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IMPROVED ADAPTIVE HINF CONTROL APPROACH FOR PIECEWISE LINEAR SYSTEMS WITH APPLICATION TO AIR-BREATHING HYPERSONIC VEHICLE

Shangmin Zhang and Xingang Liang

School of Aerospace Engineering
Tsinghua University
Haidian Dist., Beijing 100084, P. R. China
zhang-sm2016@mail.tsinghua.edu.cn

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Abstract. *This paper presents an improved adaptive Hinf control approach for piecewise linear systems (PLS) via piecewise Lyapunov function instead of common Lyapunov function. At first, by designing the projection-type piecewise adaptive law, the problem of the adaptive control of PLS can be reduced to the Hinf control problem of augmented piecewise systems. Then, we construct the piecewise linear control law for augmented systems in such a way that the piecewise quadratic Lyapunov function can be employed to guarantee the stability and Hinf performance. Most importantly, the reciprocal projection lemma is employed to formulate the synthesis condition as linear matrix inequities, which enables that the proposed PQLF approach is numerically solvable. The results are illustrated by application to control the air-breathing hypersonic vehicle, which demonstrates the efficacy and advantage of the proposed approach.*
Keywords: Piecewise linear systems, Adaptive Hinf control, Linear matrix inequalities, Hypersonic vehicle

1. **Introduction.** Piecewise linear systems (PLS) are hybrid systems [1, 2, 3, 4] with state space-partition-based switching, which often arise in practice when piecewise-linear components are encountered. These components include dead-zone, saturation, relays and hysteresis; hence many practical engineering systems can be described as PLS such as flight control systems, robotic manipulators control systems and power electronics systems. In addition, PLS can approximate nonlinear dynamical systems [5] to any degree of accuracy [6], hence providing a powerful means of analysis and synthesis for nonlinear control systems. Therefore, the investigation of control design problem of PLS is very significant for both of engineering and theory aspects.

Numerous great works have been achieved on synthesis problem of uncertain PLS with model uncertainties and disturbances [7-14] due to the requirement from practical engineering application. The feedback control synthesis problem of uncertain PLS was firstly considered in [7], the piecewise affine control law was designed by choosing a piecewise quadratic Lyapunov function (PQLF) to guarantee the robust stability and Hinf performance. Song et al. [8] investigated the robust H_∞ control problem for PLS with linear fractional uncertainties and disturbances using PQLF approach, and the LMI-based results were provided using the cone complementarity linearization (CCL) method. Furthermore, Zhang and Tang [9, 10] extended the PQLF synthesis framework to the output feedback Hinf control problem of PLS, and the mixed algorithm was presented to solve the controller and piecewise quadratic Lyapunov function. More recently, Samadi and

Rodrigues [11, 12] considered the control design of a certain kind of uncertain PLS described by piecewise-linear differential inclusions, and a dual parameter set convex relation approach was presented to formulate the PQLF synthesis conditions as LMIs.

However, to our best knowledge, no synthesis issue has been considered for PLS with polyhedral parametric uncertainties and disturbances except [13, 14]. On the other hand, both [13, 14] employ the common quadratic Lyapunov function (CQLF) approach to avoid the difficulty for formulating the synthesis condition as LMIs in the PQLF framework, which brings more conservatism.

Motivated by the above observations, we revisit the synthesis problem of PLS with polyhedral parametric uncertainties and disturbances. By designing a piecewise projection-style adaptive law, the problem of the adaptive control of PLS can be reduced to the Hinf control problem of augmented piecewise systems. Then, we construct the piecewise affine control law for augmented piecewise systems in such a way that the piecewise quadratic Lyapunov function can be employed to establish the stability and Hinf performance. Particularly, the reciprocal projection lemma is employed to formulate the synthesis condition as linear matrix inequities, which enables the proposed PQLF approach is numerically solvable.

The rest of the paper is organized as follows. The system model is described and the control design problems are formulated in Section 2. In Section 3, we present a piecewise projection-style adaptive law design method to convert the problem to the control problem of augmented piecewise systems. Furthermore, in Section 4 the control design method has been provided for augmented piecewise systems to realize H_∞ performance. The results are illustrated by application to control the air-breathing hypersonic vehicle in Section 5 and conclusions are drawn in Section 6.

2. Problem Statement.

The PLS considered in this paper can be described as

$$\begin{cases} \dot{x} = \mathcal{A}_i(\theta)x + \mathcal{B}_i(\theta)u + \mathcal{D}_i(\theta)\omega \\ z = \mathcal{C}_i(\theta)x \end{cases} \quad x \in \mathcal{R}_i, \tag{1}$$

where $\cup_{i\in\mathcal{I}}\mathcal{R}_i \subseteq \mathbb{R}^n$ denotes the state space is divided by many closed polyhedral regions; $\mathcal{I}$ denotes the index set of polyhedral regions. $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\theta \in \mathbb{R}^q$, $\omega \in \mathbb{R}^r$ and $z \in \mathbb{R}^p$ denote the state, input, uncertain parameter, disturbance and output vector, respectively. The symbols $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$ denote the system matrix, input matrix, output matrix and disturbance matrix in the description of state-space form. Assuming origin is one of vertexes for all polyhedral regions, so we can always find the matrix E_i for each closed polyhedral region $\mathcal{R}_i$ satisfying $\mathcal{R}_i = \{x|E_i x \geq 0\}$. Meanwhile considering the PLS suffering from the polyhedral uncertainties, here the dependency relationship with respect to $\theta = (\theta_1, \theta_2, \ldots, \theta_q)$ is affine, i.e.,

$$\begin{bmatrix} \mathcal{A}_i(\theta) & \mathcal{B}_i(\theta) \\ \mathcal{C}_i(\theta) & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{i0} & \mathcal{B}_{i0} \\ \mathcal{C}_{i0} & 0 \end{bmatrix} + \sum_{j=1}^{q} \theta_j \begin{bmatrix} \mathcal{A}_{ij} & \mathcal{B}_{ij} \\ \mathcal{C}_{ij} & 0 \end{bmatrix}. \tag{2}$$

Assumption 2.1. *Assume the uncertain parametric vector θ belongs to a bounded set Ω_θ satisfying*

$$\begin{array}{l} \Omega_\theta := \{\theta|\theta_{j\min} \leq \theta_j \leq \theta_{j\max}\} \\ \overline{\Omega}_\theta := \{\theta|\theta_j \in \{\theta_{j\min}, \theta_{j\max}\}\} \end{array}, \ \forall j \in \{1, \ldots, q\}, \tag{3}$$

where $\overline{\Omega}_\theta$ denotes the set containing 2^q vertexes of Ω_θ.

Definition 2.1. *Let $\hat{\theta}(t)$ denote the estimation value of θ, the adaptive Hinf control objective is to design piecewise control law*

$$u(t) = u_i\left(x, \hat{\theta}\right), \quad x \in \mathcal{R}_i, \tag{4}$$

and the related piecewise adaptive law for the uncertain parameter θ,

$$\dot{\hat{\theta}} = v_i\left(x, \hat{\theta}\right), \quad x \in \mathcal{R}_i, \tag{5}$$

to guarantee closed-loop PLS satisfying

(i) Asymptotical stability without external disturbance, $\lim_{t\to\infty} x(t) = 0$;

(ii) Disturbance rejection, there exist $\gamma > 0$ and $\epsilon > 0$ to realize

$$\|z(t)\|_2 < \gamma\|\omega(t)\|_2 + \epsilon. \tag{6}$$

3. **Piecewise Adaptive Law Design.** By designing a piecewise projection-style adaptive law in this section, the adaptive Hinf control problem of PLS can be reduced to Hinf control problem of augmented systems. Similar in [15], we first construct a projection-style adaptive law structure to achieve a well controlled adaptation process as follows

$$\dot{\hat{\theta}} = \mathrm{Proj}_{\hat{\theta}}(\Lambda), \quad \hat{\theta}(0) \in \Omega_\theta \tag{7}$$

$$\mathrm{Proj}_{\hat{\theta}}(\Lambda) = \begin{cases} 0 & \hat{\theta}_j \geq \theta_{j\max} \text{ and } \Lambda_j > 0 \\ 0 & \hat{\theta}_j \leq \theta_{j\min} \text{ and } \Lambda_j < 0 \\ \Lambda & \text{else} \end{cases}, \tag{8}$$

where $\tilde{\theta}$ denotes the estimation error $\hat{\theta} - \theta$, and Λ is an adaptive function that needs to be synthesized. It has been proposed in [15] that the adaptive law owns the following good properties. For all t

$$\begin{array}{ll} (1) & \hat{\theta}(t) \in \Omega_\theta \\ (2) & \tilde{\theta}^T\left(\mathrm{Proj}_{\hat{\theta}}(\Lambda) - \Lambda\right) \leq 0. \end{array} \tag{9}$$

By employing the above vector projection structure, we provide the piecewise projection-style adaptive law for PLS as the following theorem.

Theorem 3.1. *If there exists the positive definite function $V(x)$ and piecewise controller*

$$u = u_i\left(x, \hat{\theta}\right), \quad x \in \mathcal{R}_i, \ \hat{\theta} \in \Omega_\theta \tag{10}$$
$$u\left(0, \hat{\theta}\right) = 0$$

and for any $\hat{\theta}, \theta \in \Omega_\theta$,

$$\begin{aligned} &\frac{dV}{dx}\left[\mathcal{A}_i\left(\hat{\theta}\right)x + \mathcal{B}_i\left(\hat{\theta}\right)u_i\left(x, \hat{\theta}\right)\right] + x^T\mathcal{C}_i(\theta)^T\mathcal{C}_i(\theta)x \\ &+ \frac{\gamma^{-2}}{4}\frac{dV}{dx}\mathcal{D}_i(\theta)\mathcal{D}_i(\theta)^T\frac{dV}{dx}^T < 0, \quad x \in \mathcal{R}_i. \end{aligned} \tag{11}$$

Then the AHC problem can be solved with the following piecewise adaptive law

$$\dot{\hat{\theta}} = \mathit{Proj}_{\hat{\theta}}(\Lambda_i(x, u)), \quad \hat{\theta}(0) \in \Omega_\theta, \quad x \in \mathcal{R}_i, \tag{12}$$

where

$$\Lambda_i(x, u) = \frac{1}{2\eta}\begin{bmatrix} \frac{dV}{dx}[\mathcal{A}_{i1}x + \mathcal{B}_{i1}u] \\ \frac{dV}{dx}[\mathcal{A}_{i2}x + \mathcal{B}_{i2}u] \\ \vdots \\ \frac{dV}{dx}[\mathcal{A}_{iq}x + \mathcal{B}_{iq}u] \end{bmatrix}, \quad \eta > 0. \tag{13}$$

(1) Proof of asymptotical stability

Given positive definite $V(x)$, define

$$\vartheta\left(x,\hat{\theta}\right)=V(x)+\eta\left(\hat{\theta}-\theta\right)^{T}\left(\hat{\theta}-\theta\right),\quad \eta>0. \tag{14}$$

where $\eta>0$ can be designed by the following principle

$$\max_{\left(\hat{\theta},\theta\right)\in\Omega_\theta\times\Omega_\theta}\eta\left(\hat{\theta}-\theta\right)^{T}\left(\hat{\theta}-\theta\right)\le\epsilon,\quad \epsilon>0. \tag{15}$$

Utilizing $\vartheta\left(x,\hat{\theta}\right)$ as the Lyapunov candidate of adaptive Hinf control problem, so for all $x\in\mathcal{R}_i$, $\hat{\theta}\in\Omega_\theta$ we have

$$\begin{aligned}
\frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}&=\frac{dV}{dx}\left(\mathcal{A}_i(\theta)x+\mathcal{B}_i(\theta)u\right)+2\eta\tilde{\theta}^{T}\dot{\hat{\theta}}\\
&=\frac{dV}{dx}\left[\left(\mathcal{A}_i\left(\hat{\theta}\right)-\mathcal{A}_i\left(\tilde{\theta}\right)+\mathcal{A}_{i0}\right)\mathrm{x}+\left(\mathcal{B}_i\left(\hat{\theta}\right)-\mathcal{B}_i\left(\tilde{\theta}\right)+\mathcal{B}_{i0}\right)u\right]+2\eta\tilde{\theta}^{T}\dot{\hat{\theta}}\\
&=\frac{dV}{dx}\left(\mathcal{A}_i\left(\hat{\theta}\right)x+\mathcal{B}_i\left(\hat{\theta}\right)u\right)-\sum_{j=1}^{\mathrm{q}}\tilde{\theta}_j^{T}\frac{dV}{dx}\left[\mathcal{A}_{ij}x+\mathcal{B}_{ij}u\right]+2\eta\tilde{\theta}^{T}\dot{\hat{\theta}}\\
&=\frac{dV}{dx}\left(\mathcal{A}_i\left(\hat{\theta}\right)x+\mathcal{B}_i\left(\hat{\theta}\right)u\right)+2\eta\tilde{\theta}^{T}\left(\dot{\hat{\theta}}-\frac{1}{2\eta}\begin{bmatrix}\frac{dV}{dx}\left[\mathcal{A}_{i1}x+\mathcal{B}_{i1}u\right]\\ \frac{dV}{dx}\left[\mathcal{A}_{i2}x+\mathcal{B}_{i2}u\right]\\ \vdots\\ \frac{dV}{dx}\left[\mathcal{A}_{i\mathrm{q}}x+\mathcal{B}_{i\mathrm{q}}u\right]\end{bmatrix}\right)\\
&=\frac{dV}{dx}\left(\mathcal{A}_i\left(\hat{\theta}\right)x+\mathcal{B}_i\left(\hat{\theta}\right)u\right)+2\eta\tilde{\theta}^{T}\left(\mathrm{Proj}_{\hat{\theta}}\left(\Lambda_i(x,u)\right)-\Lambda_i(x,u)\right).
\end{aligned} \tag{16}$$

Exploiting the vector projection property function (9) and the condition (11), we have

$$\frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}\le 0,\quad x\in\mathcal{R}_i, \tag{17}$$

where '=' holds if and only if $x=0$.

The result (17) implies that both of solution $x(t)$ and $\hat{\theta}$ are bounded, further consider the augmented system consisted of (1), (10), (12), define

$$\triangle=\left\{\left(x,\hat{\theta}\right)\in\left(\mathbb{R}^n,\Omega_\theta\right)\left|\frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}=0\right.\right\}, \tag{18}$$

noting that in (17), "=" holds if and only if $x=0$, that implies

$$\triangle=\left\{\left(0,\hat{\theta}\right)\left|\hat{\theta}\in\Omega_\theta\right.\right\}. \tag{19}$$

In addition, considering $\hat{\theta}(0)\in\Omega_\theta$ implies $\hat{\theta}(t)\in\Omega_\theta$ for all $t\ge 0$, it can be well verified that $\triangle$ is an invariant set. By LaSalle's Invariant theorem [16], then we can conclude that the augmented system state $\left(x(t),\hat{\theta}(t)\right)$ will converge to $\triangle$ from any initial value $\left(x(0),\hat{\theta}(0)\right)\in\left(\mathbb{R}^n,\Omega_\theta\right)$, that is

$$\lim_{t\to\infty}x(t)=0,\quad\forall\left(x(0),\hat{\theta}(0)\right)\in\left(\mathbb{R}^n,\Omega_\theta\right). \tag{20}$$

(2) Proof of disturbance rejection

For any given $x(0)$, $\omega(t)$ and θ, let $t_0 = 0$ and $\{t_k\}_1^{\mathrm{N}_t}$ denotes the switch times, in other words at each t_k, the solution transfers from region $\mathcal{R}_{i_k}$ to $\mathcal{R}_{i_{k+1}}$.

Taking the integral of $\frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}$ from zero to infinity get that

$$
\begin{aligned}
&\int_0^\infty \frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}dt \\
&= \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[\frac{dV}{dx}\left(\mathcal{A}_{i_k}(\theta)x+\mathcal{B}_{i_k}(\theta)u+\mathcal{D}_{i_k}(\theta)\omega\right)+2\eta\tilde{\theta}^T\dot{\hat{\theta}}\right]dt \\
&= \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[\frac{dV}{dx}\left(\mathcal{A}_{i_k}\left(\hat{\theta}\right)x+\mathcal{B}_{i_k}\left(\hat{\theta}\right)u+\mathcal{D}_{i_k}(\theta)\omega\right)\right. \\
&\quad \left. -\sum_{j=1}^{\mathrm{q}}\tilde{\theta}_j^T\frac{dV}{dx}\left(\mathcal{A}_{i_kj}x+\mathcal{B}_{i_kj}u\right)+2\eta\tilde{\theta}^T\dot{\hat{\theta}}\right]dt \\
&= \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[\frac{dV}{dx}\left(\mathcal{A}_{i_k}\left(\hat{\theta}\right)x+\mathcal{B}_{i_k}\left(\hat{\theta}\right)u+\mathcal{D}_{i_k}(\theta)\omega\right)+2\eta\tilde{\theta}^T\left(\dot{\hat{\theta}}-\Lambda_{i_k}(x,u)\right)\right]dt \\
&\le \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[\frac{dV}{dx}\left(\mathcal{A}_{i_k}\left(\hat{\theta}\right)x+\mathcal{B}_{i_k}\left(\hat{\theta}\right)u+\mathcal{D}_{i_k}(\theta)\omega\right)\right]dt
\end{aligned}
$$

utilizing condition (11) to show that for all $\hat{\theta}, \theta \in \Omega_\theta$,

$$
\begin{aligned}
&\int_0^\infty \frac{d\vartheta\left(x,\hat{\theta}\right)}{dt}dt \\
&< \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[-x^T\mathcal{C}_{i_k}(\theta)^T\mathcal{C}_{i_k}(\theta)x+\frac{dV}{dx}\mathcal{D}_{i_k}(\theta)\omega-\frac{\gamma^{-2}}{4}\frac{dV}{dx}\mathcal{D}_{i_k}(\theta)\mathcal{D}_{i_k}(\theta)^T\frac{dV}{dx}^T\right]dt \\
&= \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[-x^T\mathcal{C}_{i_k}(\theta)^T\mathcal{C}_{i_k}(\theta)x+\gamma^2\omega^T\omega\right. \\
&\quad \left.-\left(\frac{1}{2\gamma}\frac{dV}{dx}\mathcal{D}_{i_k}(\theta)-\gamma\omega^T\right)\left(\frac{1}{2\gamma}\frac{dV}{dx}\mathcal{D}_{i_k}(\theta)-\gamma\omega^T\right)^T\right] \\
&\le \sum_{k=1}^{N_t+1}\int_{t_{k-1}}^{t_k}\left[-x^T\mathcal{C}_{i_k}(\theta)^T\mathcal{C}_{i_k}(\theta)x+\gamma^2\omega^T\omega\right]dt \\
&= \int_0^\infty\left[-z^Tz+\gamma^2\omega^T\omega\right]dt
\end{aligned}
$$

This implies

$$
\vartheta\left(x(\infty),\hat{\theta}(\infty)\right)-\vartheta\left(x(0),\hat{\theta}(0)\right)\le\int_0^\infty\left[-z^Tz+\gamma^2\omega^T\omega\right]dt. \tag{21}
$$

Further by noting the fact $\vartheta\left(x(\infty),\hat{\theta}(\infty)\right)\ge 0$ conclude

$$
\int_0^\infty z^Tzdt\le\gamma^2\int_0^\infty\omega^T\omega dt+V(x(0))+\eta\left(\hat{\theta}(0)-\theta\right)^T\left(\hat{\theta}(0)-\theta\right). \tag{22}
$$

Employing (15) and (35) get

$$\int_0^\infty z^T z dt \le \gamma^2 \int_0^\infty \omega^T \omega dt + V(x(0)) + \epsilon, \tag{23}$$

yielding for $x(0) = 0$

$$\|z(t)\|_2 < \gamma \|\omega(t)\|_2 + \epsilon. \tag{24}$$

This completes the proof.

Remark 3.1. *The meaning of Theorem 3.1 lies in the problem simplification, which helps to reformulate the adaptive Hinf control problem of PLS as the pure Hinf control problem of piecewise augmented system consisted of (1), (10), (12). In other words, the parametric uncertainties do not need to be considered in the piecewise augmented system anymore.*

4. **PQLF Synthesis Framework.**

To enable the PQLF approach to the problem (1), (10), (12), we introduce the next two lemmas.

Lemma 4.1. [17] *(Reciprocal Projection-style Lemma) Let Φ denote any known positive definite matrix. The next two issues are equivalent:*

$$\begin{aligned} &(1)\ \Psi + \Xi + \Xi^T < 0; \\ &(2)\ \textit{Existing matrixes } H \textit{ satisfying} \\ &\begin{bmatrix} \Psi + \Phi - \left(H + H^T\right) & \Xi^T + H^T \\ * & -\Phi \end{bmatrix} < 0. \end{aligned} \tag{25}$$

Lemma 4.2. [18] *For any given positive definite matrices P, the next inequality is satisfied with*

$$G^T P^{-1} G \ge G^T + G - P. \tag{26}$$

Now, we are ready to provide the PQLF based control synthesis approach.

Theorem 4.1. *For given constant $\gamma > 0$, if there exist symmetric matrices T, U_i, W_i and general matrices V_i, R_i, where U_i, W_i have nonnegative entries, and with $P_i = F_i^T T F_i$, the LMIs (29) are satisfied for all $\hat\theta', \hat\theta'', \theta', \theta'' \in \overline{\Omega}_\theta$, where $\Psi_i\left(\hat\theta', \hat\theta''\right) = \mathcal{A}_i\left(\hat\theta'\right) V_i + \mathcal{B}_i\left(\hat\theta'\right) R_i\left(\hat\theta''\right)$, then the following PQLF*

$$V(x) = x^T P_i x, \quad x \in \mathcal{R}_i \tag{27}$$

and piecewise linear controller

$$u = K_i\left(\hat\theta\right) x = \left[K_{i0} + \sum_{j=1}^q \hat\theta_j(t) K_{ij}\right] x, \quad x \in \mathcal{R}_i \tag{28}$$

is solvable with the condition (11) of Theorem 3.1, where $K_{ij} = R_{ij} V_i^{-1}$.

$$\begin{aligned} &P_i - E_i^T W_i E_i > 0 \\ &\begin{bmatrix} -\left(V_i + V_i^T\right) & \Psi_i^T\left(\hat\theta', \hat\theta''\right) + P_i & V_i^T & \gamma^{-1} P_i \mathcal{C}_i^T\left(\hat\theta'\right) & \mathcal{D}_i(\theta') \\ \Psi_i\left(\hat\theta', \hat\theta''\right) + P_i & -P_i & 0 & 0 & 0 \\ V_i & 0 & E_i^T U_i E_i - P_i & 0 & 0 \\ \gamma^{-1}\mathcal{C}_i\left(\hat\theta''\right) P_i & 0 & 0 & -I & 0 \\ \mathcal{D}_i^T(\theta'') & 0 & 0 & 0 & -I \end{bmatrix} < 0. \end{aligned} \tag{29}$$

Proof: Employ $V(x) = x^T P_i x$ as system Lyapunov candidate. First, it can be implied that $V(x)$ is positive definite from the first inequality of (29) by S-procedure. That means our rest objective is to prove $V(x)$ and piecewise controller (28) will satisfy (11) of Theorem 3.1, that is, for all $\hat{\theta}, \theta \in \Omega_\theta$,

$$\frac{dV}{dx}\left[\mathcal{A}_i\left(\hat{\theta}\right)x + \mathcal{B}_i\left(\hat{\theta}\right)u_i\left(x,\hat{\theta}\right)\right] + x^T\mathcal{C}_i(\theta)^T\mathcal{C}_i(\theta)x + \frac{\gamma^{-2}}{4}\frac{dV}{dx}\mathcal{D}_i(\theta)\mathcal{D}_i(\theta)^T\frac{dV}{dx}^T < 0. \quad (30)$$

Substituting $V(x)$ by $x^T P_i x$, the condition (30) can be rewritten by S-procedure, that is for all $\hat{\theta}, \theta \in \Omega_\theta$

$$\begin{aligned}&\left(\mathcal{A}_i\left(\hat{\theta}\right) + \mathcal{B}_i\left(\hat{\theta}\right)K_i\left(\hat{\theta}\right)\right)^T P_i + P_i\left(\mathcal{A}_i\left(\hat{\theta}\right) + \mathcal{B}_i\left(\hat{\theta}\right)K_i\left(\hat{\theta}\right)\right)\\ &+ \mathcal{C}_i^T\left(\hat{\theta}\right)\mathcal{C}_i\left(\hat{\theta}\right) + \gamma^{-2}P_i\mathcal{D}_i(\theta)\mathcal{D}_i^T(\theta)P_i + E_i^T U_i E_i < 0, \quad x \in \mathcal{R}_i.\end{aligned} \quad (31)$$

Note that all the $\hat{\theta}, \theta \in \Omega_\theta$ can be expressed by the convex hull of the parametric values in vertexes, the required condition (31) is equivalent to the following condition

$$\begin{aligned}&\left(\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{A}_i(\theta_\lambda) + \left[\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{B}_i(\theta_\lambda)\right]\left[\sum_{\lambda=1}^{2^q}\alpha_\lambda K_i\left(\hat{\theta}\right)\right]\right)^T P_i\\ &+ P_i\left(\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{A}_i(\theta_\lambda) + \left[\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{B}_i(\theta_\lambda)\right]\left[\sum_{\lambda=1}^{2^q}\alpha_\lambda K_i(\theta_\lambda)\right]\right)\\ &+ \left[\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{C}_i(\theta_\lambda)\right]^T\left[\sum_{\lambda=1}^{2^q}\alpha_\lambda\mathcal{C}_i(\theta_\lambda)\right]\\ &+ \gamma^{-2}P_i\left[\sum_{\lambda=1}^{2^q}\beta_\lambda\mathcal{D}_i(\theta_\lambda)\right]\left[\sum_{\lambda=1}^{2^q}\beta_\lambda\mathcal{D}_i(\theta_\lambda)\right]^T P_i + E_i^T U_i E_i < 0, \quad x \in \mathcal{R}_i,\end{aligned} \quad (32)$$

where $\{\theta_\lambda\}_{\lambda=1}^{2^q} \in \overline{\Omega}_\theta$ denotes the vertexes of Ω_θ, $\{\alpha_\lambda, \beta_\lambda\}_{\lambda=1}^{2^q}$ denotes the convex hull coefficients, which satisfies $0 \le \alpha_\lambda \le 1$, $\sum_{\lambda=1}^{2^q}\alpha_\lambda = 1$ and $0 \le \beta_\lambda \le 1$, $\sum_{\lambda=1}^{2^q}\beta_\lambda = 1$. Note that $0 \le \alpha_\lambda, \beta_\lambda \le 1$, thus a sufficient condition for inequality (32) to hold is

$$\begin{aligned}&\left(\mathcal{A}_i\left(\hat{\theta}'\right) + \mathcal{B}_i\left(\hat{\theta}'\right)K_i\left(\hat{\theta}''\right)\right)^T P_i + P_i\left(\mathcal{A}_i\left(\hat{\theta}'\right) + \mathcal{B}_i\left(\hat{\theta}'\right)K_i\left(\hat{\theta}''\right)\right)\\ &+ \mathcal{C}_i^T\left(\hat{\theta}'\right)\mathcal{C}_i\left(\hat{\theta}''\right) + \gamma^{-2}P_i\mathcal{D}_i(\theta')\mathcal{D}_i^T(\theta'')P_i + E_i^T U_i E_i < 0,\end{aligned} \quad (33)$$

for all $\hat{\theta}', \hat{\theta}'', \theta', \theta'' \in \overline{\Omega}_\theta$.

Let $Q_i = P_i^{-1}$, $\mathcal{A}_i^{cl} = \mathcal{A}_i\left(\hat{\theta}'\right) + \mathcal{B}_i\left(\hat{\theta}'\right)K_i\left(\hat{\theta}''\right)$ $S_i = C_i^T\left(\hat{\theta}'\right)C_i\left(\hat{\theta}''\right) + \gamma^{-2}P_i\mathcal{D}_i(\theta')$ $\mathcal{D}_i^T(\theta'')P_i + E_i^T U_i E_i$, and then the sufficient condition (33) can be rewritten as

$$Q_i\left(\mathcal{A}_i^{cl}\right)^T + \left(\mathcal{A}_i^{cl}\right)Q_i + Q_i S_i Q_i < 0. \quad (34)$$

The use of Lemma 4.1 with $\Psi_i = Q_i S_i Q_i$ and $\Xi_i = Q_i\left(\mathcal{A}_i^{cl}\right)^T$ yields,

$$\begin{bmatrix} Q_i S_i Q_i + \Phi_i - \left(H_i + H_i^T\right) & \mathcal{A}_i^{cl}Q_i + H_i^T \\ * & -\Phi_i \end{bmatrix} < 0. \quad (35)$$

Making congruence transformation $\begin{bmatrix} V_i & 0 \\ * & P_i \end{bmatrix}$ with $V_i = H_i^{-1}$, the inequality (35) becomes

$$\begin{bmatrix} V_i^T\left(P_i^{-1}S_iP_i^{-1} + \Phi_i\right)V_i - \left(V_i + V_i^T\right) & V_i^T\mathcal{A}_i^{cl} + P_i \\ * & -P_i\Phi_iP_i \end{bmatrix} < 0. \quad (36)$$

By applying Schur complement argument [19] on $V_i^T\left(P_i^{-1}S_iP_i^{-1}+\Phi_i\right)V_i$, we know that the proposed sufficient condition is well equivalent to the next inequality

$$\begin{bmatrix} -\left(V_i+V_i^T\right) & V_i^T\mathcal{A}_i^{cl}+P_i & V_i^T \\ * & -P_i\Phi_iP_i & 0 \\ * & * & -(P_i^{-1}S_iP_i^{-1}+\Phi_i)^{-1} \end{bmatrix}<0. \tag{37}$$

Exploring Lemma 4.2 with $\Phi_i=P_i^{-1}$ get

$$\left(P_i^{-1}S_iP_i^{-1}+\Phi_i\right)^{-1}=P_i(S_i+P_i)^{-1}P_i\geq P_i-S_i. \tag{38}$$

This inequality illustrates that the following condition can imply the inequality (37),

$$\begin{bmatrix} -\left(V_i+V_i^T\right) & V_i^T\mathcal{A}_i^{cl}+P_i & V_i^T \\ * & -P_i & 0 \\ * & * & \Upsilon_i \end{bmatrix}<0, \tag{39}$$

where

$$\Upsilon_i=S_i-P_i=\mathcal{C}_i^T\left(\hat{\theta}'\right)\mathcal{C}_i\left(\hat{\theta}''\right)+\gamma^{-2}P_i\mathcal{D}_i(\theta')\mathcal{D}_i^T(\theta'')P_i+E_i^TU_iE_i-P_i. \tag{40}$$

The dual of (39) (replacing $\mathcal{A}_i^{cl}$ by $(\mathcal{A}_i^{cl})^T$) is

$$\begin{bmatrix} -\left(V_i+V_i^T\right) & V_i^T\left(\mathcal{A}_i^{cl}\right)^T+P_i & V_i^T \\ * & -P_i & 0 \\ * & * & \Omega_i \end{bmatrix}<0, \tag{41}$$

where

$$\Omega_i=\mathcal{D}\left(\theta'\right)\mathcal{D}^T\left(\theta''\right)+\gamma^{-2}P_i\mathcal{C}_i^T\left(\hat{\theta}'\right)\mathcal{C}_i\left(\hat{\theta}''\right)P_i+E_i^TU_iE_i-P_i. \tag{42}$$

Substituting $\mathcal{A}_i^{cl}=\mathcal{A}_{i\alpha}+\mathcal{B}K_i$, $R_i=K_iV_i$ into (41), then twice Schur complement arguments on the term $\mathcal{D}(\theta')\mathcal{D}^T(\theta'')+\gamma^{-2}P_i\mathcal{C}_i^T\left(\hat{\theta}'\right)C_i\left(\hat{\theta}''\right)P_i$ shows that the condition (29) implies the inequality (11). In other words, the condition (11) of Theorem 3.1 is satisfied by piecewise quadratic Lyapunov function (27) and piecewise controller (28).

Remark 4.1. *Incorporating Theorem 3.1 and Theorem 4.1 together, the adaptive Hinf synthesis framework is presented, the piecewise affine controllers and related adaptive laws can be synthesized by solving LMIs (29).*

Remark 4.2. *From synthesis condition (33) we can find that, if employing common quadratic Lyapunov fucntion (CQLF) approach and neglecting S-procedure, this synthesis condition can be easily formulated as LMIs by multiplying the inversion of the Lyapunov matrix and applying Schur complement argument. If employing PQLF approach to achieve the less conservatism, the proposed easy approach cannot work, which brings some difficulties when formulating the synthesis condition as LMIs, which is actually the main contribution of Theorem 4.1.*

5. **Air-Breathing Hypersonic Vehicle Control.** Air-breathing hypersonic vehicles may eventually allow dramatic reductions in flight times for both commercial and military applications. Direct access to Earth orbit without the use of separate boosting stages may also become possible as scramjet powered aircraft enter service. Although numerous challenges remain, past successes with X-43 and X51 renewed research activities throughout the aerospace community suggest that this technology may be on its way to assuming a role in the next generation of aviation. The design of guidance and control systems for

air-breathing hypersonic vehicles requires the control engineer to deal with strong couplings between propulsive and aerodynamic effects while also addressing the significant flexibility associated with the slender geometries required for these aircraft.

Consider the dynamic model of the air-breathing hypersonic vehicles [20] described in the following.

$$\begin{aligned}
\dot{h} &= V\sin(\theta) \\
\dot{V} &= \frac{T\cos(\alpha) - D - mg\sin(\theta)}{m} \\
\dot{\alpha} &= \frac{-T\sin(\alpha) - L + mg\cos(\theta)}{mV} + \omega_z \\
\dot{\omega_z} &= \frac{M_z}{J} + \omega_d
\end{aligned}$$

where h, V, α and ω_z denote the altitude, velocity, attack angle and pitch angular rate, respectively. δ is the elevator angular deflection, and ω_d is the external disturbance.

$$\begin{aligned}
&L = C_L(\alpha,\delta)S\frac{1}{2}\rho V^2; \quad D = C_D(\alpha,\delta)S\frac{1}{2}\rho V^2 \\
&M_z = z_T T + \left(C_{M,\alpha}\alpha + \lambda_t C_{M,\delta}\delta\right) Sc\frac{1}{2}\rho V^2 \\
&T = C_T^{\alpha^3}\alpha^3 + C_T^{\alpha^2}\alpha^2 + C_T^{\alpha}\alpha + C_T^0
\end{aligned}$$

and

$$\begin{aligned}
&C_L(\alpha,\delta) = C_L^{\alpha}\alpha + C_L^{\delta}\delta + C_L^0 \\
&C_D(\alpha,\delta) = C_D^{\alpha^2}\alpha^2 + C_D^{\alpha}\alpha + C_D^{\delta^2}\delta^2 + C_D^{\delta}\delta + C_D^0 \\
&C_{M,\alpha} = C_{M,\alpha}^{\alpha^2}\alpha^2 + C_{M,\alpha}^{\alpha}\alpha + C_{M,\alpha}^0, \quad C_{M,\delta} = C_e\delta \\
&C_T^{\alpha^3} = \beta_1\Phi + \beta_2, \quad C_T^{\alpha^2} = \beta_3\Phi + \beta_4 \\
&C_T^{\alpha} = \beta_5\Phi + \beta_6, \quad C_T^0 = \beta_7\Phi + \beta_8
\end{aligned}$$

The detail values of the aerodynamic coefficient (lift and moment coefficients) and thrust coefficient employed in the above mathematical description can be found in [20]. Moreover, the following table illustrates all the parameter values and physical meanings, where $\theta \in [-0.1, 0.1]$ denotes the unknown parameter. From Table 1, the unknown parameter θ will affect the tail rotor torque coefficient directly, which reflects the control efficiency of the hypersonic vehicle. With the larger amplitude of θ, the parametric uncertainty is larger, the control synthesis conservatism will be enlarged, and the control performance will be reduced. It is worth pointing out that, all the parameter values in Table 1 are set the same with the research work [20], specially the parameter uncertainty of the tail rotor torque coefficient is always chosen within 10%, respecting to $\theta \in [-0.1, 0.1]$.

The synthesis problem is to seek the feedback control law satisfying the gain constraint $\|K\|_\infty \le 10$ that forces the hypersonic vehicle to reach the required attack angle, and minimize the impact of disturbance ω_d on the output vector α.

Given the possible initial angle $\alpha_0 \in [-6^\circ, 6^\circ]$, the nonlinear function in the attitude loop can be approximated by piecewise affine function yielding the PLS with three polytopic regions to approximately describe the original nonlinear system, i.e.,

$$\begin{cases}
\mathcal{R}_1 = \{\alpha | \alpha \in [-6^\circ, -2^\circ]\}, \\
\mathcal{R}_2 = \{\alpha | \alpha \in [-2^\circ, 2^\circ]\}, \\
\mathcal{R}_3 = \{\alpha | \alpha \in [2^\circ, 6^\circ]\},
\end{cases}$$

TABLE 1. Simulation parameters

$\lambda_t = 1 + \theta$	Tail rotor torque coefficient	effect on the control efficiency
$m = 30$	vehicle mass (t)	effect on the variation of Velocity
$S = 158$	reference area (m^2)	effect on the lift and moment
$J = 45 \times 10^5$	moment of inertia ($\mathrm{kg{\cdot}m}^2$)	effect on the variation of ω_z
$h_0 = 25.9$	altitude (km)	flight envelope $20\mathrm{km} < h < 40\mathrm{km}$
$V_0 = 2348$	Velocity (m/s)	flight envelope Mach > 5
$Z_t = 2.54$	thrust to moment coupling coefficient (m)	additional pitch moment from engine
Φ=1	fuel-to-air ratio	effect on the engine thrust

Following Theorem 3.1 and Theorem 4.1, the piecewise control law and its related adaptive law

$$\begin{cases} u = \left(K_{i0} + \hat{\theta} K_{i1}\right) x \\ \dot{\hat{\theta}} = \mathrm{Proj}\left(x^T Q_i^{-1} [A_{i1} x + B_{i1} u]\right) \end{cases} \quad x \in \mathcal{R}_i$$

are employed, where K_{i0}, K_{i1} are solved by computing the LMIs proposed in Theorem 4.1.

Assuming the true value of parameter is $\theta = 0.1$ and the dynamic system is suffered from the external disturbances described by

$$w(t) = -0.1 \sin(4\pi t). \tag{43}$$

Using designed piecewise control law and adaptive law, we carry out the simulation experiments with initial value $x(0) = (-6°, 0)^T$ and $x(0) = (6°, 0)^T$, respectively. It

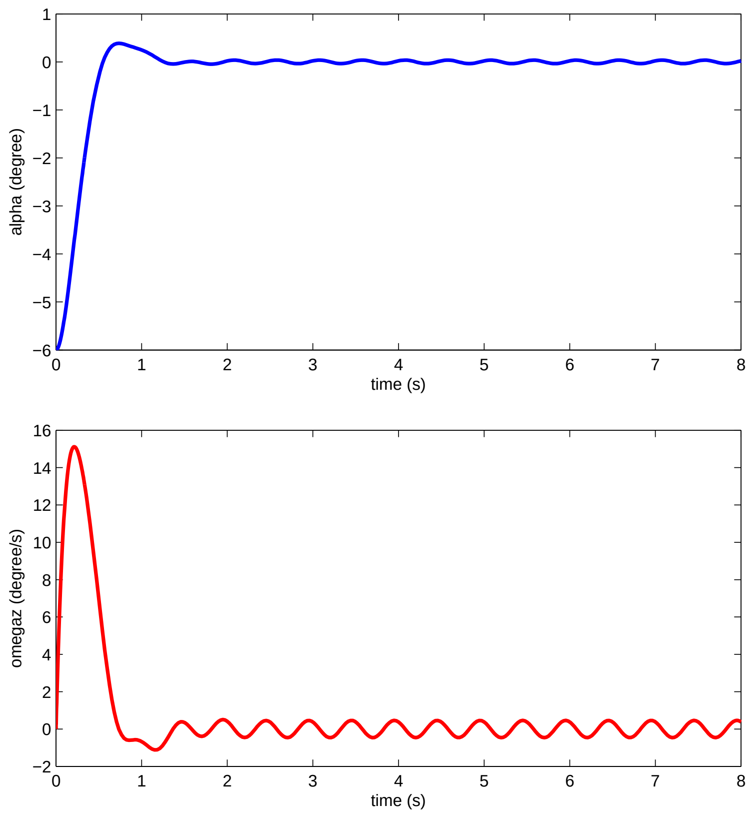

FIGURE 1. Case 1

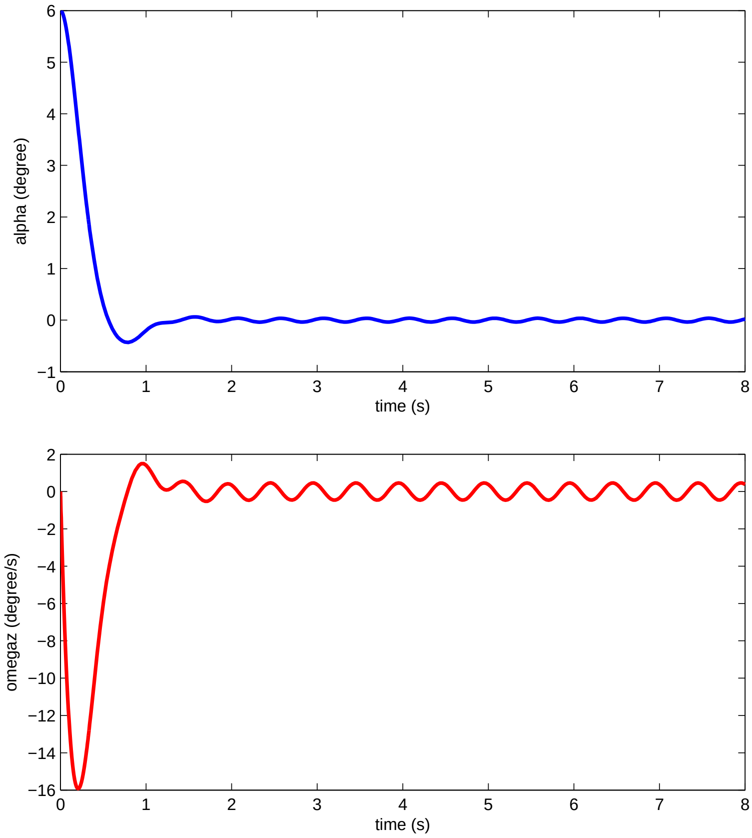

FIGURE 2. Case 2

TABLE 2. Comparison results

	PQLF approach	CQLF approach [13]	CQLF approach [14]
Hinf Performance	0.2730	0.6106	0.6956

can be observed in Figures 1 and 2 that, the system solution of closed-loop PLS is stable, which converges to origin as disturbance converges to zero, in other words the disturbance is attenuated. Moreover, for the tracking of attack angle, the overshoot can be controlled under 6% and the ascent time can be reduced within 0.4s, which illustrates the control performance of the proposed synthesis approach for the air-breathing hypersonic vehicles.

In addition, the control synthesis using the CQLF based adaptive Hinf control approach [13, 14] has been done for the same PLS. As illustrated in Table 2, the closed loop systems synthesized by the proposed PQLF approach has the minus Hinf norm value 0.2730, and both of the Hinf norm values obtained using [13, 14] are larger than the Hinf norm value using the PQLF approach proposed in this paper, which states the designed controller using PQLF approach can minimize the impact of disturbance ω_d on the output vector α better, this obviously illustrates the advantage of the proposed approach.

6. **Conclusions.** In this paper, an improved adaptive Hinf synthesis framework is presented for piecewise-affine systems, the common quadratic Lyapunov function based synthesis approach is extended to piecewise quadratic Lyapunov function by using reciprocal projection lemma to achieve less conservatism. The synthesis conditions are formulated as LMIs and hence can be solved efficiently, and simulation results well illustrate the efficacy and advantage of the improved adaptive Hinf synthesis approach.

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