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STRUCTURE OF BCK-ALGEBRAS: A NEW APPROACH BASED ON COMPLETE RESIDUATED LATTICE-VALUED LOGIC

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Abstract. *Under the semantic frame of complete residuated lattice-valued logic, we establish BCK-algebra theory based on complete residuated lattice-valued logic. In BCK-algebra, the concepts of ideals, positive implicative ideals, implicative ideals and filters have ever been depicted by classical set theory, but now, they are redefined by a unary predicate on complete residuated lattice-valued logic, and their properties and relations between them are discussed.*

Keywords: Complete residuated lattice-valued logic, L-fuzzifying ideal, L-fuzzifying positive implicative ideal, L-fuzzifying implicative ideal, L-fuzzifying filter

1. **Introduction.** One significant function of artificial intelligence is to make a computer simulate a human being in dealing with uncertain information, and logic establishes the foundations for it. However, certain information process is based on the classic logic. However, there are many uncertainties (such as fuzziness, incomparabilities, and randomness) in real world, we can handle these uncertainties by using non-classical logics [1,2], such as lattice-valued logics. As the matter of fact, non-classical logics have been proved to be a formal and useful technique for computer science to deal with fuzzy and uncertain information. Many-valued logic, being an extension and development of classical logic, has always been a crucial direction in non-classical logic. Various logical algebras, such as residuated lattices [3,4], lattice implication algebras [5], BL-algebras, MV-algebras, and MTL-algebras, have been proposed for semantical systems of non-classical logic systems. Among these logical algebras, residuated lattices are the basic and important algebraic structure because residuated lattices have close links with various important logics and the other mentioned logical algebras are all particular cases of residuated lattices. For example, if the multiplication adjoint to the residuation coincides with the meet, then residuated lattices reduce to Heyting algebras, which plays an important role in the investigation of intuitional formal theories. Pavelka [6-8] used residuated lattice logic as a tool to cope with inexact reasoning and as a basis of fuzzy logic.

Imai et al. [9,10] introduced BCK-algebras which generalizes the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory [11-16]. Moreover, in the process of developing the algebraic structure of the BCK-subalgebras, a theory of filters, a concept dual to that of ideals, has been developed for BCK-algebras [17,18]. Actually, the filters introduced by Deeba [17] are dual to lattice ideals in BCK-algebras, so Meng called them lattice filters in his paper [18] and presented a new notion of BCK-filters which are dual to BCK-ideals. Zadeh [19] introduced the concept of fuzzy sets. At present these ideas have been applied to many mathematical branches, such as groups, rings, functional analysis, vector spaces, probability theory, and topology. In 1991, Xi [20] applied this concept to BCK-algebras, Jun and Roh [21], Jun et al. [22] investigated fuzzy commutative ideals and fuzzy positive implicative ideals in BCK-algebras, respectively.

Notably, Rosser and Turquette [23] in 1952 emphatically proposed that if there exist many-valued theories beyond the level of predicate calculus, then what are the details of such theories? Ying [25,26] has made an attempt to give a partial answer in case of point set topology to the question raised above by Rosser and Turquette. He used semantical method of continuous-valued logic $Ł_{\aleph_1}$ to develop fuzzy topology from a completely different direction. In 2001, Zhang et al. [27] used this method to develop BCK-algebras based on continuous-valued logic $Ł_{\aleph_1}$. They presented the concepts of fuzzifying subalgebras, fuzzifying ideals and fuzzifying implicative ideals in BCK-algebras based on continuous-valued logic. In 1993, Ying [28] adopted the semantical method of complete residuated lattice-valued logic to build elementary L-fuzzifying topology, and generalized some results obtained in [24-26]. After then, Qin [29,30] and Peng [31] applied complete residuated lattice-valued logic to automata and grammar theory, and have obtained plenty of interesting results.

In this paper, we develop a new approach to structure of BCK-algebras, which is based on complete residuated lattice-valued logic and extend the structure of BCK-algebras based on continuous-valued logic $Ł_{\aleph_1}$ proposed by Zhang et al. [27] in two directions: firstly, BCK-algebra live in general complete residuated lattice-valued logic, which reduces to continuous-valued logic $Ł_{\aleph_1}$, when the set of truth values is Łukasiewicz interval; secondly, we use the unary fuzzy predicate and the semantic method of complete residuated lattice-valued logic [28-37] to be BCK-algebra theory, and generalize the results of [20-22,27]. Previous fuzzy structures of BCK-algebras are mainly introduced by defining a number of specific fuzzy sets, which are just some simple fuzzifications for classical problems, and also it is a lack of hierarchical structure. In the present paper, the notions of BCK-algebras structures are defined by fuzzy predicates. Therefore, we develop fuzzy BCK-algebras from a completely different direction, and features of our work are that the truth degree of formulas, axioms and inference rules are graded. This grade takes the valued in a lattice as a sign, and the results obtained form this study are systemic and beautiful.

This paper is organized as follows. In Section 2, we recall some basic notions and results of BCK-algebras and residuated lattice-valued logic. In Section 3, the concepts of L-fuzzifying subalgebras and L-fuzzifying left reduced ideals are introduced, and their homomorphic image and inverse image are studied. In Section 4, we introduce the notion of L-fuzzifying ideals and investigated its properties. L-fuzzifying ideals are an important algebraic structure of BCK-algebras; in particular, it is an L-fuzzifying implicative ideals and L-fuzzifying positive implicative ideals are introduced in Section 5, which are two special L-fuzzifying ideals. And their relations, properties and product algebras are discussed. In Section 6, the L-fuzzifying filters, a notion dual to that of L-fuzzifying ideals, are introduced. Further, the concept of the L-fuzzifying BCK-filters is propsed, the relations between L-fuzzifying filter and L-fuzzifying BCK-filters are given, and their properties are considered. Finally, we conclude the paper with a summary in Section 7.

2. **Preliminaries.**

This section aims at recalling some preliminaries concerning BCK-algebras and residuated lattice-valued logic, which can be used in other sections.

Definition 2.1. [32] *A $(2,0)$-type algebra $(X,*,0)$ is called a BCK-algebra if it satisfies the following conditions: for any $x,y,z \in X$*

(1) $((x*y)*(x*z))*(z*y)=0$,

(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $0 * x = 0$,
(5) $x * y = y * x = 0 \Longrightarrow x = y$.

Define a binary relation $\leq$ on X: $x \leq y$ if and only if $x * y = 0$, where $x, y \in X$. Then $(X, \leq)$ is a partially ordered set with the least element 0. In any BCK-algebra X, the following relations [17,18] hold for all $x, y, z \in X$:
(a) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
(b) $x \leq y$ and $y \leq z$ imply $x \leq z$,
(c) $(x * y) * z = (x * z) * y$,
(d) $x * (x * (x * y)) = x * y$,
(e) $x * y \leq x$.

A BCK-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. A non-empty subset S of a BCK-algebra X is said to be a $\sqcap$-closed (a $\sqcup$-closed) system if for any $x, y \in S$, there exists a greatest lower bound $x \sqcap y$ (a least upper bound $x \sqcup y$) in S. A BCK-algebra X is said to be a commutative BCK-algebra X if $x \dot{\wedge} y = y \dot{\wedge} x$ for all $x, y \in X$, where $x \dot{\wedge} y = y * (y * x)$ (see [17]). Let 1 be unit element in a bounded BCK-algebra X, and define an operator N:

$$N : X \to X, x \mapsto 1 * x.$$

For any $x \in X$, we denote $1 * x$ by Nx.

In any bounded commutative BCK-algebra X (see [18]) we have that
a) $N0 = 1$ and $N1 = 0$,
b) $NNx \leq x$,
c) $x \leq y$ implies $Ny \leq Nx$,
d) $Nx * y = Ny * x$,
e) $Nx \dot{\vee} Ny = N(x \dot{\wedge} y)$,
f) $Nx \dot{\wedge} Ny = N(x \dot{\vee} y)$,
where $x \dot{\vee} y = N(Nx \dot{\wedge} Ny)$.

A non-empty subset S of BCK-algebra X is called a subalgebra [14] of X if, for any $x, y \in S$, we have $x * y \in S$.

A non-empty subset I of BCK-algebra X is said to be an ideal [33] of X if it satisfies: for any $x, y \in X$
(1) $0 \in I$,
(2) $x * y \in I, y \in I \Longrightarrow x \in I$.

A non-empty set F of a BCK-algebra X is said to be a filter [17] of X if:
(1) $x \in F$ and $y \geq x$ imply that $y \in F$,
(2) $x \in F$ and $y \in F$ imply that $x \sqcap y \in F$.

A non-empty set F of a bounded BCK-algebra X is said to be a BCK-filter [18] of X if it satisfies the following conditions:
(1) $1 \in F$,
(2) $N(Nx * NY) \in F$ and $y \in F$ imply $x \in F$.

If $(X, *, 0)$ and $(X', *', 0')$ are BCK-algebras, then a map f of X onto X' is called a homomorphism [32] if, for any $x, y \in X$, we have

$$f(x * y) = f(x) *' f(y).$$

A BCK-algebra X is called a quasi-right alternate BCK-algebra [34] if $x * (y * y) = (x * y) * y$ for any $x, y \in X$, $x \neq y$. X is called a positive implicative BCK-algebra [33] if for any $x, y, z \in X$, we have $(x * z) * (y * z) = (x * y) * z$.

Suppose X is a BCK-algebra, a non-empty subset M is called an implicative ideal [14] of X if for any $x, y, z \in X$

(1) $0 \in M$,

(2) $(x * y) * z \in M, y * z \in M \Longrightarrow x * z \in M$.

For convenience, we make some explanations and conventions of notation for complete residuated lattice-valued logic. Let $\mathcal{L} = \langle L, +, \cdot, \odot, \alpha \rangle$ be a complete residuated lattice, i.e., $\langle L, +, \cdot \rangle$ a complete lattice, whose least and greatest element are 0, 1 respectively, and $\odot$, α two binary operations on L such that $\odot$ is isotone and $\langle L, \odot, 1 \rangle$ a commutative monoid, α is antitone in the first and isotone in the second variable and couple with $\odot$ as $a \odot b \leq c$ if and only if $a \leq b\alpha c$ for all $a, b, c \in L$. We call an L-valued logic if its true value is acquired in complete residuated lattice $\mathcal{L}$, still denoting it as $\mathcal{L}$. The L-valued logic $\mathcal{L}$ possesses nullary connectives $\mathbf{a}$ $(a \in L)$, the usual connectives $\vee$, $\wedge$ and $\to$, and an additional binary connective $\natural$ as well as usual ones. The symbols $\forall$ and $\exists$ are quantifier symbols in L-valued logic $\mathcal{L}$. If x and B are respectively individual and set variable element, then x and B are atomic formulas in L-valued logic $\mathcal{L}$. All of atomic formulas are formulas, and $\varphi \vee \psi$, $\varphi \wedge \psi$, $\phi \to \psi$, $\phi \natural \psi$, $(\forall x)\phi(x)$ and $(\exists x)\phi(x)$ are formulas if φ and ψ are two formulas. In an L-valued logic $\mathcal{L}$, the degree of truth value of a proposition φ is represented as $[\varphi]$. The only designated truth value is 1, in other words, a formula φ is valid, we write $\models^{\mathcal{L}} \varphi$, if and only if $[\varphi] = 1$ for every interpretation, and the rules of truth value of predicate logical formulae and set theoretical formulae are listed as follows:

(a) $[\mathbf{a}] = a$ $(a \in L)$, $[\varphi \vee \psi] = [\varphi] + [\psi]$, $[\varphi \wedge \psi] = [\varphi] \cdot [\psi]$, $[\varphi \natural \psi] = [\varphi] \odot [\psi]$, $[\varphi \to \psi] = [\varphi]\alpha[\psi]$;

(b) if X is the universe, then

$$[(\exists x)\varphi(x)] = \sum_{x \in X} [\varphi(x)], \quad [(\forall x)\varphi(x)] = \prod_{x \in X} [\varphi(x)];$$

(c) if X is the universe, A is a fuzzy set of X, x $(x \in X)$ is an individual variable element, then

$$[x \in A] = A(x).$$

In addition, the following derived formulae are necessary:

(i) $\neg\varphi := \varphi \to \mathbf{0}$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\varphi \sim \psi := (\varphi \to \psi) \natural (\psi \to \varphi)$;

(ii) $A \subseteq B := (\forall x)((x \in A) \to (x \in B))$, $A \equiv B := (A \subseteq B) \wedge (B \subseteq A)$, $A \approx B := (A \subseteq B) \natural (B \subseteq A)$.

For more detailed account of complete residuated lattice-valued logic, see [6-8].

In this paper, $\mathcal{F}(X)$ denotes the L-fuzzy power set of X, and $\mathcal{F}(\mathcal{F}(X))$ stands for the set of fuzzy predicates on X (see [27,28]), where the universe X is an arbitrary BCK-algebra.

3. *L*-Fuzzifying Subalgebras.

In this section, the concept of L-fuzzifying subalgebras of a BCK-algebra will be introduced under the frame of lattice-valued logic. The equivalent characterizations of these fuzzy subalgebras will be given.

Definition 3.1. *Let X be a BCK-algebra, a unary fuzzy predicate $s \in \mathcal{F}(\mathcal{F}(X))$ is said to be an L-fuzzifying subalgebra of X, if it is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,*

$$A \in s := (\forall x)(\forall y)((x \in A) \wedge (y \in A) \to (x * y \in A)).$$

Remark 3.1. *For any $A \in \mathcal{F}(X)$, the truth value $[A \in s]$ of $A \in s$ can be taken to any one of the possible values of L. Now, we consider a special case $[A \in s] = 1$: it means that $1 = [A \in s] = \prod_{x,y \in X}(A(x) \cdot A(y))\alpha A(x * y)$, i.e., $A(x * y) \geq A(x) \cdot A(y)$ for any $x, y \in X$. Obviously, this is the condition of the definition of the classical fuzzy subalgebra of X. In particular, when L is real unit interval $[0, 1]$, this condition becomes* *that $A(x * y) \geq \min\{A(x), A(y)\}$ for any $x, y \in X$, a condition of Xi's definition of the classical fuzzy subalgebra in [20]. Therefore, our Definition 3.1 is a generalization of the definition of the classical fuzzy subalgebra of X. Similarly, the following definitions and results in our paper are generalizations of the classical fuzzy algebraic structure of BCK-algebras.*

In fact, $[A \in s] = 1$ means that $A \in s$ is a tautology in complete residuated lattice-valued logic $\mathcal{L}$ system. As discussed above, L-fuzzifying algebraic structures (they are characterized by introducing a unary fuzzy predicate and by adopting the semantic method of complete residuated lattice-valued logic $\mathcal{L}$) are different from L-fuzzy algebraic structures (they mean that the associated membership function takes its values from the complete residuated lattice L under Xi's idea [20]), for example, all the formulas of the latter are only the tautologies of the former. From a methodological point of view, it is obvious that the semantic method is completely different from Xi's method of membership function of the classical fuzzy set. On the basis of Ying's idea [28], we use semantic method of complete residuated lattice-valued logic to develop fuzzy algebraic structure of BCK-algebras from a completely different direction in this paper and thereby establish elementary L-fuzzifying algebraic structure of BCK-algebras which is dual to the existing L-fuzzy algebraic structure of BCK-algebras. When the set of truth values is Łukasiewicz interval, the underlying logic of L-fuzzifying algebraic structures is the continuous-valued logic $L_{\aleph_1}$.

Theorem 3.1. $\models^{\mathcal{L}} A \in s \to (\forall x)\ (x \in A \to 0 \in A)$.

Proof: By Definition 3.1 and $b \cdot b = b$ for every $b \in L$, we have

$$[A \in s] = \prod_{x,y \in X} (A(x) \cdot A(y)) \alpha A(x * y) \leq \prod_{x \in X} (A(x) \cdot A(x)) \alpha A(x * x) = \prod_{x \in X} (A(x) \alpha A(0)).$$

And so Theorem 3.1 is valid. □

Theorem 3.2. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$, and then*

$$\models^{\mathcal{L}} A \in s \to (\forall x)(x \in A \to (\underbrace{x * (\cdots (x * (x * x))}_{2k} \cdots)) \in A),\ k = 1, 2, \cdots;$$

$$\models^{\mathcal{L}} A \in s \to (\forall x)(x \in A \leftrightarrow (\underbrace{x * (\cdots (x * (x * x))}_{2k+1} \cdots)) \in A),\ k = 0, 1, 2, \cdots;$$

$$\models^{\mathcal{L}} A \in s \to (\forall x)(x \in A \to ((\cdots \underbrace{((x * x) * x) \cdots) * x}_{n}) \in A),\ n = 1, 2, \cdots.$$

Proof: Since $x * x = 0$ and $x * 0 = x$ in BCK-algebras, then

$$\begin{aligned} & [(\forall x)(x \in A \to (\underbrace{x * (\cdots (x * (x * x))}_{2k} \cdots)) \in A)] \\ = & [(\forall x)(x \in A \to (\underbrace{x * (\cdots (x * (x}_{2k-2} * 0)) \cdots)) \in A)] \\ = & \cdots \\ = & [(\forall x)(x \in A \to (x * x \in A)] \\ = & \prod_{x \in X} A(x) \alpha A(0) \\ \geq & [A \in s]. \end{aligned}$$

The first is proved. Similarly, the second formula can be proved. Using $x*x = 0$, $x*0 = x$, the third can be proved too. □

Corollary 3.1. *Let X be a BCK-algebra, $A \in \mathcal{F}(X)$, and $x_1, x_2, \cdots, x_n$ be arbitrary elements of X. If at least one x_k of the set $\{x_2, x_3, \cdots, x_n\}$ equals x_1, then*

$$\models^{\mathcal{L}} A \in s \to (\forall x)(x \in A \to ((\cdots((x_1 * x_2) * x_3) \cdots) * x_n) \in A).$$

Proof: For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * y$. So we can exchange x_k to the position of x_2. Using $x_1 * x_1 = 0$, $0 * x_i = 0$, we have

$$\begin{aligned}\prod_{x \in X} A(x) \alpha A((\cdots((x_1 * x_2) * x_3) \cdots) * x_n) &= \prod_{x \in X} A(x) \alpha A((\cdots((x_1 * x_1) * x_2) \cdots) * x_n) \\ &= \cdots \\ &= \prod_{x \in X} A(x) \alpha A(0) \\ &\geq [A \in s].\end{aligned}$$

□

Definition 3.2. *Let X be a BCK-algebra. A unary fuzzy predicate $I_l \in \mathcal{F}(\mathcal{F}(X))$, called an L-fuzzifying left reduced ideal of X, is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,*

$$A \in I_l := (\forall x)(\forall y)(y \in A \to x * y \in A).$$

Similarly, we can define an L-fuzzifying right reduced ideal $I_r \in \mathcal{F}(\mathcal{F}(X))$ of X. For an L-fuzzifying left reduced ideal, we have

Theorem 3.3. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} A \in I_l \to (\forall x)(x \in A \leftrightarrow 0 \in A).$$

Proof: By Definition 3.2, $x * 0 = x$ and $x * x = 0$, we have

$$[A \in I_l] = \prod_{x,y \in X} (A(y) \alpha A(x * y)) \leq \prod_{y \in X} (A(x) \alpha A(x * x)) = \prod_{x \in X} (A(x) \alpha A(0))$$

and

$$[A \in I_l] = \prod_{x,y \in X} A(y) \alpha A(x * y) \leq \prod_{x \in X} A(0) \alpha A(x * 0) = \prod_{x \in X} A(0) \alpha A(x).$$

So

$$\begin{aligned}[A \in I_l] &\leq \left(\prod_{x \in X} A(x) \alpha A(0) \right) \cdot \left(\prod_{x \in X} A(0) \alpha A(x) \right) \\ &= \prod_{x \in X} \{ (A(x) \alpha A(0)) \cdot (A(0) \alpha A(x)) \} \\ &= [(\forall x)(x \in A \leftrightarrow 0 \in A)],\end{aligned}$$

and the proof is completed. □

Similarly, we have

Theorem 3.4. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} A \in I_r \to (\forall x)((x \in A) \to 0 \in A).$$

Theorem 3.5. *Let the lattice $< L, +, \cdot >$ be completely distributive, and let X and Y be BCK-algebras, $A \in \mathcal{F}(X)$. If $f : X \to Y$ is a surjective homomorphism, then*

$$\begin{aligned}&\models^{\mathcal{L}} A \in s \to f(A) \in s, \\ &\models^{\mathcal{L}} A \in I_l \to f(A) \in I_l, \\ &\models^{\mathcal{L}} A \in I_r \to f(A) \in I_r.\end{aligned}$$

Proof: From Theorem 7.1.10 in [35], we have $\left(\sum_i a_i\right)\alpha b = \prod_i (a_i \alpha b)$. Since the lattice $< L, +, \cdot >$ is completely distributive, it holds that

$$\begin{aligned}
[f(A) \in s] &= \prod_{z,w \in Y} ((f(A)(z) \cdot f(A)(w))\alpha f(A)(z * w)) \\
&= \prod_{z,w \in Y} \left(\left(\sum_{x \in f^{-1}(z)} A(x) \cdot \sum_{y \in f^{-1}(w)} A(y) \right) \alpha \sum_{\substack{x \in f^{-1}(z), \\ y \in f^{-1}(w)}} A(x * y) \right) \\
&\geq \prod_{z,w \in Y} \sum_{\substack{x \in f^{-1}(z), \\ y \in f^{-1}(w)}} ((A(x) \cdot A(y))\alpha A(x * y)) \\
&= \prod_{z,w \in Y} \prod_{\substack{x \in f^{-1}(z), \\ y \in f^{-1}(w)}} ((A(x) \cdot A(y))\alpha A(x * y)) \\
&= \prod_{x,y \in X} ((A(x) \cdot A(y))\alpha A(x * y)).
\end{aligned}$$

The first formula is proved. Similarly, we can prove others. □

Theorem 3.6. *Let X and Y be BCK-algebras, $B \in \mathcal{F}(Y)$. If $f : X \to Y$ is a surjective homomorphism, then*

$$\begin{aligned}
&\models^{\mathcal{L}} B \in s \leftrightarrow f^{-1}(B) \in s, \\
&\models^{\mathcal{L}} B \in I_l \leftrightarrow f^{-1}(B) \in I_l, \\
&\models^{\mathcal{L}} B \in I_r \leftrightarrow f^{-1}(B) \in I_r.
\end{aligned}$$

Proof: We only prove the second formula, and the proofs for others are similar.

$$\begin{aligned}
\left[f^{-1}(B) \in I_l\right] &= \prod_{x,y \in X} \left(f^{-1}(B)(y)\alpha f^{-1}(B)(x * y)\right) \\
&= \prod_{x,y \in X} (B(f(y))\alpha B(f(x * y))) \\
&= \prod_{x,y \in X} (B(f(y))\alpha B(f(x) *' f(y))) \\
&= \prod_{z,w \in Y} (B(w)\alpha B(z * w)) \\
&= [B \in I_l],
\end{aligned}$$

hence, the proof is completed. □

In the following, we use the symbol $\bigcap$ to denote the intersection of some fuzzy sets.

Theorem 3.7. *Let X be a BCK-algebra, $A_\lambda (\lambda \in \Lambda) \in \mathcal{F}(X)$. Then for every $\lambda \in \Lambda$,*

$$\models^{\mathcal{L}} A_\lambda \in s \to \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) \in s,$$

$$\models^{\mathcal{L}} A_\lambda \in I_l \to \left(\bigcap_{\lambda \in \Lambda} \right) A_\lambda \in I_l,$$

$$\models^{\mathcal{L}} A_\lambda \in I_r \to \left(\bigcap_{\lambda\in\Lambda}\right) A_\lambda \in I_r.$$

Proof: With the formulas (R_0) and (R_1) in [35], it is easy to see that $\prod_i a_i\alpha b \geq a_i\alpha b$, $a\alpha \prod_i b_i = \prod_i (a\alpha b_i)$. By Definition 3.1, we have

$$\begin{aligned}
\left[\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in s\right] &= \prod_{x,y\in X}\left(\left(\left[x \in \bigcap_{\lambda\in\Lambda} A_\lambda\right] \cdot \left[y \in \bigcap_{\lambda\in\Lambda} A_\lambda\right]\right) \alpha \left[x * y \in \bigcap_{\lambda\in\Lambda} A_\lambda\right]\right) \\
&= \prod_{x,y\in X}\left\{\left(\prod_{\lambda\in\Lambda} A_\lambda(x) \cdot \prod_{\lambda\in\Lambda} A_\lambda(y)\right) \alpha \prod_{\lambda\in\Lambda} A_\lambda(x*y)\right\} \\
&= \prod_{x,y\in X}\left\{\prod_{\lambda\in\Lambda}(A_\lambda(x) \cdot A_\lambda(y))\alpha \prod_{\lambda\in\Lambda} A_\lambda(x*y)\right\} \\
&\geq \prod_{x,y\in X}\left\{(A_\lambda(x) \cdot A_\lambda(y))\alpha \prod_{\lambda\in\Lambda} A_\lambda(x*y)\right\} \\
&= \prod_{x,y\in X}\prod_{\lambda\in\Lambda}\{(A_\lambda(x) \cdot A_\lambda(y))\alpha A_\lambda(x*y)\} \\
&= \prod_{\lambda\in\Lambda}\prod_{x,y\in X}\{(A_\lambda(x) \cdot A_\lambda(y))\alpha A_\lambda(x*y)\} \\
&= \prod_{\lambda\in\Lambda}[A_\lambda \in s].
\end{aligned}$$

Similarly, we can prove others. □

4. ***L*-Fuzzifying Ideals.** In this section, the concept of L-fuzzifying ideals of a BCK-algebra will be introduced under the frame of lattice-valued logic. The equivalent characterizations of this fuzzy ideal will be given.

Definition 4.1. *Let X be a BCK-algebra. For an arbitrary $A \in \mathcal{F}(X)$, we set*

$$A \in I_1 := (\forall x)(x \in A \to 0 \in A);$$
$$A \in I_2 := (\forall x)(\forall y)((x * y \in A) \wedge (y \in A) \to (x \in A)).$$

A unary fuzzy predicate $I \in \mathcal{F}(\mathcal{F}(X))$, called an L-fuzzifying ideal of X, is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,

$$A \in I := (A \in I_1) \wedge (A \in I_2).$$

Proposition 4.1. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} (\forall x)(\forall y)(\forall z)((((x*y)*y)*z \in A) \wedge (z \in A) \to x*y \in A) \to A \in I_2.$$

Proof: By Definition 4.1 and $x * 0 = x$, we have

$$\begin{aligned}
&[(\forall x)(\forall y)(\forall z)((((x*y)*y)*z \in A) \wedge (z \in A) \to x*y \in A)] \\
&= \prod_{x,y,z\in X}\{(A(((x*y)*y)*z) \cdot A(z))\alpha A(x*y)\} \\
&\leq \prod_{x,z\in X}\{(A(((x*0)*0)*z) \cdot A(z))\alpha A(x*0)\} \\
&= \prod_{x,z\in X}\{(A(x*z) \cdot A(z))\alpha A(x)\}
\end{aligned}$$

$$= [A \in I_2].$$

□

Proposition 4.2. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in I_1$, then*
(1) *for all $x, y \in X$ with $x \leq y$, $\models^{\mathcal{L}} A \in I \to (y \in A \to x \in A)$;*
(2) *$\models^{\mathcal{L}} A \in I \to (\forall x)(\forall y)(\forall z)(x * y \in A \to (x * z) * (y * z) \in A)$;*
(3) *$\models^{\mathcal{L}} A \in I \to (\forall x)(\forall y)(\forall z)((x * y) * z \in A \to ((x * z) * (y * z)) * z \in A)$.*

Proof: By $\models^{\mathcal{L}} A \in I_1$, we have $\prod_{x \in X}(A(x) \alpha A(0)) = 1$, and so $A(x) \leq A(0)$ for any $x \in X$. From $x \leq y$, we have $x * y = 0$. Thus

$$\begin{aligned}[A \in I_2] &= \prod_{x,y \in X} ((A(x * y) \cdot A(y)) \alpha A(x)) \\ &\leq \prod_{x,y \in X, x \leq y} ((A(x * y) \cdot A(y)) \alpha A(x)) \\ &= \prod_{x,y \in X, x \leq y} ((A(0) \cdot A(y)) \alpha A(x))\end{aligned}$$

Therefore, $[A \in I] \leq [A \in I_2] \leq [y \in A \to x \in A]$ for all $x, y \in X$ with $x \leq y$, and the first is proved.

With (8) and (12) in [36], we have $(x*z)*(y*z) \leq x*y$, $((x*z)*(y*z))*z \leq (x*y)*z$. So

$$\begin{aligned}&\prod_{x,y \in X, x \leq y} [y \in A \to x \in A] \\ &= [(\forall x)(\forall y)(\forall z)(x * y \in A \to (x * z) * (y * z) \in A)] \\ &= [(\forall x)(\forall y)(\forall z)((x * y) * z \in A \to ((x * z) * (y * z)) * z \in A)].\end{aligned}$$

Combined with the proof of the first formula, the second and the third formulas have been proved too. □

Proposition 4.3. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in I$, then for all $x, y, z \in X$ with $x * y \leq z$,*

$$\models^{\mathcal{L}} ((y \in A) \wedge (z \in A)) \to x \in A.$$

Proof: Form $\models^{\mathcal{L}} A \in I$, we have $A(0) \geq A(x) \geq A(x * y) \cdot A(y)$ for all $x, y \in X$. For any $x, y, z \in X$, the inequality $x * y \leq z$ in X implies that $(x * y) * z = 0$. Hence $A(x) \geq A(x * y) \cdot A(y) \geq A((x * y) * z) \cdot A(z) \cdot A(y) = A(0) \cdot A(z) \cdot A(y) = A(z) \cdot A(y)$ for all $x, y, z \in X$ with $x * y \leq z$. That is, $\prod_{x,y,z \in X, x*y \leq z}[((y \in A) \wedge (z \in A)) \to x \in A]$. □

Theorem 4.1. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. Then for all $x, y, z \in X$ with $x * y \leq z$,*

$$\models^{\mathcal{L}} (((y \in A) \wedge (z \in A)) \to x \in A) \to A \in I.$$

Proof: For any $x \in X$, $0 * x \leq x$ and so $(0 * x) * x = 0$, thus

$$\begin{aligned}\prod_{\substack{x,y,z \in X,\\ x*y \leq z}} [((y \in A) \wedge (z \in A)) \to x \in A] &= \prod_{\substack{x,y,z \in X,\\ x*y \leq z}} ((A(y) \cdot A(z)) \alpha A(x)) \\ &\leq \prod_{x \in X} ((A(x) \cdot A(x)) \alpha A(0)) \\ &= [A \in I_1].\end{aligned}$$

For $x, y \in X$, $x * (x * y) \leq y$, so

$$\prod_{\substack{x,y,z\in X,\\ x*y\leq z}} [((y \in A) \wedge (z \in A)) \to x \in A] = \prod_{\substack{x,y,z\in X,\\ x*y\leq z}} ((A(y) \cdot A(z))\alpha A(x)) \leq \prod_{x,y\in X} ((A(x*y) \cdot A(y))\alpha A(x)) = [A \in I_2].$$

Therefore, $[((y \in A) \wedge (z \in A)) \to x \in A] \leq [A \in I_1] \cdot [A \in I_2] = [A \in I]$ for all $x, y, z \in X$ with $x * y \leq z$. □

Theorem 4.2. *Let X be a BCK-algebra and $\forall A_\lambda (\lambda \in \Lambda) \in \mathcal{F}(X)$, and then for all $\lambda \in \Lambda$,*

$$\models^{\mathcal{L}} A_\lambda \in I \to \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in I.$$

Proof: It is easy to know that $\prod_i (a_i \alpha b_i) \leq (\prod_i a_i)\alpha(\prod_i b_i)$ with the formulas (A) and (A'') in [28]. By Definition 4.1, we have

$$\left[\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in I_1\right] = \prod_{x\in X}\left(\left[x \in \bigcap_{\lambda\in\Lambda} A_\lambda\right] \alpha \left[0 \in \bigcap_{\lambda\in\Lambda} A_\lambda\right]\right) = \prod_{x\in X}\left(\prod_{\lambda\in\Lambda} A_\lambda(x) \alpha \prod_{\lambda\in\Lambda} A_\lambda(0)\right) \geq \prod_{\lambda\in\Lambda}\prod_{x\in X}(A_\lambda(x)\alpha A_\lambda(0)) = \prod_{\lambda\in\Lambda}[A_\lambda \in I_1]$$

and

$$\left[\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in I_2\right] = \prod_{x,y\in X}\left(\left[x*y \in \bigcap_{\lambda\in\Lambda} A_\lambda\right] \cdot \left[y \in \bigcap_{\lambda\in\Lambda} A_\lambda\right] \alpha \left[x \in \bigcap_{\lambda\in\Lambda} A_\lambda\right]\right)$$
$$= \prod_{x,y\in X}\left(\left(\prod_{\lambda\in\Lambda} A_\lambda(x*y) \cdot \prod_{\lambda\in\Lambda} A_\lambda(y)\right) \alpha \prod_{\lambda\in\Lambda} A_\lambda(x)\right)$$
$$= \prod_{x,y\in X}\left(\left(\prod_{\lambda\in\Lambda} A_\lambda(x*y) \cdot A_\lambda(y)\right) \alpha \prod_{\lambda\in\Lambda} A_\lambda(x)\right)$$
$$\geq \prod_{\lambda\in\Lambda}\prod_{x,y\in X}((A_\lambda(x*y) \cdot A_\lambda(y))\alpha A_\lambda(x))$$
$$= \prod_{\lambda\in\Lambda}[A_\lambda \in I_2],$$

therefore

$$\left[\left(\bigcap_{\lambda\in\Lambda} A_\lambda \in I\right)\right] = \left[I_1\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)\right] \cdot \left[I_2\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)\right] \geq \prod_{\lambda\in\Lambda}[I_1(A_\lambda)] \cdot \prod_{\lambda\in\Lambda}[I_2(A_\lambda)]$$

$$= \prod_{\lambda\in\Lambda}([I_1(A_\lambda)] \cdot [I_2(A_\lambda)])$$
$$= \prod_{\lambda\in\Lambda}[I_1(A_\lambda) \wedge I_2(A_\lambda)]$$
$$= \prod_{\lambda\in\Lambda}[A_\lambda \in I],$$

so we complete the proof. □

Theorem 4.3. *Let X and Y be two BCK-algebras and $A \in \mathcal{F}(X)$, and let the lattice $< L, +, \cdot >$ be completely distributive. If $f : X \to Y$ is a surjective homomorphism, then*

$$\models^{\mathcal{L}} A \in I \to f(A) \in I.$$

Proof: By Theorem 7.1.10 in [35], we have $\left(\sum_i a_i\right)\alpha b = \prod_i (a_i \alpha b)$. Since the lattice $< L, +, \cdot >$ is completely distributive, we have

$$\begin{aligned}
[f(A) \in I_1] &= \prod_{z\in Y} (f(A)(z)\alpha f(A)(0')) \\
&= \prod_{z\in Y}\left(\sum_{x\in f^{-1}(z)} A(x)\alpha \sum_{x\in f^{-1}(0')} A(x)\right) \\
&\geq \prod_{z\in Y}\left(\sum_{x\in f^{-1}(z)} A(x)\alpha A(0)\right) \\
&= \prod_{z\in Y}\prod_{x\in f^{-1}(z)} (A(x)\alpha A(0)) \\
&= \prod_{x\in X}(A(x)\alpha A(0)) \\
&= [A \in I_1]
\end{aligned}$$

where the zero element $0'$ of Y is the image of zero element 0 of X under f, and

$$\begin{aligned}
[f(A) \in I_2] &= \prod_{z,w\in Y} ((f(A)\,(z *' w) \cdot f(A)(w))\alpha f(A)(z)) \\
&= \prod_{z,w\in Y}\left(\left(\sum_{\substack{x\in f^{-1}(z),\\ y\in f^{-1}(w)}} A(x*y) \cdot \sum_{y\in f^{-1}(w)} A(y)\right)\alpha \sum_{x\in f^{-1}(z)} A(x)\right) \\
&\geq \prod_{z,w\in Y}\left(\sum_{\substack{x\in f^{-1}(z),\\ y\in f^{-1}(w)}} (A(x*y)\cdot A(y))\alpha A(x)\right) \\
&= \prod_{z,w\in Y}\prod_{\substack{x\in f^{-1}(z),\\ y\in f^{-1}(w)}} ((A(x*y)\cdot A(y))\alpha A(x)) \\
&= \prod_{x,y\in X} ((A(x*y)\cdot A(y))\alpha A(x)) \\
&= [A \in I_2].
\end{aligned}$$

Therefore, $[f(A) \in I] = [f(A) \in I_1] \cdot [f(A) \in I_2] \geq [A \in I_1] \cdot [A \in I_2] = [A \in I]$. □

Theorem 4.4. *Let X and Y be two BCK-algebras and $B \in \mathcal{F}(Y)$. If $f : X \to Y$ is a surjective homomorphism, then*

$$\models^{\mathcal{L}} B \in I \leftrightarrow f^{-1}(B) \in I.$$

Proof: With Definition 4.1, we have

$$\begin{aligned}
[f^{-1}(B) \in I_1] &= \prod_{x \in X} \left(f^{-1}(B)(x) \alpha f^{-1}(B)(0)\right) \\
&= \prod_{x \in X} (B(f(x)) \alpha B(f(0))) \\
&= \prod_{y \in Y} (B(y) \alpha B(0')) \\
&= [B \in I_1]
\end{aligned}$$

and

$$\begin{aligned}
[f^{-1}(B) \in I_2] &= \prod_{x,y \in X} \left(\left(f^{-1}(B)(x * y) \cdot f^{-1}(B)(y)\right) \alpha f^{-1}(B)(x)\right) \\
&= \prod_{x,y \in X} ((B(f(x) *' f(y)) \cdot B(f(y))) \alpha B(f(x))) \\
&= \prod_{z,w \in Y} ((B(z *' w) \cdot B(w)) \alpha B(z)) \\
&= [B \in I_2].
\end{aligned}$$

Therefore, $[f^{-1}(B) \in I] = [B \in I]$. □

Theorem 4.5. *Let X be a quasi-right alternate BCK-algebra and $A \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} A \in s \to A \in I.$$

Proof: By Theorem 4.1, we have $[A \in s] \leq [A \in I_1]$. Otherwise, for any $x, y, z \in X$, $\models^{\mathcal{L}} \{(x \in A) \wedge (y \in A) \to (x * y) \in A\} \leftrightarrow \{(z \in A) \wedge (y \in A) \to (z * y) \in A\}$. With the formula (A'') in [35], we have $(a \alpha b) \odot a \leq b$ for all $a, b \in L$. Notice that X is a quasi-right alternate,

$$\begin{aligned}
&\{(A(x) \cdot A(y)) \alpha A(x * y)\} \odot (A(x * y) \cdot A(y)) \\
=\ &\{(A(x * y) \cdot A(y)) \alpha A((x * y) * y)\} \odot (A(x * y) \cdot A(y)) \\
\leq\ &A((x * y) * y) \\
=\ &A(x * (y * y)) \\
=\ &A(x * 0) \\
=\ &A(x).
\end{aligned}$$

By formula (A) in [35], we have $a \odot b \leq c$ if and only if $a \leq b \alpha c$. Thus

$$(A(x) \cdot A(y)) \alpha A(x * y) \leq (A(x * y) \cdot A(y)) \alpha A(x),$$

it implies that $[A \in s] \leq [A \in I_2]$, and so $[A \in s] \leq [A \in I]$.

Conversely, for any $x, y \in X$, we have

$$\begin{aligned}
&\{(A(x * y) \cdot A(y)) \alpha A(x)\} \odot \{A(x) \cdot A(y)\} \\
=\ &\{(A((x * y) * y) \cdot A(y)) \alpha A(x * y)\} \odot \{A(x * (y * y)) \cdot A(y)\} \\
=\ &\{(A((x * y) * y) \cdot A(y)) \alpha A(x * y)\} \odot \{A((x * y) * y) \cdot A(y)\}
\end{aligned}$$

$$\leq A(x * y).$$

It follows that $(A(x*y) \cdot A(y))\alpha A(x) \leq (A(x) \cdot A(y))\alpha A(x*y)$, and so $[A \in I_2] \leq [A \in s]$. Therefore, $[A \in I] \leq [A \in I_2] \leq [A \in s]$. By combining with the above conclusions we complete the proof. □

5. ***L*-Fuzzifying Implicative Ideals.** In this section, the concept of L-fuzzifying implicative ideals of a BCK-algebra will be introduced under the frame of lattice-valued logic. The equivalent characterizations of this fuzzy implicative ideal will be given.

Definition 5.1. *Let X be a BCK-algebra, and a unary fuzzy predicate $II \in \mathcal{F}(\mathcal{F}(X))$ is called an L-fuzzifying positive implicative ideal, if it is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,*

$$A \in II := (A \in I_1) \wedge (A \in I_3),$$

where $A \in I_3 := (\forall x)(\forall y)(\forall z)((((x * y) * z) \in A) \wedge ((y * z) \in A) \to (x * z) \in A)$.

Theorem 5.1. *Let X be a BCK-algebra, $A \in \mathcal{F}(X)$, and then $\models^{\mathcal{L}} A \in II \to A \in I$.*

Proof: From $x * 0 = x$, we have

$$\begin{aligned}
[A \in I_3] &= \prod_{x,y,z \in X} ((A((x*y)*z) \cdot A(y*z))\alpha A(x*z)) \\
&\leq \prod_{x,y \in X} ((A((x*y)*0) \cdot A(y*0))\alpha A(x*0)) \\
&= \prod_{x,y \in X} ((A(x*y) \cdot A(y))\alpha A(x)) \\
&= [A \in I_2].
\end{aligned}$$

Therefore, $[A \in II] = [A \in I_1] \cdot [A \in I_3] \leq [A \in I_1] \cdot [A \in I_2] = [A \in I]$. □

Theorem 5.2. *Let X be a positive implicative BCK-algebra, $A \in \mathcal{F}(X)$, and then $\models^{\mathcal{L}} A \in I \to A \in II$.*

Proof: Since X is a positive implicative BCK-algebra, we have $(x*z)*(y*z) = (x*y)*z$ for any $x, y, z \in X$. So

$$\begin{aligned}
&\{(A(x*y) \cdot A(y))\alpha A(x)\} \odot (A((x*y)*z) \cdot A(y*z)) \\
= &\{(A((x*z)*(y*z)) \cdot A(y*z))\alpha A(x*z)\} \odot (A((x*y)*z) \cdot A(y*z)) \\
= &\{(A((x*z)*(y*z)) \cdot A(y*z))\alpha A(x*z)\} \odot (A((x*z)*(y*z)) \cdot A(y*z)) \\
\leq &A(x*z),
\end{aligned}$$

i.e., $(A(x*y) \cdot A(y))\alpha A(x) \leq (A((x*y)*z) \cdot A(y*z))\alpha A(x*z)$, and hence $[A \in I_2] \leq [A \in I_3]$. It follows that $[A \in I] \leq [A \in II]$, and hence the conclusion holds. □

Proposition 5.1. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in I$, then*

$$\begin{aligned}
&\models^{\mathcal{L}} A \in II \to (\forall x)(\forall y)((x*y)*y \in A \to x*y \in A), \\
&\models^{\mathcal{L}} (\forall x)(\forall y)((x*y)*y \in A \to x*y \in A) \to (\forall x)(\forall y)(\forall z)((x*y)*z \in A \\
&\to (x*z)*(y*z) \in A), \\
&\models^{\mathcal{L}} A \in II \to (\forall x)(\forall y)(\forall z)(((x*z)*(y*z))*z \in A \to (x*z)*(y*z) \in A).
\end{aligned}$$

Proof: By $\models^{\mathcal{L}} A \in I$, we have $\prod_{x\in X}(A(x)\alpha A(0)) = 1 = \prod_{x\in X}((A(x*y)\cdot A(x))\alpha A(0))$. It follows that $A(0) \geq A(x) \geq A(x*y)\cdot A(y)$ for any $x, y, z \in X$. With $y*y = 0$, we have

$$\begin{aligned}[A \in I_3] &= \prod_{x,y,z\in X} \{(A((x*y)*z)\cdot A(y*z))\alpha A(x*z)\} \\ &\leq \prod_{x,y\in X} \{(A((x*y)*y)\cdot A(y*y))\alpha A(x*y)\} \\ &= \prod_{x,y\in X} (A((x*y)*y)\alpha A(x*y)).\end{aligned}$$

Therefore, $\models^{\mathcal{L}} A \in II \to (\forall x)(\forall y)((x*y)*y \in A \to x*y \in A)$.

With $\models^{\mathcal{L}} A \in I$ we obtain $A(x) \geq A(y)$ when $x \leq y$ in X. Applying this conclusion to

$$((x*z)*(y*z))*z \leq (x*y)*z,$$

we have $A(((x*z)*(y*z))*z) \geq A((x*y)*z)$. Consequently,

$$\begin{aligned}&[(\forall x)(\forall y)((x*y)*y \in A \to x*y \in A)] \\ &= [(\forall x)(\forall y)(\forall z)(((x*(y*z))*z)*z \in A \to (x*(y*z))*z \in A)] \\ &= \prod_{x,y,z\in X} (A(((x*(y*z))*z)*z)\alpha A((x*(y*z))*z)) \\ &\leq \prod_{x,y,z\in X} (A((x*y)*z)\alpha A((x*(y*z))*z)) \\ &= \prod_{x,y,z\in X} (A((x*y)*z)\alpha A((x*z)*(y*z))).\end{aligned}$$

The second is proved. By $((x*(y*z))*z)*z = ((x*z)*(y*z))*z$, we have $[(\forall x)(\forall y)((x*y)*y \in A \to x*y \in A)] = [(\forall x)(\forall y)(\forall z)(((x*z)*(y*z))*z \in A \to (x*z)*(y*z) \in A)]$, and so the third formula is proved. □

Definition 5.2. *Let X be a BCK-algebra, and a unary fuzzy predicate $III \in \mathcal{F}(\mathcal{F}(X))$, called an L-fuzzifying implicative ideal, is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,*

$$A \in III := (A \in I_1) \wedge (A \in I_4),$$

where $A \in I_4 := (\forall x)(\forall y)(\forall z)((((x*(y*x))*z \in A) \wedge (z \in A)) \to x \in A)$.

Theorem 5.3. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$. Then $\models^{\mathcal{L}} A \in III \to A \in I$.*

Proof: From Definition 5.2, we have

$$\begin{aligned}[A \in I_4] &= \prod_{x,y,z\in X} \{(A((x*(y*x))*z)\cdot A(z))\alpha A(x)\} \\ &\leq \prod_{x,z\in X} \{(A((x*(x*x))*z)\cdot A(z))\alpha A(x)\} \\ &= \prod_{x,z\in X} \{(A((x*0)*z)\cdot A(z))\alpha A(x)\} \\ &= \prod_{x,z\in X} \{(A(x*z)\cdot A(z))\alpha A(x)\} \\ &= [A \in I_2].\end{aligned}$$

Combined with $A \in I_1$, the proof is completed. □

In the following theorem, we can see that the converse of Theorem 5.3 also holds in an implicative BCK-algebra.

Theorem 5.4. *Let X be an implicative BCK-algebra (see [10]), $A \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} A \in I \leftrightarrow A \in III.$$

Proof: Since X is an implicative BCK-algebra, it follows that $x = x * (y * x)$ for all $x, y \in X$. Then

$$\begin{aligned}[A \in I_2] &= \prod_{x,z \in X} ((A(x * z) \cdot A(z))\alpha A(x)) \\ &= \prod_{x,y,z \in X} ((A((x * (y * x)) * z) \cdot A(z))\alpha A(x)) \\ &= [A \in I_4].\end{aligned}$$

Combined with $A \in I_1$, we have proved the theorem. □

Theorem 5.5. *Let X be a BCK-algebra, $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in I$, then*

$$\models^{\mathcal{L}} A \in III \to A \in II.$$

Proof: Since $\models^{\mathcal{L}} A \in I$ and $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$, we have $A((x * y) * z) \cdot A(y * z) \leq A(((x * z) * z) * (y * z)) \cdot A(y * z) \leq A((x * z) * z)$. Noting that $(x * z) * (x * (x * z)) = (x * (x * (x * z))) * z = (x * z) * z$ and the binary operation α on L is antitone in the first and isotone in the second variable, we have

$$\begin{aligned}[A \in I_4] &= \prod_{x,y,z \in X} \{(A(((x * z) * (x * (x * z))) * y) \cdot A(y))\alpha A(x * z)\} \\ &\leq \prod_{x,z \in X} \{(A(((x * z) * (x * (x * z))) * 0) \cdot A(0))\alpha A(x * z)\} \\ &= \prod_{x,z \in X} \{A((x * z) * (x * (x * z)))\alpha A(x * z)\} \\ &= \prod_{x,z \in X} \{A((x * z) * z)\alpha A(x * z)\} \\ &\leq \prod_{x,z \in X} \{(A((x * y) * z) \cdot A(y * z))\alpha A(x * z)\} \\ &= [A \in I_3].\end{aligned}$$

Therefore, $[A \in III] = [A \in I_1] \cdot [A \in I_4] \leq [A \in I_1] \cdot [A \in I_3] = [A \in II]$. □

Proposition 5.2. *Let X be a BCK-algebra, $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in I_1$, then*

$$\models^{\mathcal{L}} A \in III \to (\forall x)(\forall y)(y * (y * x) \in A \to x * (x * y) \in A).$$

Proof: By $\models^{\mathcal{L}} A \in I_1$, we have $A(0) \geq A(x)$ for all $x \in X$. From Proposition 4.2, it holds that $[A \in I] \leq [y \in A \to x \in A]$ for all $x, y \in X$ with $x \leq y$, and so $A(x) \geq A(y)$ for any $x, y \in X$ with $x \leq y$. Since $x * (x * y) \leq x$, it follows from (8) in [33] that $y * (x * (x * y)) \geq y * x$, and thus

$$\begin{aligned}(x * (x * y)) * (y * (x * (x * y))) &\leq (x * (x * y)) * (y * x) \\ &= (x * (y * x)) * (x * y) \\ &\leq y * (y * x).\end{aligned}$$

Therefore, $A((x * (x * y)) * (y * (x * (x * y)))) \geq A(y * (y * x))$.

With the formula (R_0) in [35], we have

$$[A \in I_4] = \prod_{x,y,z \in X} \{(A((x * (y * x)) * z) \cdot A(z))\alpha A(x)\}$$

$$= \prod_{x,y,z\in X}\{(A(((x*(x*y))*(y*(x*(x*y))))*z)\cdot A(z))\alpha A(x*(x*y))\}$$
$$\leq \prod_{x,y\in X}\{(A(((x*(x*y))*(y*(x*(x*y))))*0)\cdot A(0))\alpha A(x*(x*y))\}$$
$$= \prod_{x,y\in X}\{A((x*(x*y))*(y*(x*(x*y))))\alpha A(x*(x*y))\}$$
$$\leq \prod_{x,y\in X}\{A(y*(y*x))\alpha A(x*(x*y))\}$$
$$= [(\forall x)(\forall y)(y*(y*x)\in A\rightarrow x*(x*y)\in A)],$$

hence, the proposition is proved. □

Proposition 5.3. *Let X be a BCK-algebra, $A\in\mathcal{F}(X)$. If $\models^{\mathcal{L}} A\in I$, then*

$$\models^{\mathcal{L}} A\in III\leftrightarrow(\forall x)(\forall y)(x*(y*x)\in A\rightarrow x\in A).$$

Proof: It follows from $\models^{\mathcal{L}} A\in I$ that $A(0)\geq A(x)$ for all $x\in X$. So

$$[A\in I_4] = \prod_{x,y,z\in X}\{(A((x*(y*x))*z)\cdot A(z))\alpha A(x)\}$$
$$\leq \prod_{x,y\in X}\{(A((x*(y*x))*0)\cdot A(0))\alpha A(x)\}$$
$$= \prod_{x,y\in X}\{A(x*(y*x))\alpha A(x)\}$$
$$= [(\forall x)(\forall y)(x*(y*x)\in A\rightarrow x\in A)].$$

Conversely, by $\models^{\mathcal{L}} A\in I$, we have $A(x*(y*x))\geq A((x*(y*x))*z)\cdot A(z)$ for all $x,y,z\in X$. So, for any $z\in X$,

$$[(\forall x)(\forall y)(x*(y*x)\in A\rightarrow x\in A)] = \prod_{x,y\in X}\{A(x*(y*x))\alpha A(x)\}$$
$$\leq \prod_{x,y\in X}\{(A((x*(y*x))*z)\cdot A(z))\alpha A(x)\}$$
$$= [A\in I_4].$$

Consequently, $[A\in III]=[A\in I_1]\cdot[A\in I_4]=[A\in I_4]=[(\forall x)(\forall y)(x*(y*x)\in A\rightarrow x\in A)]$. □

Similar to the proofs of Theorem 4.2, Theorem 4.3 and Theorem 4.4, we can have the following three theorems.

Theorem 5.6. *Let $A_\lambda\in\mathcal{F}(X)$ for all $\lambda\in\Lambda$, and then for every $\lambda\in\Lambda$,*

$$\models^{\mathcal{L}} A_\lambda\in II\rightarrow\left(\bigcap_{\lambda\in\Lambda}A_\lambda\right)\in II,$$
$$\models^{\mathcal{L}} A_\lambda\in III\rightarrow\left(\bigcap_{\lambda\in\Lambda}A_\lambda\right)\in III.$$

Theorem 5.7. *Let the lattice $< L,+,\cdot >$ be completely distributive, $A\in\mathcal{F}(X)$, and let $f: X\rightarrow Y$ be a surjective homomorphism. Then*

$$\models^{\mathcal{L}} A\in II\rightarrow f(A)\in II,$$
$$\models^{\mathcal{L}} A\in III\rightarrow f(A)\in III.$$

Theorem 5.8. *Let $B \in \mathcal{F}(Y)$, and $f : X \to Y$ a surjective homomorphism. Then*

$$\models^{\mathcal{L}} B \in II \leftrightarrow f^{-1}(B) \in II,$$
$$\models^{\mathcal{L}} B \in III \leftrightarrow f^{-1}(B) \in III.$$

Definition 5.3. *For any $A \in \mathcal{F}(X \times X)$ and $B \in \mathcal{F}(X)$, a binary fuzzy predicate $R_B \in \mathcal{F}(\mathcal{F}(X \times X))$ is called an L-fuzzifying relation on B if*

$$A \in R_B := (\forall x)(\forall y)((x, y) \in A \to (x \in B) \wedge (y \in B)).$$

Clearly, we have

Proposition 5.4. *Let $A, B \in \mathcal{F}(X)$, then*

$$\models^{\mathcal{L}} (\forall x)(\forall y)((x, y) \in A \times B \leftrightarrow (x \in A) \wedge (y \in B)).$$

Theorem 5.9. *Let X be a BCK-algebra, and $A, B \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} (A \in I_l) \wedge (B \in I_l) \to A \times B \in I_l,$$
$$\models^{\mathcal{L}} (A \in I_r) \wedge (B \in I_l) \to A \times B \in I_r,$$
$$\models^{\mathcal{L}} (A \in I) \wedge (B \in I) \to A \times B \in I,$$
$$\models^{\mathcal{L}} (A \in II) \wedge (B \in I_l) \to A \times B \in II,$$
$$\models^{\mathcal{L}} (A \in III) \wedge (B \in III) \to A \times B \in III.$$

Proof: We only prove the third formula, and the others are similar. With the formulas (R_0) and (R_1) in [35], we have

$$\begin{aligned}
[A \times B \in I_1] &= \prod_{x,y \in X} \{(A \times B)(x, y) \alpha (A \times B)(0, 0)\} \\
&= \prod_{x,y \in X} \{(A(x) \cdot B(y)) \alpha (A(0) \cdot B(0))\} \\
&= \prod_{x,y \in X} \{\{(A(x) \cdot B(y)) \alpha A(0)\} \cdot \{(A(x) \cdot B(y)) \alpha B(0)\}\} \\
&\geq \prod_{x,y \in X} \{(A(x) \alpha A(0)) \cdot (B(y) \alpha B(0))\} \\
&= \prod_{x \in X} (A(x) \alpha A(0)) \cdot \prod_{y \in X} (B(y) \alpha B(0)) \\
&= [A \in I_1] \cdot [B \in I_1]
\end{aligned}$$

and

$$\begin{aligned}
&[A \times B \in I_2] \\
&= \prod_{x_1,x_2,y_1,y_2 \in X} \{((A \times B)((x_1, x_2) * (y_1, y_2)) \wedge (A \times B)(y_1, y_2)) \alpha (A \times B)(x_1, x_2)\} \\
&= \prod_{x_1,x_2,y_1,y_2 \in X} \{((A \times B)(x_1 * y_1, x_2 * y_2) \wedge (A \times B)(y_1, y_2)) \alpha (A \times B)(x_1, x_2)\} \\
&= \prod_{x_1,x_2,y_1,y_2 \in X} \{(A(x_1 * y_1) \cdot B(x_2 * y_2) \cdot A(y_1) \cdot B(y_2)) \alpha (A(x_1) \cdot B(x_2))\} \\
&\geq \prod_{x_1,x_2,y_1,y_2 \in X} \{((A(x_1 * y_1) \cdot A(y_1)) \alpha A(x_1)) \cdot ((B(x_2 * y_2) \cdot B(y_2)) \alpha B(x_2))\} \\
&= \prod_{x_1,y_1 \in X} ((A(x_1 * y_1) \cdot A(y_1)) \alpha A(x_1)) \cdot \prod_{x_2,y_2 \in X} ((B(x_2 * y_2) \cdot B(y_2)) \alpha B(x_2))
\end{aligned}$$

$= [A \in I_2] \cdot [B \in I_2]$.

Therefore, $[A \times B \in I] = [A \times B \in I_1] \cdot [A \times B \in I_2] \geq [A \in I_1] \cdot [B \in I_1] \cdot [A \in I_2] \cdot [B \in I_2] = [A \in I] \cdot [B \in I]$, and the third statement is proved. □

6. ***L*-Fuzzifying Filters.** In this section, the concept of L-fuzzifying filters of a BCK-algebra will be introduced under the frame of lattice-valued logic. The equivalent characterizations of this fuzzy filter will be given.

Definition 6.1. *Let X be a BCK-algebra. For an arbitrary $A \in \mathcal{F}(X)$, we set*
(1) $A \in F_1 := x \in A \to y \in A$ *for all* $x, y \in X$ *with* $y \geq x$;
(2) $A \in F_2 := (\forall x)(\forall y)((x \in A) \wedge (y \in A) \to x \sqcap y \in A)$.
A unary fuzzy predicate $f_l \in \mathcal{F}(\mathcal{F}(X))$, called an L-fuzzifying filter of X, is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,

$$A \in f_l := (A \in F_1) \wedge (A \in F_2).$$

Proposition 6.1. *Let X be a bounded BCK-algebra, $A \in \mathcal{F}(\mathcal{F}(X))$. Then*

$$\models^{\mathcal{L}} A \in f_l \to (\forall x)(x \in A \to 1 \in A).$$

Proof: Since X is a bounded BCK-algebra, $1 \geq x$ for all $x \in X$, and so

$$[A \in F_1] = \prod_{x,y \in X, y \geq x} (A(x) \alpha A(y)) \leq \prod_{x \in X} A(x) \alpha A(1) = [(\forall x)(x \in A \to 1 \in A)].$$

Therefore, $[A \in f_l] \leq [A \in F_1] \leq [(\forall x)(x \in A \to 1 \in A)]$. □

Theorem 6.1. *Let X be a BCK-algebra and $A \in \mathcal{F}(X)$, and then*

$$\models^{\mathcal{L}} A \in f_l \leftrightarrow (\forall x)(\forall y)((x \in A) \wedge (y \in A) \leftrightarrow x \sqcap y \in A).$$

Proof: Since $x, y \geq x \sqcap y$ for all $x, y \in X$, we have

$$[A \in F_1] = \prod_{x,z \in X, z \geq x} (A(x) \alpha A(z)) \leq \prod_{x,y \in X} (A(x \sqcap y) \alpha A(x)).$$

Similarly, $[A \in F_1] \leq \prod_{x,y \in X} (A(x \sqcap y) \alpha A(y))$. Thus

$$\begin{aligned} [A \in F_1] &\leq \prod_{x,y \in X} (A(x \sqcap y) \alpha A(x)) \cdot \prod_{x,y \in X} (A(x \sqcap y) \alpha A(y)) \\ &= \prod_{x,y \in X} (A(x \sqcap y) \alpha A(x)) \cdot (A(x \sqcap y) \alpha A(y)) \\ &= \prod_{x,y \in X} (A(x \sqcap y) \alpha (A(x) \cdot A(y))) \end{aligned}$$

and so $[A \in f_l] \leq [(\forall x)(\forall y)(x \sqcap y \in A \to (x \in A) \wedge (y \in A))]$. On the other hand, $[A \in f_l] \leq [A \in F_2] = [(\forall x)(\forall y)((x \in A) \wedge (y \in A) \to x \sqcap y \in A)]$. Therefore,

$$\models^{\mathcal{L}} A \in f_l \to (\forall x)(\forall y)((x \in A) \wedge (y \in A) \leftrightarrow x \sqcap y \in A).$$

Conversely, $[(\forall x)(\forall y)(x \in A \wedge y \in A \leftrightarrow x \sqcap y \in A)] \leq [(\forall x)(\forall y)(x \in A \wedge y \in A \to x \sqcap y \in A)] = [A \in F_2]$. Since the binary operation α is isotone in the second variable, we have

$$\begin{aligned} [(\forall x)(\forall y)(x \in A \wedge y \in A \leftrightarrow x \sqcap y \in A)] &\leq [(\forall x)(\forall y)(x \sqcap y \in A \to x \in A \wedge y \in A)] \\ &= \prod_{x,y \in X} A(x \sqcap y) \alpha (A(x) \cdot A(y)) \end{aligned}$$

$$\leq \prod_{x,y\in X, y\geq x} A(x)\alpha(A(x)\cdot A(y))$$
$$\leq \prod_{x,y\in X, y\geq x} (A(x)\alpha A(y))$$
$$= [A \in F_1]$$

Therefore, $[(\forall x)(\forall y)(x \in A \wedge y \in A \leftrightarrow x \sqcap y \in A)] \leq [A \in F_1] \cdot [A \in F_2] = [A \in f_l]$, i.e., $\models^{\mathcal{L}} (\forall x)(\forall y)((x \in A) \wedge (y \in A) \leftrightarrow x \sqcap y \in A) \to A \in f_l$, and we have completed the proof. □

Corollary 6.1. *Let X be a commutative BCK-algebra and $A \in \mathcal{F}(X)$, and then*

$$\models^{\mathcal{L}} A \in f_l \leftrightarrow (\forall x)(\forall y)((x \in A) \wedge (y \in A) \leftrightarrow x\dot{\wedge}y \in A).$$

Similar to the proofs of Theorem 3.5, Theorem 3.6 and Theorem 3.7, we have the following three theorems.

Theorem 6.2. *Let X be a BCK-algebra and $A_\lambda \in \mathcal{F}(X)$ for all $\lambda \in \Lambda$, and then for every $\lambda \in \Lambda$,*

$$\models^{\mathcal{L}} A_\lambda \in f_l \to \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in f_l.$$

Theorem 6.3. *Let X and Y be two BCK-algebras, $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, and let the lattice $< L, +, \cdot >$ be completely distributive. If $f : X \to Y$ is a surjective homomorphism, then*

$$\models^{\mathcal{L}} A \in f_l \to f(A) \in f_l,$$
$$\models^{\mathcal{L}} B \in f_l \leftrightarrow f^{-1}(B) \in f_l.$$

Theorem 6.4. *Let X be a BCK-algebra, and $A, B \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} (A \in f_l) \wedge (B \in f_l) \to A \times B \in f_l.$$

Definition 6.2. *Let X be a bounded BCK-algebra. For an arbitrary $A \in \mathcal{F}(X)$, we set*
(1) $A \in F_3 := (\forall x)(x \in A \to 1 \in A)$,
(2) $A \in F_4 := (\forall x)(\forall y)(N(Nx * Ny) \in A \wedge y \in A \to x \in A)$.
A unary fuzzy predicate $f_B \in \mathcal{F}(\mathcal{F}(X))$ is said to be an L-fuzzifying BCK-filter of X, if it is given as follows: for an arbitrary $A \in \mathcal{F}(X)$,

$$A \in f_B := (A \in F_3) \wedge (A \in F_4).$$

Theorem 6.5. *Let X be a bounded BCK-algebra and $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in f_B$, then for all $x, y \in X$ with $Nx \leq Ny$,*

$$\models^{\mathcal{L}} y \in A \to x \in A.$$

Proof: From $\models^{\mathcal{L}} A \in f_B$ we have that $\prod_{x\in X}(A(x)\alpha A(1)) = 1 = \prod_{x,y\in X}\{(A(N(Nx * Ny)) \cdot A(y))\alpha A(x)\}$, and so $A(x) \leq A(1)$ and $A(N(Nx * Ny)) \cdot A(y) \leq A(x)$ for all $x, y \in X$. If $Nx \leq Ny$, then $Nx * Ny = 0$, so $N(Nx * Ny) = N0 = 1$. It follows that $A(y) = A(1) \cdot A(y) = A(N(Nx * Ny)) \cdot A(y) \leq A(x)$ for all $x, y \in X$. Therefore, for all $x, y \in X$ with $Nx \leq Ny$,

$$[y \in A \to x \in A] = A(y)\alpha A(x) = 1,$$

proving the theorem. □

Corollary 6.2. *Let X be a bounded BCK-algebra and $A \in \mathcal{F}(\mathcal{F}(X))$. If $\models^{\mathcal{L}} A \in f_B$, then for all $x, y \in X$ with $y \leq x$,*

$$\models^{\mathcal{L}} y \in A \to x \in A.$$

Proof: From $y \leq x$, we have $Nx \leq Ny$, and prove the corollary using Theorem 6.5. □

Theorem 6.6. *Let X be a bounded commutative BCK-algebra and $A \in \mathcal{F}(X)$. If $\models^{\mathcal{L}} A \in f_B$, then*

$$\models^{\mathcal{L}} A \in f_l.$$

Proof: Since $\models^{\mathcal{L}} A \in f_B$, by Corollary 6.1 we have that $\models^{\mathcal{L}} A \in F_1$. That is to say, $A(y) \geq A(x)$ for all $x, y \in X$ and $x \leq y$. Because X is a bounded commutative BCK-algebra, there exists element $1 \in X$ such that $y \leq 1$ for all $y \in X$. For any $x, y \in X$ we have that $y * x \leq 1 * x = Nx$, and so

$$x = NNx \leq N(y * x) = N(y * (y * (y * x))) = N(y * (x \dot{\wedge} y)) = N(N(x \dot{\wedge} y) * Ny).$$

Hence, $A(N(N(x \dot{\wedge} y) * Ny)) \geq A(x)$ for any $x, y \in X$. By hypothesis and Corollary 6.1, we have

$$\begin{aligned} 1 &= [A \in f_B] \\ &\leq \prod_{z,w \in X} \{(A(N(Nz * Nw)) \cdot A(w))\alpha A(z)\} \\ &\leq \prod_{x,y \in X} \{(A(N(N(x \dot{\wedge} y) * Ny)) \cdot A(y))\alpha A(x \dot{\wedge} y)\} \\ &\leq \prod_{x,y \in X} \{(A(x) \cdot A(y))\alpha A(x \dot{\wedge} y)\} \\ &= [(\forall x)(\forall y)(x \in A \wedge y \in A \to x \dot{\wedge} y \in A)] \\ &= [A \in f_l]. \end{aligned}$$

The theorem is proved. □

The next theorem shows that for commutative BCK-algebras, Definition 6.2 can be simplified.

Theorem 6.7. *Let X be a bounded commutative BCK-algebra and $A \in \mathcal{F}(X)$. Then* $\models^{\mathcal{L}} A \in f_B \leftrightarrow (A \in F_3) \wedge ((\forall x)(\forall y)(y \in A \wedge N(y * x) \in A \to x \in A))$.

The proof is trivial and omitted.

Now, we consider the converse of Theorem 6.7.

Theorem 6.8. *Let X be a bounded commutative BCK-algebra and $A \in \mathcal{F}(X)$. If X is implicative and $\models^{\mathcal{L}} A \in f_l$, then*

$$\models^{\mathcal{L}} A \in f_B.$$

Proof: Since X is implicative, bounded and commutative, we have that $x \dot{\wedge} y = y * (y * x) = N(y * x) * Ny = N(y * x) \dot{\wedge} y = N(Nx * Ny) \dot{\wedge} y$ for all $x, y \in X$. From $\models^{\mathcal{L}} A \in f_l$ we have that $[A \in F_1] = 1 = [A \in F_2]$. It follows that $A(y) \geq A(x)$ for all $x, y \in X$, $x \leq y$. By $x \dot{\wedge} y \leq x$, we get that $A(x) \geq A(x \dot{\wedge} y)$ for all $x, y \in X$. Because the binary operation α is isotone in the second variable, we have

$$\begin{aligned} 1 &= [A \in F_2] \\ &= \prod_{x,y \in X} \{(A(x) \cdot A(y))\alpha A(x \dot{\wedge} y)\} \end{aligned}$$

$$\le \prod_{x,y\in X} (A(N(Nx * Ny)) \cdot A(y))\alpha A(N(Nx * Ny)\dot{\wedge} y)$$
$$= \prod_{x,y\in X} (A(N(Nx * Ny)) \cdot A(y))\alpha A(x\dot{\wedge} y)$$
$$\le \prod_{x,y\in X} (A(N(Nx * Ny)) \cdot A(y))\alpha A(x)$$
$$= [A \in F_4].$$

On the other hand,

$$1 = [A \in F_1] = \prod_{x,y\in X, x\le y} \{A(x)\alpha A(y)\} \le \prod_{x\in X}\{A(x)\alpha A(1)\} = [A \in F_3],$$

and so $[A \in F_4] = 1 = [A \in F_3]$. Therefore, $[A \in f_B] = [A \in F_3]\cdot[A \in F_4] = 1$, completing the proof. □

Similar to the proofs of Theorem 4.2, Theorem 4.3, Theorem 4.4 and Theorem 5.9, we have the following three theorems.

Theorem 6.9. *Let X be a bounded BCK-algebra and $A_\lambda \in \mathcal{F}(X)$ for all $\lambda \in \Lambda$, and then for every $\lambda \in \Lambda$,*

$$\models^{\mathcal{L}} A_\lambda \in f_B \to \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \in f_B.$$

Theorem 6.10. *Let X and Y be bounded BCK-algebras, $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, and let the lattice $< L, +, \cdot >$ be completely distributive. If $f : X \to Y$ is a surjective homomorphism, then*

$$\models^{\mathcal{L}} A \in f_B \to f(A) \in f_B,$$
$$\models^{\mathcal{L}} B \in f_B \leftrightarrow f^{-1}(B) \in f_B.$$

Theorem 6.11. *Let X be a bounded BCK-algebra, and $A, B \in \mathcal{F}(X)$. Then*

$$\models^{\mathcal{L}} (A \in f_B) \wedge (B \in f_B) \to A \times B \in f_B.$$

7. **Conclusions.** As a branch of general algebra, BCK-algebra originated in 1960s and is booming now. BCK-algebras appear in many branches of mathematics such as universal algebra, group theory, ring theory, lattice theory, point set topology, and topological algebra. Fuzzy generalizations of BCK-algebras were introduced by Xi et al. In this paper, we generalize the notion of Xi's fuzzy BCK-algebras [20] taking a complete residuated lattice as a basic algebraic structure.

It is well known that ideal theory plays a very important role in studying the algebraic structures of ring, lattice, BCK-algebra and so on. In this paper, we have generalized several classes of fuzzy ideals to the complete residuated lattice-valued logic scenario, presented the concepts of several classes of fuzzifying ideals such as L-fuzzifying ideal, L-fuzzifying positive implicative ideal and L-fuzzifying implicative ideal, and investigated basic properties and relations between these classes. Filter theory not only plays a very important role in studying the related algebraic structures, but also gives rise to a crucial methodology for analyzing logic systems. It is therefore natural to consider L-fuzzifying filters. In particular, an interesting problem concerns fuzzy inference rules: how to combine methods commonly applied in fuzzy logics with reasoning methods traditionally developed within the framework of filter theory. Filters correspond to the basic modus ponens. We have combined the semantical method of complete residuated lattice-valued logic and the filter theory to develop the notions of L-fuzzifying filter and L-fuzzifying

BCK-filter in BCK-algebras. At the same time, the relations between these notions are investigated. We have discussed homomorphism relations and intersections of several fuzzy structures such as L-fuzzifying subalgebras, L-fuzzifying ideals, and L-fuzzifying filters. We desperately hope that our work would serve as a foundation for enriching theory of BCK-algebras and many-valued logical system.

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REFERENCES

[1] L. Bolc and P. Borowik, *Many-Valued Logic*, Springer, Berlin, 1994.
[2] V. Novak, First order fuzzy logic, *Studia Logica*, vol.46, pp.87-109, 1982.
[3] M. Ward and R. P. Dilworth, Residuated lattices, *Trans. Amer. Math. Soc.*, vol.45, pp.335-354, 1939.
[4] R. P. Dilworth, Abstract residuation over lattices, *Bull. Amer. Math. Soc.*, vol.44, pp.262-268, 1938.
[5] Y. Xu, Lattice implication algebra, *Journal of Southwest Jiaotong Univ.*, vol.28, pp.20-27, 1993.
[6] J. Pavelka, On fuzzy logic I: Many-valued rules of inferences, *Z. Math. Logik Grundlagen Math.*, vol.25, pp.45-52, 1979.
[7] J. Pavelka, On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi, *Z. Math. Logik Grundlagen Math.*, vol.25, pp.119-134, 1979.
[8] J. Pavelka, On fuzzy logic III: Semantical completeness of some many-valued propositional calculi, *Z. Math. Logik Grundlagen Math.*, vol.25, pp.447-464, 1979.
[9] Y. Imai and K. Iseki, On axiom system of prepositional calculi XIV, *Proc. of Japan Academy*, vol.42, pp.19-22, 1966.
[10] K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japon.*, vol.23, pp.1-26, 1978.
[11] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa Co., Seoul, Korea, 1994.
[12] M. Palasinski, Ideals in BCK-algebras which are lower semilattice, *Math. Japon.*, vol.26, pp.245-250, 1981.
[13] J. Meng, On ideals of BCK-algebras, *Math. Japon.*, vol.40, pp.143-154, 1994.
[14] K. Iseki and S. Tanaka, Ideal theory of BCK-algebras, *Math. Japon.*, vol.21, pp.351-366, 1976.
[15] K. Iseki, On some ideals in BCK-algebras, *Math. Seminar Notes*, vol.3, pp.65-70, 1975.
[16] S. K. Goel and A. K. Arora, Obstinate ideals in BCK-algebras, *Math. Japon.*, vol.32, pp.559-567, 1987.
[17] E. Y. Deeba, Filter theory of BCK-algebras, *Math. Japon.*, vol.25, pp.631-639, 1980.
[18] J. Meng, BCK-filters, *Math. Japon.*, vol.44, pp.119-129, 1996.
[19] L. A. Zadeh, Fuzzy sets, *Inform. Control*, vol.8, pp.338-353, 1965.
[20] O. G. Xi, Fuzzy BCK-algebras, *Math. Japon.*, vol.36, pp.935-942, 1991.
[21] Y. B. Jun and E. H. Roh, Fuzzy commutative ideals of BCK-algebras, *Fuzzy Sets and Systems*, vol.64, pp.401-405, 1994.
[22] Y. B. Jun, S. M. Hong, J. Meng and X. L. Xin, Characterizations of fuzzy positive implicative ideals in BCK-algebras, *Math. Japon.*, vol.40, pp.503-507, 1994.
[23] J. B. Rosser and A. R. Turquette, *Many-Valued Logics*, North-Holland, Amsterdam, New York, 1952.
[24] Y. W. Yang, P. F. He and X. L. Xin, Fuzzy ideals of soft BCK-algebras, *Fuzzy Systems and Mathematics*, vol.28, pp.31-38, 2014.
[25] M. S. Ying, A new approach for fuzzy topology (II), *Fuzzy Sets and Systems*, vol.47, pp.221-232, 1992.
[26] M. S. Ying, A new approach for fuzzy topology (III), *Fuzzy Sets and Systems*, vol.55, pp.193-207, 1993.
[27] G. J. Zhang, C. Zhang and K. Q. Zou, Fuzzifying BCK-algebras, *Fuzzy Systems and Mathematics*, vol.15, pp.33-36, 2001.

[28] M. S. Ying, Fuzzifying topology based on complete residuated lattice-valued logic, *Fuzzy Sets and Systems*, vol.56, pp.337-373, 1993.
[29] D. W. Qin, Automata theory based on complete residuated lattice-valued logic I, *Science in China: Series E*, vol.33, pp.137-146, 2003.
[30] D. W. Qin, Automata theory based on complete residuated lattice-valued logic II, *Science in China: Series E*, vol.33, pp.340-349, 2003.
[31] J. Y. Peng, Automata and grammar theory based on complete residuated lattice-valued logic, *Pattern Recognition and Artificial Intelligence*, vol.24, pp.610-617, 2011.
[32] Y. Arai, K. Iséki and S. Tanaka, Characterizations of BCI, BCK-algebras, *Proc. of Japan Academy*, vol.42, pp.105-107, 1996.
[33] K. Iséki and S. Tanaka, Ideal theory of BCK-algebras, *Math. Japon.*, vol.2, pp.351-366, 1976.
[34] T. D. Lei and Y. S. Pu, *BCK-Algebra and BCI-Algebra*, Lecture in Shanxi Normal University, 1983.
[35] G. J. Wang, *Non-Classical Mathematical Logic and Approximate Reasoning*, Science Press, Beijing, 1994.
[36] J. Y. Peng, On fuzzifying ideals in BCK-algebras, *Journal of Sichuan Normal University*, vol.33, 592-597, 2010.
[37] X. D. Liu, $(\in, \in \vee q_{(\lambda,\mu)})$-fuzzy ideals in sub-BCI-algebras, *Fuzzy Systems and Mathematics*, vol.30, pp.38-47, 2016.