results show that the new Gaussian approximation filter method has higher estimation accuracy than the existing methods. In the future, we will reduce the computation complexity of our method using more advanced Gaussian filter method. And we would apply it into actual engineering projects.

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