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ROBUST FAULT-TOLERANT CONTROL FOR FUZZY DELAY SYSTEMS WITH UNMEASURABLE PREMISE VARIABLES VIA UNCERTAIN SYSTEM APPROACH

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Abstract. *This paper addresses the problem of fault estimation (FE) and fault-tolerant control (FTC) for a class of fuzzy systems with unmeasurable premise variables, random time delays, actuator faults and external disturbances, simultaneously. Firstly, by using improved delay partitioning approach, a robust adaptive FE observer under H_∞ constraint is constructed. Then, based on the online estimation information, a novel fuzzy dynamic output feedback controller with unmeasurable premise variables is designed to compensate for the impact of actuator faults by using uncertain system approach, while guaranteeing that the closed-loop system is stable with the prescribed H_∞ performance. Compared with the existing results, the proposed design scheme is with less conservative and a wilder application range. Finally, the simulation results show the effectiveness of the proposed approach.*
Keywords: Fault estimation, Fault-tolerant control, Random time delays, Asymptotically stable, Unmeasurable premise variables, Linear matrix inequalities (LMIs)

1. Introduction.

It is well known that most physical systems and processes are nonlinear, and the Takagi-Sugeno (T-S) fuzzy models have been extensively used to describe nonlinear systems over the past decade [1]. Because it provides a general framework to represent a nonlinear plant by using a set of local linear models which are smoothly connected through nonlinear fuzzy membership functions, many issues related to the stability analysis and controller design of T-S fuzzy systems have been reported in [2, 3, 4, 5] and references therein.

Recently, due to an increasing demand for higher performances, as well as for higher safety and reliability, fault estimation (FE) and fault-tolerant control (FTC) have been an active field of research over the past decades. Under the T-S fuzzy system framework, lots of research into FE and FTC have been carried out and various methods have been proposed. Since measuring all of the internal states of physical systems may be difficult and costly, and only their outputs are available for control purpose, output feedback fuzzy controllers are preferred. In output feedback control design scheme, static output feedback [6, 7], dynamic output feedback [8, 9, 10, 11, 12] and observer-based feedback control [13, 14, 15, 16, 17, 18] have been employed in much literature. However, in many controller design methods reported in the existing literature, it is always assumed that the premise variables of fuzzy rules are measurable [19, 20, 21, 22, 23, 24], allowing to select the premise variables the same as those of T-S fuzzy observer and controller gains independently. However, due to the fact that the premise variables generally depend on the unmeasurable state variables and the separation principle does not hold in this case, such approaches are not valid and the T-S output feedback design is much more complex. In order to have a real output feedback design, one must consider the case that the premise variables of the fuzzy observer or controller depend on the estimated state variables by the fuzzy observer. More recently, the control design schemes for the unmeasurable premise variables case have been addressed in some papers [25, 26, 27, 28, 29]. Especially, in [25], the problem of fault-tolerant tracking control was studied for the vehicle dynamics represented by an uncertain T-S model with unmeasurable premise variables. A robust unknown input observer for the joint state and fault estimation in discrete-time T-S fuzzy systems with unmeasurable premise variables is presented in [28]. Using descriptor approach, [29] discusses the fault-tolerant trajectory tracking control for T-S fuzzy systems with unmeasurable premise variables, and a fuzzy observer is designed to estimate the system states and the sensor faults. To the best of our knowledge, the design of dynamic output feedback fault-tolerant controller with unmeasurable premise variables has not been fully investigated. Thus, it is necessary to deal with the FE and FTC for T-S fuzzy system with unmeasurable premise variables, which are very common in practical engineering.

On the other hand, it is generally known that time delay is often one of the main sources of poor performance or oscillations for systems. As a result, there are some recent results of FTC for fuzzy system with time delays, see, for example [20, 21, 22, 23, 24, 30]. More recently, in [21], a fuzzy descriptor learning observer is constructed to achieve simultaneous reconstruction of system states and actuator faults for T-S fuzzy descriptor systems with time delays. Based on the $(k-1)$th fault estimation information, a k-step fault estimation observer is proposed to estimate the actuator fault of time delay T-S fuzzy systems in [22]. In [23], the adaptive fault estimation problem is studied for a class of T-S fuzzy stochastic Markovian jumping systems with time delays and nonlinear parameters. Based on a multiple Lyapunov function and the slack variables, in [30], the fault tolerant saturated control problem for discrete-time T-S fuzzy systems with delay is studied.

Although the influence of delay has been taken into account in the controller design process, it should be pointed out that time delay in a real system often exists in a stochastic way [31], and its probabilistic characteristic can be calculated by statistical methods, so the random delay on T-S fuzzy systems cannot be ignored. Moreover, the type of time delay considered in all the aforementioned works is constant $\tau(t) = \tau$ or $0 < \tau(t) < \tau$, and the lower bound of delay is restricted to 0, which is not more general. The interval time-varying delay, $0 < \tau_1 \leq \tau(t) \leq \tau_2$, has been identified from many practical systems, especially the networked control systems. Based on the above discussion, solving the problem of FE and FTC for fuzzy systems with unmeasurable premise variables, actuator faults, random interval time-varying delays and external disturbances simultaneously is a meaningful research and motivates our study.

The aim of this paper is to develop an FTC design scheme for a class of T-S fuzzy systems with unmeasurable premise variables subject to random time delays and external disturbances. The problem is a complicated one as it requires using the estimated premise variables in the structure of the observer and controller to have a practical design. The main contribution of this paper lies in the following aspects. (1) By using the improved delay partitioning approach, a robust adaptive fuzzy fault estimation observer under the unmeasurable premise variables and H_∞ performance constraint is constructed to achieve the estimation of actuator faults, and the less conservative sufficient conditions for the existence of observer are explicitly provided. (2) A fuzzy dynamic output feedback fault-tolerant controller with unmeasurable premise variables and random time delay is designed by using uncertain system approach, which guarantee the closed-loop system is asymptotically stable with the prescribed H_∞ performance. Finally, simulation examples demonstrate the effectiveness of the proposed approaches.

The rest of this paper is organized as follows. The system description and problem formulations are presented in Section 2. Sections 3 and 4 present the main results on robust fault estimation observer and fault-tolerant controller design scheme. In Section 5, simulation results of numerical example are presented to demonstrate the effectiveness and merits of the proposed methods. Finally, Section 6 concludes the paper.

Notations: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space; I denotes the identity matrix; the superscripts "T" and "-1" stand for the matrix transpose and inverse, respectively; notation $X > 0$ ($X \geq 0$) means that matrix X is real symmetric positive definite (positive semi-definite); $\|\cdot\|$ is the spectral norm. If not explicitly stated, all matrices are assumed to have compatible dimensions for algebraic operations. The symbol "$*$" stands for matrix block induced by symmetry. For any square matrix M, $\mathrm{Sym}(M)$ is defined by $\mathrm{Sym}(M) = M + M^T$.

2. **Problem Statement and Preliminaries.**

In this paper, we consider a nonlinear system which can be represented by the following extended T-S fuzzy time-delay model with exogenous disturbance and actuator faults simultaneously. The ith rule of the T-S fuzzy model on a compact region $\mathbb{D}$ is given below:

IF $\xi_1(t)$ is M_{i1} and $\ldots$ and $\xi_p(t)$ is M_{ip}, **THEN**

$$\begin{cases} \dot{x}_i(t) = A_i x(t) + A_{\tau i} x(t-\tau(t)) + B(u(t) + f(t)) + B_{di} d(t) \\ y_i(t) = C_i x(t) + C_{\tau i} x(t-\tau(t)) + D_{di} d(t) \\ x_i(t) = \phi_i(t), \ \forall t \in [-\tau_2, 0], \ i = 1, 2, \ldots, r \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ are the states of the local model and nonlinear model, respectively. $u(t) \in \mathbb{R}^q$ denotes the input vector, and $y(t) \in \mathbb{R}^l$ stands for output vector. $d(t) \in \mathbb{R}^m$ is the exogenous disturbance input that belongs to $L_2[0, \infty)$, and $f(t) \in \mathbb{R}^q$ represents the possible actuator fault. M_{ij} ($i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, p$) are fuzzy sets, and $\xi(t) = [\xi_1(t), \ldots, \xi_p(t)]^T$ is the premise variable vector that does not depend on the input variables $u(t)$. A_i, $A_{\tau i}$, B_{di}, C_i, $C_{\tau i}$, D_{di} and B are constant real matrices of appropriate dimensions. It is assumed that the pairs (A_i, B) are controllable, and the pairs (A_i, C_i) are observable, where $i = 1, 2, \ldots, r$. The time delay $\tau(t)$ is assumed to be a random one and satisfies

$$0 < \tau_1 \leq \tau(t) \leq \tau_2 \tag{2}$$

where τ_1 and τ_2 are lower and upper bounds of state delay $\tau(t)$, respectively. $\phi_i(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau_2, 0]$.

It is noted that, in practice, some values of the delay are very large but the probabilities of the delays taking such large values are very small. Taking this point into consideration, in order to describe the probability distribution of the random time delays, we define the following two mapping functions:

$$\tau_1(t) = \begin{cases} \tau(t), & t \in \mathcal{D}_1 \\ \bar{\tau}_1, & t \in \mathcal{D}_2 \end{cases} \quad \tau_2(t) = \begin{cases} \tau(t), & t \in \mathcal{D}_2 \\ \bar{\tau}_\rho, & t \in \mathcal{D}_1 \end{cases} \tag{3}$$

where $\mathcal{D}_1 = \{t | \tau(t) \in [\tau_1, \tau_\rho]\}$, $\mathcal{D}_2 = \{t | \tau(t) \in [\tau_\rho, \tau_2]\}$, $\bar{\tau}_1 = [\tau_1, \tau_\rho]$ and $\bar{\tau}_\rho = [\tau_\rho, \tau_2]$. The scalar τ_ρ satisfies $\tau_1 \leq \tau_\rho \leq \tau_2$, where $\tau_\rho = \tau_1 + \rho\delta$, $0 < \rho < 1$, $\delta = \tau_2 - \tau_1$. And a Bernoulli distributed stochastic variable is defined as

$$\delta(t) = \begin{cases} 1, & t \in \mathcal{D}_1 \\ 0, & t \in \mathcal{D}_2 \end{cases} \tag{4}$$

with $\text{Prob}\{\delta(t) = 1\} = \text{Prob}\{\tau(t) \in [\tau_1, \tau_\rho]\} = \mathbb{E}[\delta(t)] = \delta_0$ and $\text{Prob}\{\delta(t) = 0\} = \text{Prob}\{\tau(t) \in [\tau_\rho, \tau_2]\} = \mathbb{E}[\delta(t)] = 1 - \delta_0$. Here, it is also assumed that delays $\tau_1(t)$ and $\tau_2(t)$ satisfy the condition

$$\tau_1 \le \tau_1(t) \le \tau_\rho, \quad \dot{\tau}_1(t) \le d_1, \quad \tau_\rho \le \tau_2(t) \le \tau_2, \quad \dot{\tau}_2(t) \le d_2 \tag{5}$$

where d_1, d_2 are positive constants. It is easy to check that $t \in \mathcal{D}_1$ implies the event $\tau(t) \in [\tau_1, \tau_\rho]$ occurs and $t \in \mathcal{D}_2$ implies the event $\tau(t) \in [\tau_\rho, \tau_2]$ occurs.

Then, by fuzzy blending of each individual plant rule, the global fuzzy system with random time delay can be inferred as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\dot{x}_i(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + \delta(t)A_{\tau i}x(t-\tau_1(t)) \\ \qquad + (1-\delta(t))A_{\tau i}x(t-\tau_2(t)) + B(u(t)+f(t)) + B_{di}d(t)\} \\ y(t) = \sum_{i=1}^{r} \mu_i(\xi(t))y_i(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{C_i x(t) + \delta(t)C_{\tau i}x(t-\tau_1(t)) \\ \qquad + (1-\delta(t))C_{\tau i}x(t-\tau_2(t)) + D_{di}d(t)\} \\ x(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\phi_i(t), \ t \in [-\tau_2, 0] \end{cases} \tag{6}$$

where fuzzy basis functions are given by $\mu_i(\xi(t)) = \beta_i(\xi(t))/\sum_{i=1}^{r}\beta_i(\xi(t))$, $\beta_i(\xi(t)) = \prod_{i=1}^{p} M_{ij}(\xi_j(t))$, and $M_{ij}(\xi_j(t))$ represents the grade of membership of $\xi_j(t)$ in M_{ij}. Here, we assume that $\beta_i(\xi(t)) \ge 0$, $i = 1, \ldots, r$, $\sum_{i=1}^{r}\beta_i(\xi(t)) > 0$ for any $\xi(t)$. Hence, $\mu_i(\xi(t))$ satisfies $\mu_i(\xi(t)) \ge 0$, $i = 1, \ldots, r$, $\sum_{i=1}^{r}\mu_i(\xi(t)) = 1$ for any $\xi(t)$.

It is well known that when state is selected as the premise variable, the fuzzy system can describe a broader range of systems [27]. Therefore, in this paper, we consider fuzzy time delay system with the premise variable depending on the state, which is unmeasurable. For the sake of notation, the following definitions are used:

$$X_\mu = \sum_{i=1}^{r}\mu_i(x(t))X_i, \ X_{\hat{\mu}} = \sum_{i=1}^{r}\mu_i\left(\hat{x}(t)\right)X_i, \ X_{\mu\hat{\mu}} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(x(t))\mu_j\left(\hat{x}(t)\right)X_{ij} \tag{7}$$

Then, the fuzzy system (6) becomes

$$\begin{cases} \dot{x}(t) = A_\mu x(t) + \delta(t)A_{\tau\mu}x(t-\tau_1(t)) + (1-\delta(t))A_{\tau\mu}x(t-\tau_2(t)) + B_{d\mu}d(t) \\ \qquad + B(u(t)+f(t)) \\ y(t) = C_\mu x(t) + \delta(t)C_{\tau\mu}x(t-\tau_1(t)) + (1-\delta(t))C_{\tau\mu}x(t-\tau_2(t)) + D_{d\mu}d(t) \\ x(t) = \phi_\mu, \ t \in [-\tau_2, 0] \end{cases} \tag{8}$$

Before proceeding further, we will introduce some lemmas to be needed in the development of main results through this paper.

Lemma 2.1. [5] *Let Ω, Γ, Σ be matrices with appropriate dimensions, and Ω is a symmetrical matrix, then for every matrix F with $F^TF \le I$, we have $\Omega + \Gamma F\Sigma + (\Gamma F\Sigma)^T \le 0$ if and only if there exists a constant $\varepsilon > 0$ such that:*

$$\Omega + \varepsilon\Gamma\Gamma^T + \varepsilon^{-1}\Sigma^T\Sigma \le 0$$

Lemma 2.2. [32] *For any constant matrix* $X \in \mathbb{R}^{n\times n}$, $X = X^T > 0$, *scalar* $r > 0$, *and vector function* $\dot{x} : [-r, 0] \to \mathbb{R}^n$ *such that the following integration is well defined, then*

$$-r\int_{-r}^{0} \dot{x}^T(t+s)X\dot{x}(t+s)ds \le \begin{bmatrix} x^T(t) & x^T(t-r)\end{bmatrix}\begin{bmatrix} -X & X \\ X & -X\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-r)\end{bmatrix}$$

Lemma 2.3. [33] *For any positive semi-definite matrices* $X = (X_{ij})_{3\times 3} \ge 0$, *the following integral inequality holds:*

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)ds$$
$$\le \int_{t-\tau(t)}^{t} \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & \dot{x}^T(s)\end{bmatrix}\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ \dot{x}(s)\end{bmatrix} ds$$

Remark 2.1. *A more general fuzzy system with unmeasurable premise variables is considered in this paper, including possible random state time delay, actuator fault and exogenous disturbance input simultaneously. If there is no state delay, i.e.,* $\tau_1(t) = \tau_2(t) = 0$, *then system* (6) *reduces to the existing one in* [9]. *Further, if* $\tau_1(t) = \tau_2(t) \ne 0$, (6) *can be transformed to* [22]. *Moreover, different from* [22], *the lower bound of delay is not restricted to zero, which is even more applicable to networked control systems and other practical systems.*

3. **Adaptive Fault Estimation Observer Design.**

In this section, in order to estimate the system faults, the following adaptive fault estimation observer is constructed, in which the premise variable $\xi(t)$ is the estimate of the state.

$$\begin{cases} \dot{\hat{x}}(t) = A_{\hat{\mu}}\hat{x}(t) + \delta(t)A_{\tau\hat{\mu}}\hat{x}(t-\tau_1(t)) + (1-\delta(t))A_{\tau\hat{\mu}}\hat{x}(t-\tau_2(t)) \\ \qquad\quad + B\left(u(t)+\hat{f}(t)\right) + L_{\hat{\mu}}(y(t)-\hat{y}(t)) \\ \hat{y}(t) = C_{\hat{\mu}}\hat{x}(t) + \delta(t)C_{\tau\hat{\mu}}\hat{x}(t-\tau_1(t)) + (1-\delta(t))C_{\tau\hat{\mu}}\hat{x}(t-\tau_2(t)) \\ \hat{f}(t) = \Gamma\displaystyle\int_{t_f}^{t} F_{\hat{\mu}}(y(s)-\hat{y}(s))ds \end{cases} \tag{9}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state, $\hat{y}(t) \in \mathbb{R}^l$ is the output, $\hat{f}(t) \in \mathbb{R}^q$ is the estimate of fault $f(t)$, symmetric positive definite matrix $\Gamma \in \mathbb{R}^{q\times q}$ is the learning rate and t_f denotes the instant when fault occurs. The matrixes $L_{\hat{\mu}} \in \mathbb{R}^{n\times l}$, $F_{\hat{\mu}} \in \mathbb{R}^{q\times l}$ are the gain matrices based on the notation given in (7) with L_i and F_i, which are the appropriate dimensions matrices to be determined by the designer to achieve the fault estimation objectives.

Then, let us define $e_x(t) = \hat{x}(t) - x(t)$, $e_f(t) = \hat{f}(t) - f(t)$, $e_y(t) = \hat{y}(t) - y(t)$, and by adding and subtracting the term $\sum_{i=1}^{r}\mu_i(\hat{x}(t))\dot{x}_i(t)$ and $\sum_{i=1}^{r}\mu_i(\hat{x}(t))y_i(t)$ to the state and output residual dynamical from (8) and (9), the error dynamic of state and output can be obtained as follows:

$$\begin{cases} \dot{e}_x(t) = \left(A_{\hat{\mu}} - L_{\hat{\mu}}C_{\hat{\mu}}\right)e_x(t) + \delta(t)\left(A_{\tau\hat{\mu}} - L_{\hat{\mu}}C_{\tau\hat{\mu}}\right)e_x(t-\tau_1(t)) \\ \qquad\quad + (1-\delta(t))\left(A_{\tau\hat{\mu}} - L_{\hat{\mu}}C_{\tau\hat{\mu}}\right)e_x(t-\tau_2(t)) + Be_f(t) \\ \qquad\quad + \left(B_{d\mu} - L_{\hat{\mu}}D_{d\mu}\right)d(t) + \delta_1(\hat{x},x,t) - L_{\hat{\mu}}\delta_2(\hat{x},x,t) \\ e_y(t) = C_{\hat{\mu}}e_x(t) + \delta(t)C_{\tau\hat{\mu}}e_x(t-\tau_1(t)) + D_{d\mu}d(t) + \delta_2(\hat{x},x,t) \\ \qquad\quad + (1-\delta(t))C_{\tau\hat{\mu}}e_x(t-\tau_2(t)) \end{cases} \tag{10}$$

where $\delta_1(\hat{x}, x, t) = f_1(\hat{x}, x, t) - f_1(x, x, t), \delta_2(\hat{x}, x, t) = f_2(\hat{x}, x, t) - f_2(x, x, t)$ with

$$f_1(\hat{x}, x, t) = A_{\hat{\mu}}x(t) + \delta(t)A_{\tau\hat{\mu}}x(t - \tau_1(t)) + (1 - \delta(t))A_{\tau\hat{\mu}}x(t - \tau_2(t)) \tag{11}$$

$$f_2(\hat{x}, x, t) = C_{\hat{\mu}}x(t) + \delta(t)C_{\tau\hat{\mu}}x(t - \tau_1(t)) + (1 - \delta(t))C_{\tau\hat{\mu}}x(t - \tau_2(t)) \tag{12}$$

Assumption 3.1. *The functions $f_1(\hat{x}, x, t)$ and $f_2(\hat{x}, x, t)$ in (11) and (12) are Lipschitz with respect to its first variable. Then, there exist positive scalars η_1, η_2, such that*

$$\delta_1^T(\hat{x}, x, t)\delta_1(\hat{x}, x, t) \leq \eta_1^2 e_x^T(t)e_x(t), \quad \delta_2^T(\hat{x}, x, t)\delta_2(\hat{x}, x, t) \leq \eta_2^2 e_x^T(t)e_x(t)$$

Remark 3.1. *It should be noted that this assumption is a mild condition when trying to design an observer for fuzzy systems with unmeasurable premise variables. According to* [34, 35]*, this condition is satisfied if $\mu_i(t)$ is differentiable w.r.t $x(t)$ almost everywhere and has a bounded first derivative for almost all $x(t)$ that is satisfied by most membership functions in practice.*

Based on this transformation, the H_∞ fault estimation observer design problem to be addressed in this paper can be formulated as follows: (i) The error dynamic system (10) with $d(t) = 0$ is asymptotically stable for any time-delay satisfying (2)-(5); (ii) For a given scalar γ, the following H_∞ performance is satisfied:

$$\mathbb{E}\left\{\int_0^\infty \|e_x(t)\|^2 dt\right\} \leq \gamma^2 \int_0^\infty \|d(t)\|^2 dt \tag{13}$$

for all $d(t) \in L_2[0, \infty)$ under zero initial conditions and the FE algorithm can realize $\lim_{t\to\infty} e_f(t) = 0$.

Theorem 3.1. *For the given scalars τ_1, τ_2, η, γ and $0 < \rho < 1$, the error dynamic system* (10) *is asymptotically stable with $d(t) = 0$ for any time-varying delay $\tau(t)$ defined in* (2)-(5)*, while satisfying a prescribed H_∞ performance* (13)*, if there exist matrices $P > 0$, $Q_n > 0$, $W_n > 0$, $(n = 1, 2, \ldots, N)$, $\hat{S}_0 > 0$, $S_1 > 0$, $\hat{S}_1 > 0$, $S_2 > 0$, $S_3 > 0$, $R_1 > 0$, $R_2 > 0$, $Y = (Y_{ij})_{3\times 3} \geq 0$, $Z = (Z_{ij})_{3\times 3} \geq 0$ and Y_i, F_i, $\lambda_1, \lambda_2, \lambda_3$ such that the following LMIs hold:*

$$\Pi_{iii} < 0 \qquad i = 1, 2, \ldots, r \tag{14}$$

$$\Pi_{ijk} + \Pi_{jki} + \Pi_{kij} \leq 0 \quad 1 \leq i \leq j < k \leq r \tag{15}$$

$$\Pi_{kji} + \Pi_{jik} + \Pi_{ikj} \leq 0 \quad 1 \leq i < j \leq k \leq r \tag{16}$$

$$R_1 - Y_{33} \geq 0, \quad R_2 - Z_{33} \geq 0 \tag{17}$$

where

$$\Pi_{ijk} = \begin{bmatrix} \Pi_{ijk}^1 & \Pi_{ijk}^2 & \bar{P} & \bar{Y}_i^T & 0 \\ * & \Pi_{ijk}^3 & 0 & 0 & \bar{F}_i \\ * & * & -\lambda_1 I & 0 & 0 \\ * & * & * & -\lambda_2 I & 0 \\ * & * & * & 0 & -\lambda_3 I \end{bmatrix}$$

with

$$\Pi_{ijk}^1 = \begin{bmatrix} \Pi_{11}^{ij} & W_1 & \cdots & 0 \\ * & \Pi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Pi_{nn} \end{bmatrix}, \Pi_{ijk}^2 = \begin{bmatrix} 0 & \Pi_{(1,N2)}^{ij} & 0 & \Pi_{(1,N4)}^{ij} & 0 & \Pi_{(1,N6)}^{ij} & \Pi_{(1,N7)}^{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi^3_{ijk} = \begin{bmatrix} \Pi_{(N1,1)} & \Pi_{(N1,2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Pi_{(N2,2)} & \Pi_{(N2,3)} & 0 & 0 & \Pi_{(N2,6)} & 0 & 0 \\ * & * & \Pi_{(N3,3)} & \Pi_{(N3,4)} & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{(N4,4)} & \Pi_{(N4,5)} & \Pi_{(N4,6)} & 0 & 0 \\ * & * & * & * & \Pi_{(N5,5)} & 0 & 0 & 0 \\ * & * & * & * & * & M & \Pi_{(N6,7)} & 0 \\ * & * & * & * & * & 0 & -\gamma^2 I & 0 \\ * & * & * & * & * & 0 & 0 & \Pi_n \end{bmatrix}$$

where

$$\begin{aligned} &\Pi^{ij}_{11} = \mathrm{sym}\left(PA_i - Y_iC_j\right) + Q_1 - W_1 + S_1 + \hat{S}_0 + \hat{S}_1 + \left(\lambda_1\eta_1^2 + (\lambda_2+\lambda_3)\,\eta_2^2\right) I \\ &\Pi^{ij}_{(1,N2)} = \delta_0\left(PA_{\tau i} - Y_iC_{\tau j}\right),\ \Pi^{ij}_{(1,N4)} = (1-\delta_0)(PA_{\tau i} - Y_iC_{\tau j}) \\ &\Pi^{ij}_{(1,N6)} = PB - C_j^TF_i,\ \Pi^{ij}_{(1,N7)} = PB_{dk} - Y_iD_{dk} \\ &\Pi_{nn} = -Q_{n-1} - W_{n-1} + Q_n - W_n,\ n = 1,2,\ldots,N-1 \end{aligned} \tag{18}$$

$$\begin{aligned} &\Pi_{(N1,1)} = -Q_N - W_N + S_2 + \rho\delta Y_{11} + Y_{13} + Y_{13}^T \\ &\Pi_{(N2,2)} = -(1-d_1)S_1 + \rho\delta Y_{22} - Y_{23} - Y_{23}^T + \rho\delta Y_{11} + Y_{13} + Y_{13}^T \\ &\Pi_{(N3,3)} = S_3 - S_2 + \rho\delta Y_{22} - Y_{23} - Y_{23}^T + (1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T \\ &\Pi_{(N4,4)} = -(1-d_2)\hat{S}_1 + (1-\rho)\delta(Z_{11} + Z_{22}) - Z_{23} - Z_{23}^T + Z_{13} + Z_{13}^T \\ &\Pi_{(N5,5)} = -\hat{S}_0 - S_3 + (1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T \\ &\Pi_{(N1,2)} = \Phi_{(N2,3)} = \rho\delta Y_{12} - Y_{13} + Y_{23}^T,\ \Pi_{(N2,6)} = -\delta_0 C_j^TF_i, \\ &\Pi_{(N3,4)} = \Phi_{(N4,5)} = (1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T,\ \Pi_{(N4,6)} = (\delta_0 - 1)C_j^TF_i \\ &\Pi_{(N6,7)} = -F_iD_{dk},\ \Pi_n = \sum_{n=1}^{N} h^2W_n + \rho\delta R_1 + (1-\rho)\delta R_2 \end{aligned} \tag{19}$$

$$\bar{P} = [P \quad 0 \quad \cdots \quad 0],\ \bar{Y}_i = [Y_i \quad 0 \quad \cdots \quad 0],\ \bar{F}_i = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad F_i \quad 0 \quad 0] \tag{20}$$

Then the fuzzy fault estimation algorithm in (9) *can realize* $e_x(t)$ *and* $e_f(t)$ *uniformly ultimate bounded, and the observer gain matrices can be obtained as follows:*

$$L_i = P^{-1}Y_i \tag{21}$$

Proof: For simplicity, we introduce the following vector, $\zeta_1^T(t) = \left[e_x^T(t) \quad e_n^T(t,h) \quad e_x^T(t-\tau_1(t)) \quad e_x^T(t-\tau_\rho) \quad e_x^T(t-\tau_2(t)) \quad e_x^T(t-\tau_2) \quad e_f(t) \quad d^T(t) \quad \dot{e}_x^T(t)\right]$, where $e_n^T(t,h) = \left[e_n^T(t-h) \quad e_n^T(t-2h) \quad \cdots \quad e_n^T(t-Nh)\right]$, $h = \tau_1/n$ $(n = 1,2,\ldots,N)$ is the length of each division, and N is the number (a positive integer) of divisions of the interval $[-\tau_1, 0]$. Then, the following novel Lyapunov-Krasovskii functional candidate is constructed to prove system (10) is asymptotically stable with H_∞ performance.

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{22}$$

where

$$\begin{aligned} V_1(t) &= e_x^T(t)Pe_x(t) \\ V_2(t) &= \sum_{n=1}^{N}\int_{t-nh}^{t-(n-1)h} e_x^T(s)Q_ne_x(s)ds + \int_{t-\tau_2}^{t} e_x^T(s)\hat{S}_0e_x(s)ds + \int_{t-\tau_1(t)}^{t} e_x^T(s)S_1e_x(s)ds \\ &\quad + \int_{t-\tau_2(t)}^{t} e_x^T(s)\hat{S}_1e_x(s)ds + \int_{t-\tau_\rho}^{t-\tau_1} e_x^T(s)S_2e_x(s)ds + \int_{t-\tau_2}^{t-\tau_\rho} e_x^T(s)S_3e_x(s)ds \end{aligned}$$

$$V_3(t) = \sum_{n=1}^{N} \int_{-nh}^{-(n-1)h} \int_{t+\theta}^{t} \dot{e}_x^T(s) hW_n \dot{e}_x(s) ds d\theta + \int_{-\tau_\rho}^{-\tau_1} \int_{t+\theta}^{t} \dot{e}_x^T(s) R_1 \dot{e}_x(s) ds d\theta + \int_{-\tau_2}^{-\tau_\rho} \int_{t+\theta}^{t} \dot{e}_x^T(s) R_2 \dot{e}_x(s) ds d\theta$$

$$V_4(t) = e_f^T(t)\Gamma^{-1} e_f(t)$$

where the unknown matrices $P > 0$, $S_1 > 0$, $\hat{S}_0 > 0$, $\hat{S}_1 > 0$, $S_2 > 0$, $S_3 > 0$, $R_1 > 0$, $R_2 > 0$, $Q_n > 0$ and $W_n > 0$ $(n = 1, 2, \ldots, N)$ are to be determined. Then, the time derivatives of $V(t)$ along the trajectories of the argument systems (10) satisfy

$$\begin{aligned}\mathbb{E}\Big\{\dot{V}_1(t)\Big\} = \mathbb{E}\Big\{& e_x^T(t)\left(P\left[A_{\hat{\mu}} - L_{\hat{\mu}}C_{\hat{\mu}}\right] + \left[A_{\hat{\mu}} - L_{\hat{\mu}}C_{\hat{\mu}}\right]^T P\right) e_x(t) \\ &+ 2\delta(t)e_x^T(t)P\left[A_{\tau\hat{\mu}} - L_{\hat{\mu}}C_{\tau\hat{\mu}}\right] e_x(t-\tau_1(t)) + 2e_x^T(t)P\left[B_{d\mu} - L_{\hat{\mu}}D_{d\mu}\right] d(t) \\ &+ 2(1-\delta(t))e_x^T(t)P\left[A_{\tau\hat{\mu}} - L_{\hat{\mu}}C_{\tau\hat{\mu}}\right] e_x(t-\tau_2(t)) + 2e_x^T PBe_f(t) \\ &+ 2e_x^T P\delta_1(\hat{x}, x, t) + 2e_x^T PL_i\delta_2(\hat{x}, x, t)\Big\}\end{aligned} \tag{23}$$

With Assumption 3.1 and since it is well known that the following inequality holds

$$X^TY + Y^TX \leq X^T\Sigma X + Y^T\Sigma^{-1}Y$$

for any matrices X, Y and $\Sigma = \Sigma^T > 0$, then, for any positive λ, it follows

$$2e_x^T P\delta_1(\hat{x}, x, t) \leq \lambda_1^{-1}e_x^T(t)PPe_x(t) + \lambda_1\eta_1^2 e_x^T(t)e_x(t) \tag{24}$$

$$2e_x^T PL_i\delta_2(\hat{x}, x, t) \leq \lambda_2^{-1}e_x^T(t)(PL_i)^T(PL_i)e_x(t) + \lambda_2\eta_2^2 e_x^T(t)e_x(t) \tag{25}$$

For $V_2(t)$ and $V_3(t)$, one can obtain that

$$\begin{aligned}\mathbb{E}\Big\{\dot{V}_2(t)\Big\} = \mathbb{E}\Bigg\{& \sum_{n=1}^{N} e_x^T(t-(n-1)h)Q_n e_x(t-(n-1)h) - \sum_{n=1}^{N} e_x^T(t-nh)Q_n e_x(t-nh) \\ &+ e_x^T(t)\hat{S}_0 e_x(t) - e_x^T(t-\tau_2)\hat{S}_0 e_x(t-\tau_2) \\ &+ e_x^T(t)S_1 e_x(t) - (1-\dot{\tau}_1(t))e_x^T(t-\tau_1(t))S_1 e_x(t-\tau_1(t)) \\ &+ e_x^T(t)\hat{S}_1 e_x(t) - (1-\dot{\tau}_2(t))e_x^T(t-\tau_2(t))\hat{S}_1 e_x(t-\tau_2(t)) \\ &+ e_x^T(t-\tau_1)S_2 e_x(t-\tau_1) - e_x^T(t-\tau_\rho)S_2 e_x(t-\tau_\rho) \\ &+ e_x^T(t-\tau_\rho)S_3 e_x(t-\tau_\rho) - e_x^T(t-\tau_2)S_3 e_x(t-\tau_2)\Bigg\}\end{aligned} \tag{26}$$

$$\begin{aligned}\mathbb{E}\Big\{\dot{V}_3(t)\Big\} = \mathbb{E}\Bigg\{& \dot{e}_x^T(t)\Lambda_n \dot{e}_x(t) - \sum_{n=1}^{N} \int_{t-nh}^{t-(n-1)h} \dot{e}_x^T(s)hW_n \dot{e}_x(s) ds \\ &- \int_{t-\tau_\rho}^{t-\tau_1} \dot{e}_x^T(s)(R_1 - Y_{33})\dot{e}_x(s) ds - \int_{t-\tau_\rho}^{t-\tau_1} \dot{e}_x^T(s)Y_{33}\dot{e}_x(s) ds \\ &- \int_{t-\tau_2}^{t-\tau_\rho} \dot{e}_x^T(s)(R_2 - Z_{33})\dot{e}_x(s) ds - \int_{t-\tau_2}^{t-\tau_\rho} \dot{e}_x^T(s)Z_{33}\dot{e}_x(s) ds\Bigg\}\end{aligned} \tag{27}$$

where $\Lambda_n = \sum_{n=1}^{N} h^2W_n + \rho\delta R_1 + (1-\rho)\delta R_2$. When $\tau_1 \leq \tau_1(t) \leq \tau_\rho$ and $\tau_\rho \leq \tau_2(t) \leq \tau_2$, the following equations are true:

$$-\int_{t-\tau_\rho}^{t-\tau_1} \dot{e}_n^T(s)Y_{33}\dot{e}_n(s) ds = -\int_{t-\tau_1(t)}^{t-\tau_1} \dot{e}_n^T(s)Y_{33}\dot{e}_n(s) ds - \int_{t-\tau_\rho}^{t-\tau_1(t)} \dot{e}_n^T(s)Y_{33}\dot{e}_n(s) ds \tag{28}$$

$$-\int_{t-\tau_2}^{t-\tau_\rho} \dot{e}_n^T(s)Z_{33}\dot{e}_n(s)ds = -\int_{t-\tau_2(t)}^{t-\tau_\rho} \dot{e}_n^T(s)Z_{33}\dot{e}_n(s)ds - \int_{t-\tau_2}^{t-\tau_2(t)} \dot{e}_n^T(s)Z_{33}\dot{e}_n(s)ds \quad (29)$$

By utilizing Lemma 2.3 and the Leibniz-Newton formula, we have

$$\begin{aligned} -\int_{t-\tau_\rho}^{t-\tau_1(t)} \dot{e}_n^T(s)Y_{33}\dot{e}_n(s)ds &\le \int_{t-\tau_\rho}^{t-\tau_1(t)} e_n^T(t,s) \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & 0 \end{bmatrix} e_n(t,s)ds \\ &\le e_n^T(t-\tau_1(t)) \left[\rho\delta Y_{11} + Y_{13} + Y_{13}^T\right] e_n(t-\tau_1(t)) \\ &\quad + e_n^T(t-\tau_\rho) \left[\rho\delta Y_{22} - Y_{23} - Y_{23}^T\right] e_n(t-\tau_\rho) \\ &\quad + 2e_n^T(t-\tau_1(t)) \left[\rho\delta Y_{12} - Y_{13} - Y_{13}^T\right] e_n(t-\tau_\rho) \end{aligned} \quad (30)$$

where $e_n^T(t,s) = \left[e_n^T(t-\tau_1(t)) \; e_n^T(t-\tau_\rho) \; \dot{e}_n^T(s)\right]$. Similarly, we obtain

$$\begin{aligned} -\int_{t-\tau_1(t)}^{t-\tau_1} \dot{e}_n^T(s)Y_{33}\dot{e}_n(s)ds &\le e_n^T(t-\tau_1) \left[\rho\delta Y_{11} + Y_{13} + Y_{13}^T\right] e_n(t-\tau_1) \\ &\quad + 2e_n^T(t-\tau_1) \left[\rho\delta Y_{12} - Y_{13} + Y_{23}^T\right] e_n(t-\tau_1(t)) \\ &\quad + e_n^T(t-\tau_1(t)) \left[\rho\delta Y_{22} - Y_{23} - Y_{23}^T\right] e_n(t-\tau_1(t)) \end{aligned} \quad (31)$$

$$\begin{aligned} -\int_{t-\tau_2(t)}^{t-\tau_\rho} \dot{e}_n^T(s)Z_{33}\dot{e}_n(s)ds &\le e_n^T(t-\tau_\rho) \left[(1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T\right] e_n(t-\tau_\rho) \\ &\quad + 2e_n^T(t-\tau_\rho) \left[(1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T\right] e_n(t-\tau_2(t)) \\ &\quad + e_n^T(t-\tau_2(t)) \left[(1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T\right] e_n(t-\tau_2(t)) \end{aligned} \quad (32)$$

$$\begin{aligned} -\int_{t-\tau_2}^{t-\tau_2(t)} \dot{e}_n^T(s)Z_{33}\dot{e}_n(s)ds &\le e_n^T(t-\tau_2(t)) \left[(1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T\right] e_n(t-\tau_2(t)) \\ &\quad + 2e_n^T(t-\tau_2(t)) \left[(1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T\right] e_n(t-\tau_2) \\ &\quad + e_n^T(t-\tau_2) \left[(1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T\right] e_n(t-\tau_2) \end{aligned} \quad (33)$$

Based on Assumption 3.1 and the derivative of $f(t)$ with respect to time is norm-bounded $\|\dot{f}(t)\| \le f_1$, it is easy to show that for a symmetric positive definite matrix M, it can obtain that

$$2e_f^T(-F_{\hat{\mu}})\delta_2(\hat{x},x,t) \le \lambda_3^{-1}e_f^T(t)F_{\hat{\mu}}^T F_{\hat{\mu}} e_f(t) + \lambda_3\eta_2^2 e_x^T(t)e_x(t) \quad (34)$$

and

$$\begin{aligned} 2e_f^T(t)\Gamma^{-1}\dot{f}(t) &\le e_f^T(t)Me_f(t) + \dot{f}^T(t)\Gamma^{-1}M^{-1}\Gamma^{-1}\dot{f}(t) \\ &\le e_f^T(t)Me_f(t) + f_1^2\lambda_{\max}\left(\Gamma^{-1}M^{-1}\Gamma^{-1}\right) = e_f^T(t)Me_f(t) + \delta \end{aligned} \quad (35)$$

Therefore,

$$\begin{aligned} \mathbb{E}\Big\{\dot{V}_4(t)\Big\} &= \mathbb{E}\Big\{2e_f^T(t)\Gamma^{-1}\dot{e}_f(t)\Big\} = \mathbb{E}\Big\{2e_f^T(t)\Gamma^{-1}\left(\dot{f}(t) - \dot{\hat{f}}(t)\right)\Big\} \\ &= \mathbb{E}\Big\{-2e_f^T(t)F_{\hat{\mu}}\big[C_{\hat{\mu}}e_x(t) + \delta(t)C_{\tau\hat{\mu}}e_x(t-\tau_1(t)) + (1-\delta(t))C_{\tau\hat{\mu}}e_x(t-\tau_2(t)) \\ &\quad + D_{d\mu}d(t) + \delta_2(\hat{x},x,t)\big] + 2e_f^T(t)\Gamma^{-1}\dot{f}(t)\Big\} \end{aligned} \quad (36)$$

In order to study the H_∞ performance of error dynamic system (10) and minimize the external disturbance effect on fault estimation, we introduce the following relation

$$\begin{aligned} J(t) &= \mathbb{E}\left\{ \int_0^\infty \left[e_x^T(t)e_x(t) - \gamma^2 d^T(t)d(t)\right] dt\right\} \; \forall t > 0 \\ &= \mathbb{E}\left\{ \int_0^\infty \left[e_x^T(t)e_x(t) - \gamma^2 d^T(t)d(t) + \dot{V}(t, x(t)) + V(0, x(0)) - V(\infty, x(\infty))\right] dt\right\} \end{aligned} \tag{37}$$

Under the zero initial condition, substituting (23)-(36) into (37), by using Lemma 2.2, a straightforward computation gives

$$\begin{aligned} J(t) \le \mathbb{E}\Bigg\{ &\sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r \mu_i(\hat{x}(t))\mu_j(\hat{x}(t))\mu_k(x(t))\Bigg\{\zeta_1^T(t)\bar{\Pi}_{ijk}\zeta_1(t) + \delta \\ &- \int_{t-\tau_\rho}^{t-\tau_1} \dot{e}_x^T(s)(R_1 - Y_{33})\dot{e}_x(s)ds - \int_{t-\tau_2}^{t-\tau_\rho} \dot{e}_x^T(s)(R_2 - Z_{33})\dot{e}_x(s)ds\Bigg\}\Bigg\} \end{aligned} \tag{38}$$

where $\bar{\Pi}_{ijk} = \begin{pmatrix} \bar{\Pi}_{ijk}^1 & \bar{\Pi}_{ijk}^2 \\ * & \Pi_{ijk}^3 \end{pmatrix}$, and Π_{ijk}^3 is defined as in (19), where $\Pi_{(N6,6)} = M + \lambda_3^{-1}F_i^TF_i$

$$\bar{\Pi}_{ijk}^1 = \begin{bmatrix} \bar{\Pi}_{11}^{ijk} & W_1 & \cdots & 0 \\ * & \Pi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Pi_{nn} \end{bmatrix}, \; \bar{\Pi}_{ijk}^2 = \begin{bmatrix} 0 & \bar{\Pi}_{(1,N2)}^{ij} & 0 & \bar{\Pi}_{(1,N4)}^{ij} & 0 & \bar{\Pi}_{(1,N6)}^{ij} & \bar{\Pi}_{(1,N7)}^{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ W_N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \bar{\Pi}_{11}^{ij} &= \mathrm{sym}(PA_i - PL_iC_j) + Q_1 - W_1 + S_1 + \hat{S}_0 + \hat{S}_1 + \left(\lambda_1\eta_1^2 + (\lambda_2 + \lambda_3)\,\eta_2^2\right) I \\ &\quad + \lambda_1^{-1}PP + \lambda_2^{-1}\left(PL_i^T(PL_i)\right), \quad \bar{\Pi}_{(1,N2)}^{ij} = \delta(t)\left(PA_{\tau i} - PL_iC_{\tau j}\right) \end{aligned}$$

$$\bar{\Pi}_{(1,N4)}^{ij} = (1 - \delta(t))\left(PA_{\tau i} - PL_iC_{\tau j}\right), \quad \bar{\Pi}_{(1,N6)}^{ij} = PB - C_j^TF_i, \quad \bar{\Pi}_{(1,N7)}^{ik} = PB_{dk} - Y_iD_{dk}$$

From (38), it is clear that when $R_1 - Y_{33} \ge 0$, $R_2 - Z_{33} \ge 0$, that is the last two terms in (38) are all less than 0. Then, it follows from Schur complement theorem and with the changes of variables as $Y_i = PL_i$, we can see if the inequalities (14)-(17) hold, one has $J(t) < 0$, and there exists a scalar $\phi > 0$, such that $\dot{V}(t) < -\phi\|\zeta_1(t)\|^2 + \delta$. It follows that $\dot{V}(t) < 0$ for $\phi\|\zeta_1(t)\|^2 > \delta$, which means estimation errors of both the state and the fault are uniformly ultimately bounded.

In addition, by choosing the same Lyapunov function as (22) and following the similar line in the earlier deduction under conditions (14)-(17), we can easily obtain that the time derivative of $V(t)$ along the solution of error dynamics (10) with $d(t) = 0$ satisfies $\dot{V}(t) < 0$, which indicates the asymptotic stability of systems (10). This completes the proof of Theorem 3.1.

Remark 3.2. *An important aspect to be considered in the FE problem for fuzzy systems is the unavailability of all premise variables in real time for implementation. Different from the existing results* [16, 17, 19, 20, 21, 22, 23] *the proposed method designs an adaptive FE observer with random time-varying delay when the premise variables are unmeasurable, which have a wider application range in practical engineering.*

Remark 3.3. *Motivated by the delay partitioning approach, we divide the constant part of time-varying delay* $[0, \tau_1]$ *into* n *segments, that is,* $\left[0, \frac{1}{n}\tau_1\right]$, $\left[\frac{1}{n}\tau_1, \frac{2}{n}\tau_1\right]$, $\cdots$, $\left[\frac{n-1}{n}\tau_1, \tau_1\right]$, $n = 1, 2, \ldots, N$. *Because different energy functions are defined in LKF* (22) *to correspond to different delay interval, the conservativeness of fault estimation result can be reduced as fault estimation steps* n *increase.*

Remark 3.4. *In view that time-varying delay occurs randomly, the interval* $[\tau_1, \tau_2]$ *is divided into two unequal variable subintervals* $[\tau_1, \tau_\rho]$ *and* $[\tau_\rho, \tau_2]$. *It is clear that different information on error state variable* $e_n(t-\tau_\rho)$ *is utilized when* ρ *is changed, which is more general than the one in* [19, 20, 21, 22, 23]. *Moreover, when* $\tau_1(t) = \tau_2(t)$, ρ *is a tunable parameter. By seeking an appropriate* ρ, *the result of fault estimation or stability criterion can further reduce the analysis and synthesis conservatism.*

4. **Fault-Tolerant Controller Design.** On the basis of the obtained online fault estimation information, we consider a dynamic output feedback controller where the actual values of the premise variables are the unmeasurable system states. Hence, the following controller with the premise variables being state estimates is constructed:

$$\begin{cases} \dot{x}_c(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(\hat{x}(t))\mu_j(\hat{x}(t))\big\{A_{cij}x_c(t) + \delta(t)A_{\tau cij}x_c(t-\tau_1(t)) \\ \qquad + (1-\delta(t))A_{\tau cij}x_c(t-\tau_2(t)) + B_{ci}y(t)\big\} \\ u(t) = \sum_{i=1}^{r}\mu_i(\hat{x}(t))\big\{C_{ci}x_c(t) + \delta(t)C_{\tau ci}x_c(t-\tau_1(t)) + y(t) - \hat{f}(t) \\ \qquad + (1-\delta(t))C_{\tau ci}x_c(t-\tau_2(t))\big\} \end{cases} \tag{39}$$

where $A_{cij} \in \mathbb{R}^{n\times n}$, $A_{\tau cij} \in \mathbb{R}^{n\times n}$, $B_{ci} \in \mathbb{R}^{n\times l}$, $C_{ci} \in \mathbb{R}^{q\times n}$, $C_{\tau ci} \in \mathbb{R}^{q\times n}$ are the designed controller matrices. So the closed-loop system for T-S fuzzy model (8) with this controller can be written as

$$\begin{aligned} \dot{x}(t) = {} & [A_\mu + BC_{\hat{\mu}} + EF(t)H_c]x(t) + \delta(t)[A_{\tau\mu} + BC_{\tau\hat{\mu}} + EF(t)H_{\tau c}]x(t-\tau_1(t)) \\ & + (1-\delta(t))[A_{\tau\mu} + BC_{\tau\hat{\mu}} + EF(t)H_{\tau c}]x(t-\tau_2(t)) \\ & + [B_{d\mu} + BD_{d\hat{\mu}} + EF(t)H_d]d(t) + BC_{c\hat{\mu}}x_c(t) + B\left(f(t) - \hat{f}(t)\right) \\ & + \delta(t)BC_{\tau c\hat{\mu}}x_c(t-\tau_1(t)) + (1-\delta(t))BC_{\tau c\hat{\mu}}x_c(t-\tau_2(t)) \end{aligned}$$

where

$$\begin{aligned} & F(t) = \operatorname{diag}\big\{(\mu_1(x) - \mu_1(\hat{x}))I \ \cdots \ (\mu_r(x) - \mu_r(\hat{x}))I\big\} \in \mathbb{R}^{n\times nr} \\ & E = [\, I \ \ I \ \ \cdots \ \ I \,] \in \mathbb{R}^{n\times nr}, \ H_c^T = \big[(BC_1)^T \ \cdots \ (BC_r)^T\big] \in \mathbb{R}^{n\times nr} \\ & H_{\tau c}^T = \big[(BC_{\tau 1})^T \ \cdots \ (BC_{\tau r})^T\big] \in \mathbb{R}^{n\times nr}, \ H_d^T = \big[(BD_{d1})^T \ \cdots \ (BD_{dr})^T\big] \in \mathbb{R}^{n\times nr} \end{aligned}$$

Similarly, with $H_{A\hat{\mu}}^T = \big[A_{c1\hat{\mu}}^T \ \cdots \ A_{cr\hat{\mu}}^T\big] \in \mathbb{R}^{n\times nr}$, $H_{A_\tau\hat{\mu}}^T = \left[A_{\tau c1\hat{\mu}}^T \ \cdots \ A_{\tau cr\hat{\mu}}^T\right] \in \mathbb{R}^{n\times nr}$, and denote $\tilde{x}^T(t) = \left(x^T(t), x_c^T(t)\right)$, $e_f(t) = \hat{f}(t) - f(t)$, $\tilde{\omega}^T(t) = \left(d^T(t), e_f^T(t)\right)$, then one can obtain the following closed-loop augmented system.

$$\begin{cases} \dot{\tilde{x}}(t) = \left[\tilde{A}_{\mu\hat{\mu}} + \tilde{E}\tilde{F}(t)\tilde{H}_{c\mu\hat{\mu}}\right]\tilde{x}(t) + \delta(t)\left[\tilde{A}_{\tau\mu\hat{\mu}} + \tilde{E}\tilde{F}(t)\tilde{H}_{\tau c\mu\hat{\mu}}\right]\tilde{x}(t-\tau_1(t)) \\ \qquad + (1-\delta(t))\left[\tilde{A}_{\tau\mu\hat{\mu}} + \tilde{E}\tilde{F}(t)\tilde{H}_{\tau c\mu\hat{\mu}}\right]\tilde{x}(t-\tau_2(t)) + \left[\tilde{B}_{\omega\mu\hat{\mu}} + \tilde{E}\tilde{F}(t)\tilde{H}_{d\mu}\right]\tilde{\omega}(t) \\ y(t) = \tilde{C}_\mu\tilde{x}(t) + \delta(t)\tilde{C}_{\tau\mu}\tilde{x}(t-\tau_1(t)) + (1-\delta(t))\tilde{C}_{\tau\mu}\tilde{x}(t-\tau_2(t)) + \tilde{D}_{\omega\mu}\tilde{\omega}(t) \end{cases} \tag{40}$$

where

$$\tilde{A}_{\mu\hat{\mu}} = \begin{bmatrix} A_\mu + BC_{\hat{\mu}} & BC_{c\hat{\mu}} \\ B_{c\hat{\mu}}C_\mu & A_{c\mu\hat{\mu}} \end{bmatrix}, \ \tilde{A}_{\tau\mu\hat{\mu}} = \begin{bmatrix} A_{\tau\mu} + BC_{\tau\hat{\mu}} & BC_{\tau c\hat{\mu}} \\ B_{c\hat{\mu}}C_{\tau\mu} & A_{\tau c\mu\hat{\mu}} \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} \lambda E & 0 \\ 0 & \lambda E \end{bmatrix}$$

$$\tilde{B}_{\omega\mu\hat{\mu}} = \begin{bmatrix} B_{d\mu} + B_\mu D_{d\hat{\mu}} & -B \\ B_{c\hat{\mu}}D_{d\mu} & 0 \end{bmatrix}, \ \tilde{H}_{c\mu\hat{\mu}} = \begin{bmatrix} \frac{1}{\lambda}H_{c\mu} & 0 \\ 0 & -\frac{1}{\lambda}H_{A\hat{\mu}} \end{bmatrix}, \ \tilde{H}_{\tau c\mu\hat{\mu}} = \begin{bmatrix} \frac{1}{\lambda}H_{\tau c\mu} & 0 \\ 0 & -\frac{1}{\lambda}H_{A_\tau\hat{\mu}} \end{bmatrix}$$

$$\tilde{H}_{d\mu} = \begin{bmatrix} \frac{1}{\lambda}H_{d\mu} & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{F}(t) = \begin{bmatrix} F(t) & 0 \\ 0 & F(t) \end{bmatrix},\ \tilde{C}_{\mu} = [C_{\mu}\ 0],\ \tilde{C}_{\tau\mu} = [C_{\tau\mu}\ 0],\ \tilde{D}_{\omega\mu} = [D_{d\mu}\ 0]$$

Remark 4.1. *In order to design a stabilizing output feedback controller for a fuzzy system, the PDC method is feasible only if it shares the same premise variables as those of fuzzy system* (8)*, which are often assumed to be measurable in* [7, 8, 9, 10, 11, 12, 22]*. However, in this paper, we note that although the state $x(t)$ is not measurable and $\mu_i(x(t))$, $i = 1, \ldots, r$ are unknown, a closed-loop system* (40) *can be obtained by using uncertain system approach. In addition, it is easy to see that $\tilde{F}(t)$ in* (40) *is a time varying unknown function but satisfies $\tilde{F}^T(t)\tilde{F}(t) \le I$, because $-1 \le \mu_i(x(t)) - \mu_i(\hat{x}(t)) \le 1$ for $i = 1, \ldots, r$.*

So far, the problem of robust dynamic output feedback control for system (8) is to design the gain matrices of (39) such that: (i) The closed-loop fuzzy system (40) with $\tilde{\omega}(t) = 0$ is asymptotically stable for any time-delay satisfying (2)-(5); (ii) For a given scalar $\gamma > 0$, the following H_∞ performance is satisfied:

$$\mathbb{E}\left\{ \int_0^\infty \|y(t)\|^2 dt \right\} \le \gamma^2 \int_0^\infty \|\tilde{\omega}(t)\|^2 dt \tag{41}$$

for all $\tilde{\omega}(t) \in L_2[0, \infty)$ under zero initial conditions.

In what follows, a less conservative delay dependent sufficient condition is given for the existence of controller (39) with unmeasurable premise variables.

Theorem 4.1. *For the given positive scalars τ_1, τ_2, γ and $0 < \rho < 1$, the closed-loop system* (40) *with $\tilde{\omega}(t) = 0$ is asymptotically stable for any time-varying delay $\tau(t)$ satisfying* (2)-(5)*, and the prescribed H_∞ performance* (41) *is satisfied under zero initial condition for any nonzero $\tilde{\omega}(t) \in L_2[0, \infty)$, if there exist appropriately dimensional matrices $X > 0$, $Y > 0$, $\hat{Q}_n > 0$, $\hat{W}_n > 0$, $(n = 1, 2, \ldots, N)$, $\hat{S}_k > 0$, $(k = 0, 1, 2, 3)$, $\hat{R}_1 > 0$, $\hat{R}_2 > 0$, and $\hat{A}_{ij}$, $\hat{A}_{\tau ij}$, $\hat{B}_j$, $\hat{C}_j$, $\hat{C}_{\tau j}$ such that*

$$\hat{\Psi}_{ii} < 0 \qquad i = 1, 2, \ldots, r \tag{42}$$

$$\hat{\Psi}_{ij} + \hat{\Psi}_{ji} \le 0 \qquad 1 \le i < j \le r \tag{43}$$

where

$$\hat{\Psi}_{ij} = \begin{bmatrix} \begin{pmatrix} \hat{\Psi}_{ij}^1 & \hat{\Psi}_{ij}^2 \\ * & \hat{\Psi}_{ij}^3 \end{pmatrix} & \hat{\Gamma}_{3i}^T & \varepsilon\hat{\Omega}_1^T & \hat{\Omega}_{2ij}^T \\ * & -I & 0 & 0 \\ * & * & -\varepsilon I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix},\quad \hat{\Psi}_{ij}^1 = \begin{bmatrix} \hat{\Psi}_{11}^{ij} & \hat{W}_1 & \cdots & 0 \\ * & \hat{\Psi}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \hat{\Psi}_{nn} \end{bmatrix}$$

$$\hat{\Psi}_{ij}^2 = \begin{bmatrix} 0 & \delta_0\hat{\Xi}_{A_\tau}^{ij} & 0 & (1-\delta_0)\hat{\Xi}_{A_\tau}^{ij} & 0 & 0 & \hat{\Xi}_{B_\omega}^{ij} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{W}_N & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\Psi}_{ij}^3 = \begin{bmatrix} \hat{\Psi}_{(N1,1)} & R_1 & 0 & 0 & 0 & 0 & 0 \\ * & \hat{\Psi}_{(N2,2)} & R_1 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Psi}_{(N3,3)} & R_2 & 0 & 0 & 0 \\ * & * & * & \hat{\Psi}_{(N4,4)} & R_2 & 0 & 0 \\ * & * & * & * & \hat{\Psi}_{(N5,5)} & 0 & 0 \\ * & * & * & * & * & \hat{\Psi}_{(N6,6)} & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

with

$$\begin{aligned}
&\hat{\Psi}_{11}^{ij} = \hat{\Xi}_A^{ij} + \left(\hat{\Xi}_A^{ij}\right)^T + \hat{S}_0 + \hat{S}_1 + \hat{Q}_1 - \hat{W}_1,\ \hat{\Psi}_{nn} = -\hat{Q}_{n-1} + \hat{Q}_n - \hat{W}_{n-1} - \hat{W}_n \\
&\hat{\Psi}_{(N1,1)} = \hat{S}_2 - \hat{R}_1 - \hat{Q}_N - \hat{W}_N,\ \hat{\Psi}_{(N2,2)} = -(1-d_1)\hat{S}_0 - 2\hat{R}_1 \\
&\hat{\Psi}_{(N3,3)} = \hat{S}_3 - \hat{S}_2 - \hat{R}_1 - \hat{R}_2,\ \hat{\Psi}_{(N4,4)} = -(1-d_2)\hat{S}_1 - 2\hat{R}_2 \\
&\hat{\Psi}_{(N5,5)} = -\hat{S}_3 - \hat{R}_2,\ \hat{\Psi}_{(N6,6)} = \sum_{n=1}^{N} h^2\hat{W}_n + (\rho\delta)^2\hat{R}_1 + ((1-\rho)\delta)^2\hat{R}_2 \\
&\hat{\Gamma}_{3i} = \begin{bmatrix}(C_iX \quad C_i) & 0 & \cdots & 0 & (C_{\tau i}X \quad C_{\tau i}) & 0 & (D_{di}\ 0)\end{bmatrix}^T
\end{aligned}$$

and $\hat{\Xi}_A^{ij}$, $\hat{\Xi}_{A_\tau}^{ij}$, $\hat{\Xi}_{B_\omega}^{ij}$, $\hat{\Omega}_1$, $\hat{\Omega}_{2ij}$ *are defined as* (48)-(52). *Then, the gain matrices of the dynamic output feedback fault tolerant controller are given by*

$$\begin{aligned}
A_{cij} &= N^{-1}\left(\hat{A}_{ij} - \left(YA_i - \hat{B}_iC_j\right)X\right)M^{-T} + N^{-1}YBC_{cj} \\
A_{\tau cij} &= N^{-1}\left(\hat{A}_{\tau ij} - \left(YA_{\tau i} - \hat{B}_iC_{\tau j}\right)X\right)M^{-T} + N^{-1}YBC_{\tau cj} \\
B_{cj} &= N^{-1}\hat{B}_j,\ C_{cj} = \left(\hat{C}_j - C_jX\right)M^{-T},\ C_{\tau cj} = \left(\hat{C}_{\tau j} - C_{\tau j}X\right)M^{-T}
\end{aligned}$$

where M, N *satisfy* $MN^T = I - XY$.

Remark 4.2. *Since inequalities* (42) *and* (43) *include the term* $\varepsilon\hat{\Omega}_1^T$, *it is not possible to solve them for* ε *and* Y *or* N *simultaneously as an LMI. Thus, a line search algorithm should be employed to find* ε. *After initialization of* ε *with a small positive value, if the LMI problem is infeasible,* ε *should be increased until the problem becomes feasible.*

Remark 4.3. *As in* [9, 22], *from* $F_1^TPF_1 > 0$, *we can obtain* $Y > 0$ *and* $X - Y^{-1} < 0$ *which imply that* $I - XY$ *is nonsingular. Therefore, we can always find nonsingular matrices* M *and* N *satisfying* $MN^T = I - XY$, *and they can be calculated by the QR function of Matlab toolbox.*

Remark 4.4. *The conditions* (42) *and* (43) *indicate that the robust controller design problem can be included as an optimization variable, which can be exploited to reduce the attenuation level bound. Then, the minimum attenuation level of* H_∞ *performance can be obtained by solving a convex optimization problem* ***P****:* $\min\vartheta$ *subject to* (42) *and* (43) *with* $\vartheta = \gamma^2$.

5. **Numerical Example.** We provide an illustrative example with simulation result which is based on the truck-trailer model to demonstrate the applicability of the proposed design method. To provide a realistic framework for the simulation result, we assume that the system $x_1(t)$ is perturbed by time-delay and the delayed model is given as

$$\begin{cases}
\dot{x}_1(t) = -a\dfrac{v\bar{t}}{Lt_0}x_1(t) - (1-a)\dfrac{v\bar{t}}{Lt_0}x_1(t-\tau(t)) + \dfrac{v\bar{t}}{lt_0}u(t) \\
\dot{x}_2(t) = a\dfrac{v\bar{t}}{Lt_0}x_1(t) + (1-a)\dfrac{v\bar{t}}{Lt_0}x_1(t-\tau(t)) \\
\dot{x}_3(t) = \dfrac{v\bar{t}}{t_0}\sin\left[x_2(t) + a\dfrac{v\bar{t}}{2L}x_1(t) + (1-a)\dfrac{v\bar{t}}{2L}x_1(t-\tau(t))\right]
\end{cases} \tag{44}$$

where $x_1(t)$ is the angle difference between the truck and the trailer, $x_2(t)$ is the angle of the trailer, $x_3(t)$ is the vertical position of the rear end of the trailer, and $u(t)$ is the steering angle. The constant $a = 0.7$ is the retarded coefficient, which satisfies the conditions: $a \in [0, 1]$. The limits 1 and 0 correspond to no delay term and to a completed delay term, respectively. In order to make the model have practical significance, in this example, the parameters in (44) are given as [16, 20, 21, 22], that is $a = 0.7$, $l = 2.8$, $L = 5.5$, $v = -1.0$, $\bar{t} = 2.0$, $t_0 = 0.5$, where l is the length of truck, L is the length of trailer, t is sampling time, and v is the constant speed of backing up.

Remark 5.1. *The truck-trailer example is often used in stability analysis and controller design for fuzzy systems. However, in most existing results, such as* [16, 20, 21, 22, 24]*, the premise variables of the obtained T-S fuzzy model are assumed to be measurable. It is well known that when system state is selected as the premise variable, the aforementioned results cannot be used in practical application because of the unmeasurable state. In this paper, both the observer and the controller are designed based on the premise variables depending on the estimates of state, which is more common in practice.*

To demonstrate the results in Theorems 3.1 and 4.1, we assume the delay in (44) as randomly occurring and satisfy (2)-(5). Then system (44) with actuator fault $f(t)$ and disturbance $d(t)$ can be represented by the following T-S fuzzy model

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(x(t))[A_i x(t) + \delta(t) A_{\tau i} x(t - \tau_1(t)) + (1 - \delta(t)) A_{\tau i} x(t - \tau_2(t)) + B_{di} d(t) \\ \qquad + B_i(u(t) + f(t))] \\ y(t) = \sum_{i=1}^{2} \mu_i(x(t))[C_i x(t) + \delta(t) C_{\tau i} x(t - \tau_1(t)) + (1 - \delta(t)) C_{\tau i} x(t - \tau_2(t)) + D_{di} d(t)] \\ x(t) = \sum_{i=1}^{2} \mu_i(x(t)) \phi_i(t), t \in [-\tau_2, 0] \end{cases}$$

where $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T$ and

$$A_1 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, \ A_{\tau 1} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{dv^2\bar{t}^2}{2Lt_0} & \frac{dv\bar{t}}{t_0} & 0 \end{bmatrix}, \ A_{\tau 2} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{dv^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix}$$

Here, in order to facilitate simulation, it is supposed that $C_1 = C_2 = [-0.2,\ 0.05,\ -0.15]$, $C_{\tau 1} = C_{\tau 2} = (1-a)C_1$, the disturbance matrices are $B_{d1} = B_{d2} = [0.01,\ 0.01,\ 0.01]^T$, $D_{d1} = D_{d2} = 0.05$. Meanwhile, we choose membership functions for Rules 1 and 2 are $\mu_1(x(t)) = 1/(1 + \exp(x_1(t) + 0.5))$, $\mu_2(x(t)) = 1 - \mu_1(x(t))$ with initial condition $[0.5\pi \quad 0.75\pi - 5]^T$, $d = 10 * 0.05/\pi$. Due to the fact that $x(t)$ is unmeasurable, the membership functions for observer (9) and controller (39) are selected as $\mu_1(\hat{x}(t)) = 1/(1 + \exp(\hat{x}_1(t) + 0.5))$, $\mu_2(\hat{x}(t)) = 1 - \mu_1(\hat{x}(t))$. For the random time-varying delay $\tau(t)$ satisfying (2)-(5), we assume that $\tau_1(t) = 0.5 + 0.2cos(t)$, $\tau_2(t) = 1 + 0.3\sin(t)$. Thus,

$\tau_1 = 0.3$, $d_1 = 0.2$, $\tau_2 = 1.3$, $d_2 = 0.3$, $\rho = 0.4$, $\delta = 1$. Then, we set $\lambda_1 = \lambda_2 = \lambda_3 = 0.5$, $\eta = 10$ and $\delta_0 = 0.05$, which means $\tau_1(t)$ with a small occurring probability, by solving the conditions (14)-(17) in Theorem 3.1 based on the mincx function of Matlab toolbox, one obtains the feasible solution as follows:

$$L_1 = \begin{bmatrix} -61.5852 \\ 72.0660 \\ -49.2033 \end{bmatrix}, \ F_1 = 4.0998, \ L_2 = \begin{bmatrix} -73.0435 \\ 42.2620 \\ -63.1385 \end{bmatrix}, \ F_2 = 1.5521$$

Then, a constant fault and a time-varying fault are respectively created as

$$f_1(t) = \begin{cases} 0 & 0 \le t < 5 \\ 10 & 5 \le t \le 30 \end{cases}, \quad f_2(t) = \begin{cases} 0 & 0 \le t < 5 \\ 10\sin((t+0.2)/0.5) & 5 \le t \le 30 \end{cases}$$

and the disturbance $d(t)$ is band-limited white noise with power 0.001 and sampling time 0.1s. When $u(t) = 0$, Figure 1(a) shows that all the trajectories of the system states are unstable. Then, by taking the learning rate $\Gamma = 100$ and the sample time is 0.1s, and using the obtained observer gain matrix, Figure 1(b) and Figure 2 illustrate the simulation result of the estimates of system states ($d(t) = 0$), actuator fault $f_1(t)$ and $f_2(t)$ ($d(t) \neq 0$), respectively. As shown in Figure 2, it is obvious that the robust adaptive fault estimation observer is insensitive to the exogenous disturbance and has a good performance to estimate the constant and time-varying fault $f(t)$.

Next, by solving the conditions in Theorem 4.1, we design a dynamic output feedback controller with attenuation value $\gamma = 0.5$ on the basis of the obtained fault estimation as follows:

$$A_{c11} = \begin{bmatrix} -0.3453 & 0.7082 & 0.1383 \\ 5.2522 & -4.6526 & -3.8435 \\ 0.5727 & -4.8792 & 0.6050 \end{bmatrix}, \ A_{c12} = \begin{bmatrix} -0.3489 & 0.6490 & 0.1527 \\ 5.4580 & -4.0907 & -3.7796 \\ 0.5571 & -5.0332 & 0.6649 \end{bmatrix}$$

$$A_{c21} = \begin{bmatrix} -0.6059 & 0.7382 & 0.2548 \\ 8.6252 & -5.6198 & -4.3884 \\ 1.2169 & -5.6382 & 0.4717 \end{bmatrix}, \ A_{c22} = \begin{bmatrix} -0.5891 & 0.7550 & 0.3091 \\ 8.1457 & -5.3109 & -5.6569 \\ 1.2120 & -5.3278 & 0.2228 \end{bmatrix}$$

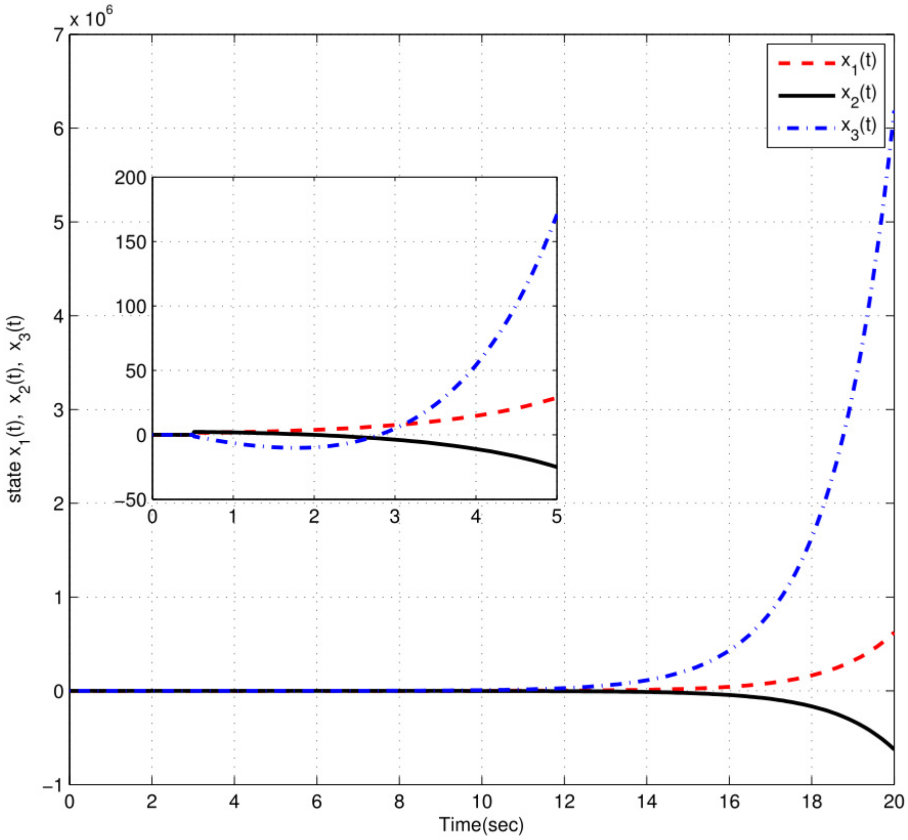
$$A_{\tau c11} = \begin{bmatrix} 0.0383 & 0.1357 & -0.0469 \\ -0.2785 & -0.8260 & 0.2851 \\ -0.2987 & -1.5179 & 0.5033 \end{bmatrix}, \ A_{\tau c12} = \begin{bmatrix} 0.0445 & 0.1313 & -0.0459 \\ -0.3120 & -0.6957 & 0.2477 \\ -0.3153 & -1.5170 & 0.5035 \end{bmatrix}$$

$$A_{\tau c21} = \begin{bmatrix} 0.0524 & 0.1473 & -0.0517 \\ -0.4526 & -0.9755 & 0.3527 \\ -0.3394 & -1.5498 & 0.5169 \end{bmatrix}, \ A_{\tau c22} = \begin{bmatrix} 0.0573 & 0.1419 & -0.0508 \\ -0.4413 & -0.7939 & 0.2981 \\ -0.3498 & -1.5457 & 0.5167 \end{bmatrix}$$

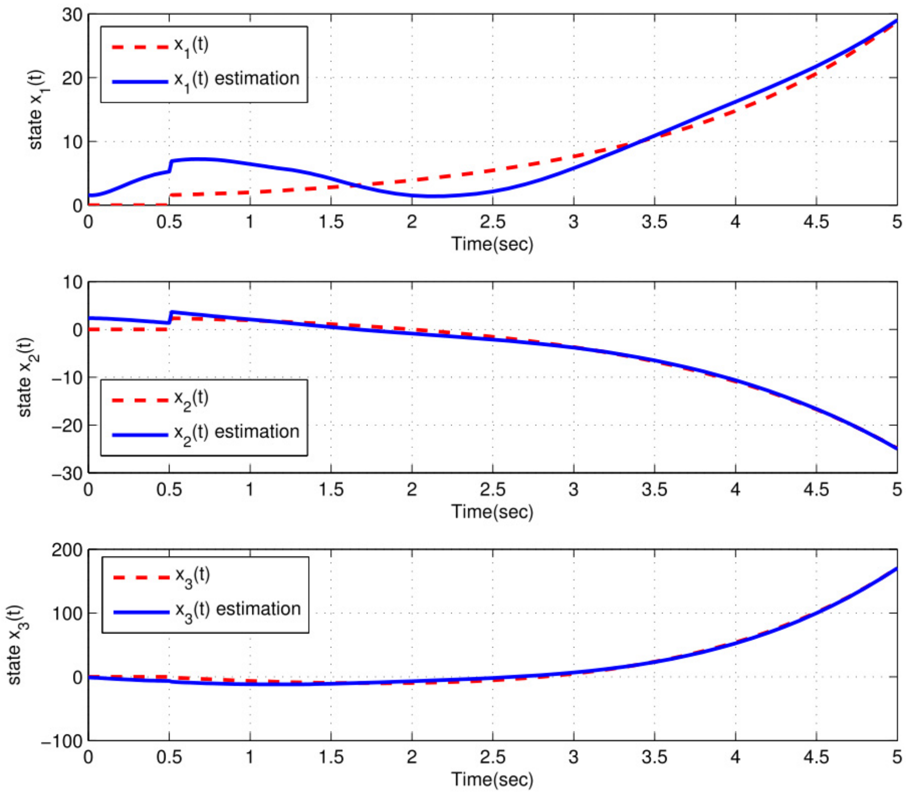
and

$$B_{c1}^T = \begin{bmatrix} -1.3570 & 7.7444 & 16.8299 \end{bmatrix}, \ B_{c2}^T = \begin{bmatrix} -1.2177 & 5.5418 & 16.6085 \end{bmatrix}$$
$$C_{c1} = \begin{bmatrix} -12.1897 & -1.6331 & -11.9848 \end{bmatrix}, \ C_{\tau c1} = \begin{bmatrix} -0.6275 & -0.6262 & 0.2567 \end{bmatrix}$$
$$C_{c2} = \begin{bmatrix} -12.9225 & -2.1864 & -14.2795 \end{bmatrix}, \ C_{\tau c2} = \begin{bmatrix} -0.5299 & -0.5249 & 0.2181 \end{bmatrix}$$

Simulation results for the stability of the closed-loop systems and the systems output response are shown in Figures 3(a) and 3(b) and Figures 4(a) and 4(b). It can be seen that although the open-loop system is unstable in Figure 1(a), the proposed fuzzy dynamic output fault tolerant control design method still achieves the performance under actuator faults and unmeasurable premise variables, and the stability of closed-loop systems is guaranteed while satisfying the prescribed H_∞ performance. As indicated by the simulation result graph, we can see that whether the random time delay fuzzy systems with constant fault or time-varying fault, the fuzzy adaptive fault estimation observer can

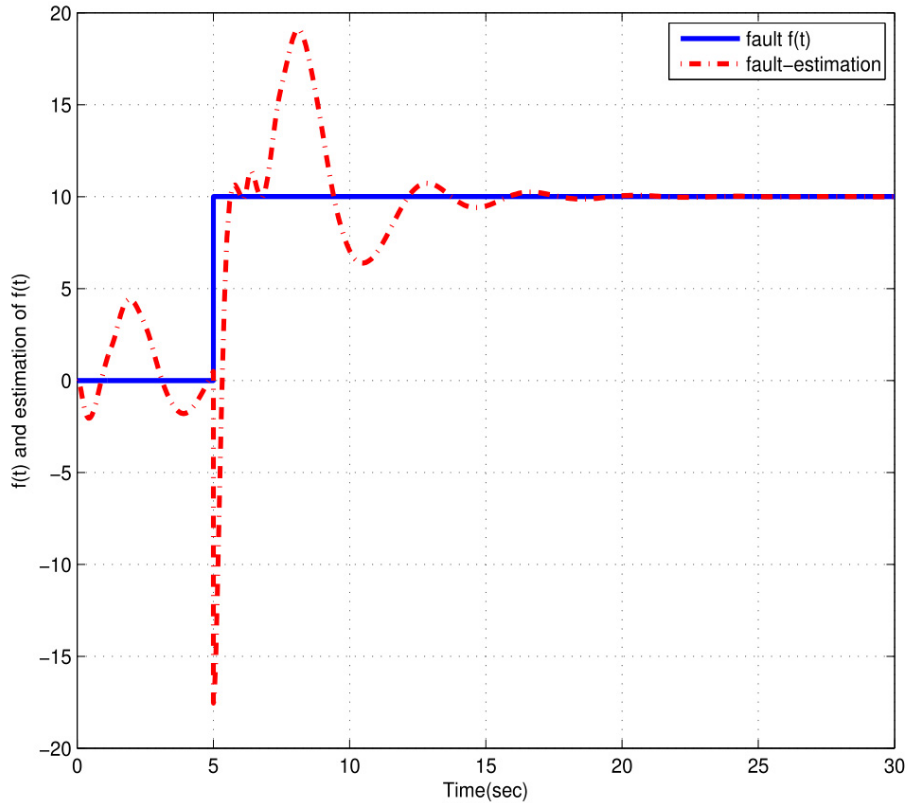


(a) System states with $u(t) = 0$

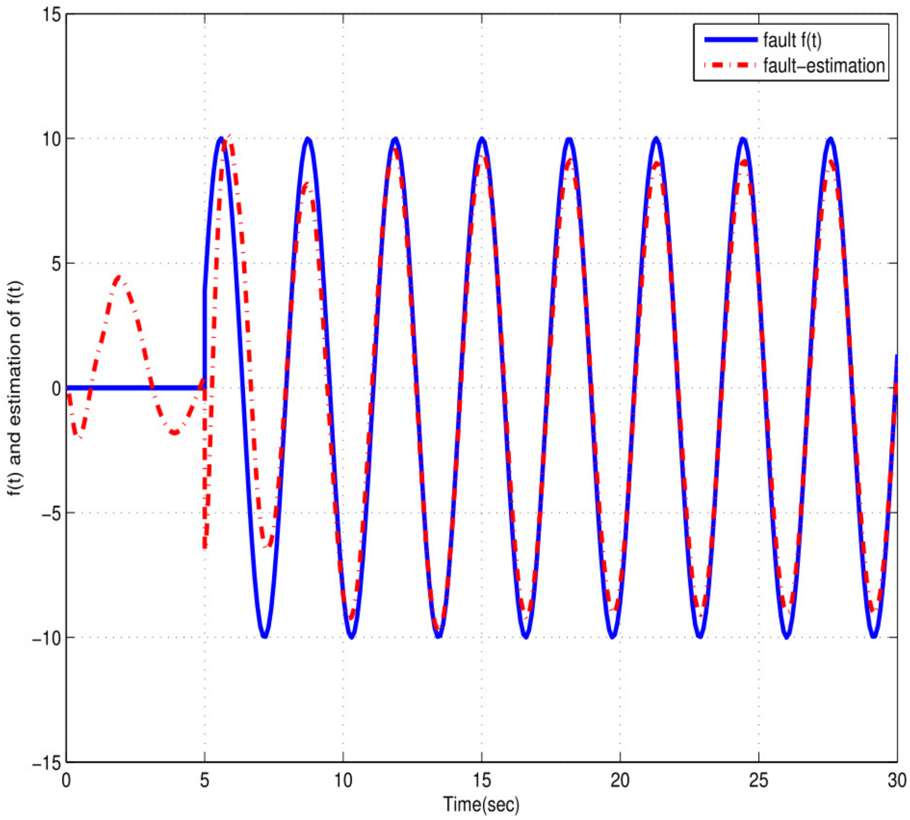


(b) Estimates of system states with $d(t) = 0$

FIGURE 1. Response curves of system states and estimates with $d(t) = 0$

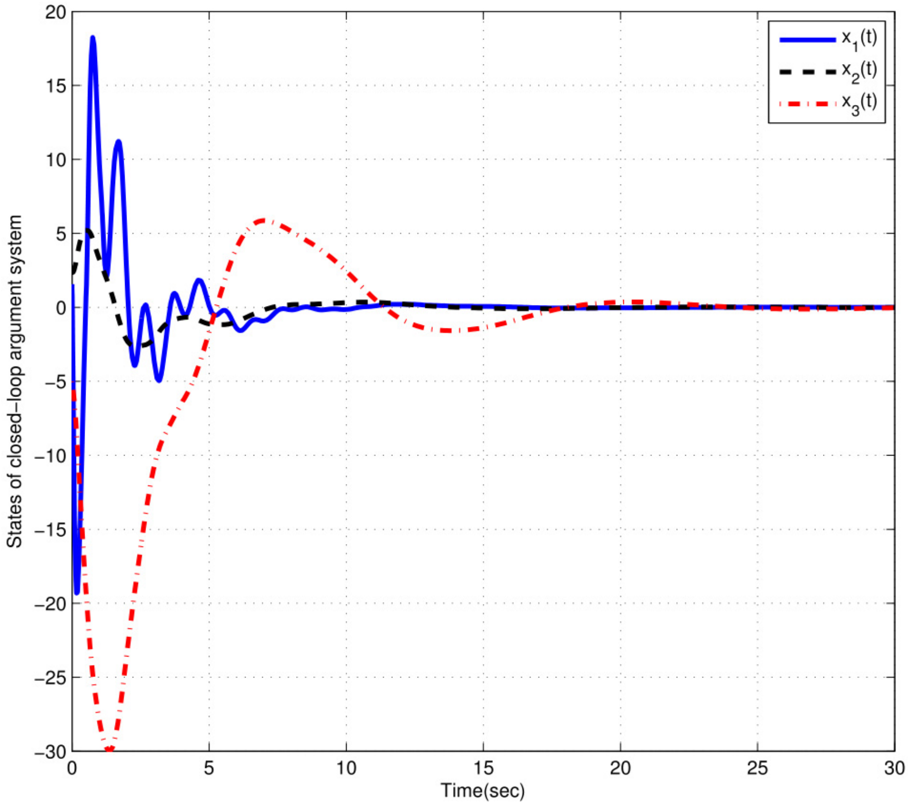


(a) Constant fault $f_1(t)$

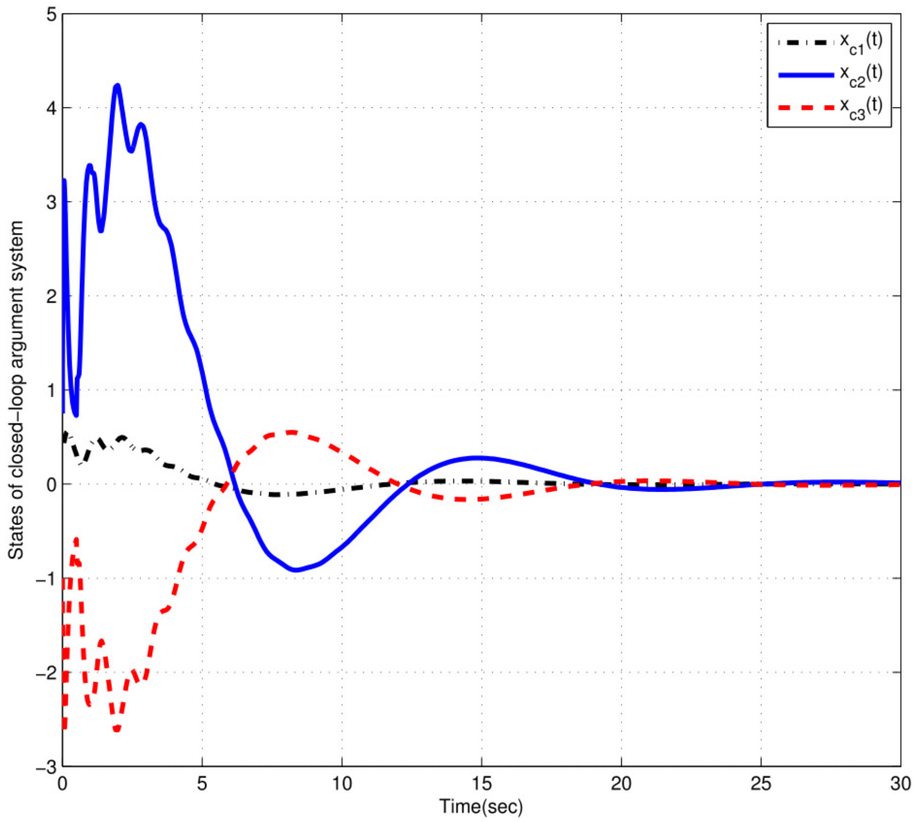


(b) Time-varying fault $f_2(t)$

FIGURE 2. Estimation result of fault $f_1(t)$ and $f_2(t)$ with $d(t) \neq 0$

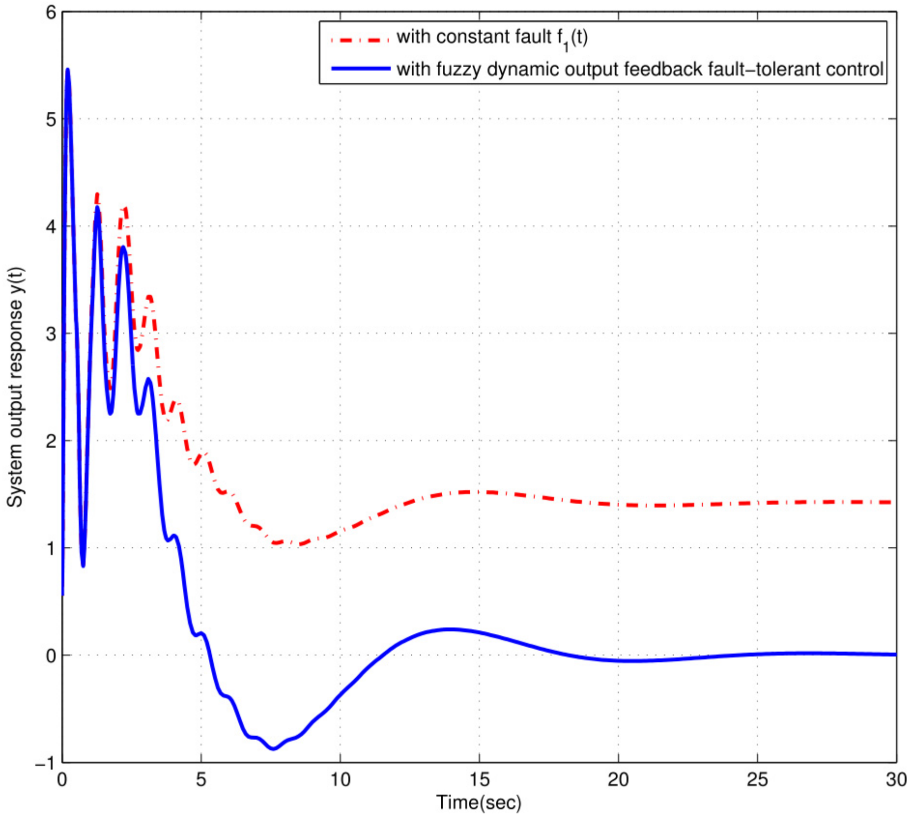


(a) Closed-loop states $x_1(t)$, $x_2(t)$, $x_3(t)$

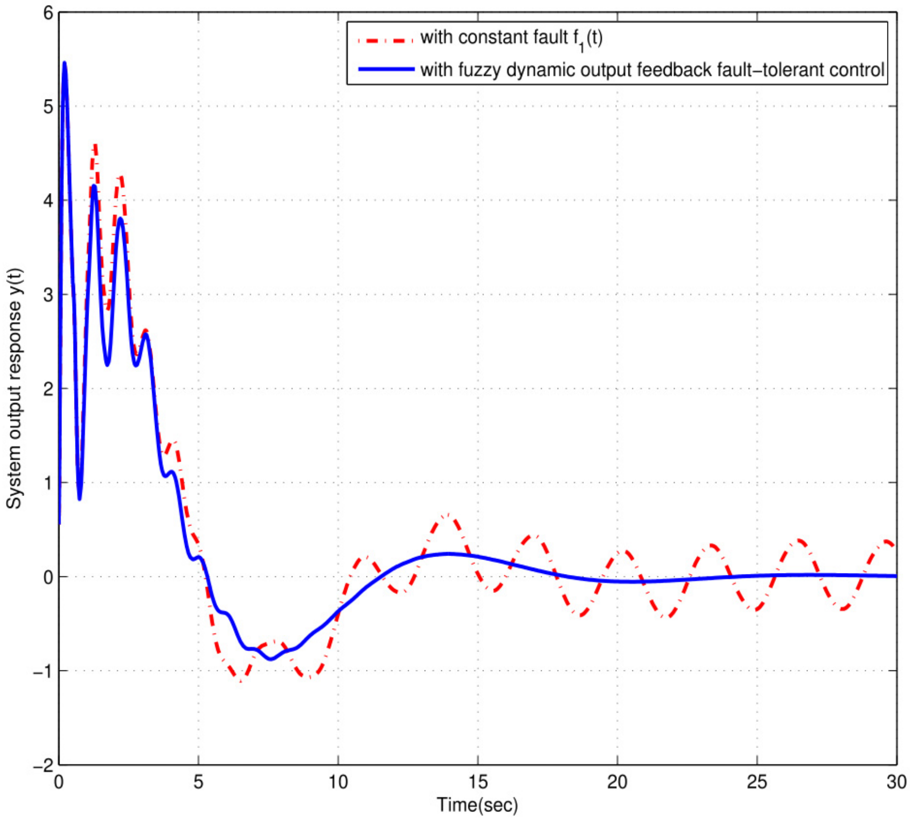


(b) Controller states $x_{c1}(t)$, $x_{c2}(t)$, $x_{c3}(t)$

FIGURE 3. Response curves of closed-loop states and controller states



(a) Using FE information with $f_1(t)$



(b) Using FE information with $f_2(t)$

FIGURE 4. Response $y(t)$ using fault estimation information shown in Figure 2

TABLE 1. The comparison with the existing results

System state $x(t)$	[16]	[20]	[22]	our method
Measurable	FS	FS	FS	FS
Unmeasurable	No FS	No FS	No FS	FS
Interval time delays	No FS	FS	No FS	FS
Random time delays	No FS	No FS	No FS	FS

almost realize accurate fault estimation, and the fuzzy dynamic output feedback control strategy can effectively accommodate the effect of actuator faults on system performance.

In addition, if time delay occurs randomly, the methods proposed in [16, 20, 22] fail to give a feasible solution. To better illustrate the advantage of the proposed method, the comparison with some existing results is given in Table 1 (FS = feasible solution), which shows that the adopted design method refers to be less conservative than other methods.

6. **Conclusions.** In this paper, by using improved delay partitioning method and uncertain system approach, the problem of FE and FTC for a class of T-S fuzzy systems with unmeasurable premise variables and random time-varying delay has been investigated. Some less conservative conditions for the existence of fault estimation observer and fault-tolerant controller are provided. Finally, simulation example has clearly verified the effectiveness of the proposed method. This paper focuses on robust FTC for T-S fuzzy systems with actuator fault, not contains sensor fault. The consideration of the system with actuator and sensor fault simultaneously will be studied in our future work.

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Appendix A. Proof: Consider Lyapunov-Krasovskii functional candidate in the following form

$$\begin{aligned} V(t) &= \tilde{x}^T(t)P\tilde{x}(t) + \sum_{n=1}^{N}\int_{t-nh}^{t-(n-1)h}\tilde{x}^T(s)Q_n\tilde{x}(s)ds + \int_{t-\tau_1(t)}^{t}\tilde{x}^T(s)S_0\tilde{x}(s)ds \\ &+ \int_{t-\tau_2(t)}^{t}\tilde{x}^T(s)S_1\tilde{x}(s)ds + \int_{t-\tau_\rho}^{t-\tau_1}\tilde{x}^T(s)S_2\tilde{x}(s)ds + \int_{t-\tau_2}^{t-\tau_\rho}\tilde{x}^T(s)S_3\tilde{x}(s)ds \\ &+ \sum_{n=1}^{N}\int_{-nh}^{-(n-1)h}\int_{t+\theta}^{t}\dot{\tilde{x}}^T(s)hW_n\dot{\tilde{x}}(s)dsd\theta + V_\rho(t) \end{aligned}$$

with

$$V_\rho(t) = \int_{-\tau_\rho}^{-\tau_1}\int_{t+\theta}^{t}\dot{\tilde{x}}^T(s)\rho\delta R_1\dot{\tilde{x}}(s)dsd\theta + \int_{-\tau_2}^{-\tau_\rho}\int_{t+\theta}^{t}\dot{\tilde{x}}^T(s)(1-\rho)\delta R_2\dot{\tilde{x}}(s)dsd\theta$$

where the unknown matrices $P > 0$, $S_0 > 0$, $S_1 > 0$, $S_2 > 0$, $S_3 > 0$, $R_1 > 0$, $R_2 > 0$, $Q_n > 0$ and $W_n > 0$ $(n = 1, 2, \ldots, N)$ are to be determined. Then, by using Lemma 2.3 to the time derivatives of $V_\rho(t)$ and following the same proof process as Theorem 3.1 with the introduced vectors as follows:

$$\begin{aligned} \zeta_2^T(t) &= \Big[\tilde{x}^T(t) \;\; \tilde{x}^T(t-h) \;\; \cdots \;\; \tilde{x}^T(t-nh) \\ &\qquad \tilde{x}^T(t-\tau_1(t)) \;\; \tilde{x}^T(t-\tau_\rho) \;\; \tilde{x}^T(t-\tau_2(t)) \;\; \tilde{x}^T(t-\tau_2) \;\; \dot{\tilde{x}}^T(t) \;\; \tilde{\omega}^T(t)\Big] \\ \Gamma_{2\mu\hat{\mu}} &= \Big[\tilde{A}_{\mu\hat{\mu}} \;\; 0 \;\; \cdots \;\; 0 \;\; \delta(t)\tilde{A}_{\tau\mu\hat{\mu}} \;\; 0 \;\; (1-\delta(t))\tilde{A}_{\tau\mu\hat{\mu}} \;\; 0 \;\; 0 \;\; \tilde{B}_{\omega\mu\hat{\mu}}\Big] \\ \Gamma_{3\mu} &= \Big[\;\tilde{C}_{\mu} \;\; 0 \;\; \cdots \;\; 0 \;\; \delta(t)\tilde{C}_{\tau\mu} \;\; 0 \;\; (1-\delta(t))\tilde{C}_{\tau\mu} \;\; 0 \;\; 0 \;\; \tilde{D}_{\omega\mu}\Big] \end{aligned}$$

One obtains that the performance (41) can be satisfied, if the following inequality is true

$$\begin{aligned} &\mathbb{E}\Big\{\dot{V}(t) + y^T(t)y(t) - \gamma^2\tilde{\omega}^T(t)\tilde{\omega}(t)\Big\} \\ &= \mathbb{E}\Big\{\zeta_2^T(t)\Big(\Psi_{\mu\hat{\mu}} + \Gamma_{3\mu}^T\Gamma_{3\mu} + \Omega_1^T\tilde{F}(t)\Omega_{2\mu\hat{\mu}} + \Omega_{2\mu\hat{\mu}}^T\tilde{F}(t)\Omega_1\Big)\zeta_2(t)\Big\} < 0 \end{aligned} \tag{45}$$

where $\Psi_{\mu\hat{\mu}} = \begin{bmatrix}\Psi_{\mu\hat{\mu}}^1 & \Psi_{\mu\hat{\mu}}^2 \\ * & \Psi_{\mu\hat{\mu}}^3\end{bmatrix}$ with $\Psi_{\mu\hat{\mu}}^1 = \begin{bmatrix}\Psi_{11} & W_1 & \cdots & 0 \\ * & \Psi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Psi_{nn}\end{bmatrix}$

$$\Psi_{\mu\hat{\mu}}^2 = \begin{bmatrix} 0 & \delta(t)P\tilde{A}_{\tau\mu\hat{\mu}} & 0 & (1-\delta(t))P\tilde{A}_{\tau\mu\hat{\mu}} & 0 & 0 & PB_{\omega\mu\hat{\mu}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi_{\mu\hat{\mu}}^3 = \begin{bmatrix} \Psi_{(N1,1)} & R_1 & 0 & 0 & 0 & 0 & 0 \\ * & \Psi_{(N2,2)} & R_1 & 0 & 0 & 0 & 0 \\ * & * & \Psi_{(N3,3)} & R_2 & 0 & 0 & 0 \\ * & * & * & \Psi_{(N4,4)} & R_2 & 0 & 0 \\ * & * & * & * & \Psi_{(N5,5)} & 0 & 0 \\ * & * & * & * & * & \Psi_{(N6,6)} & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$\Psi_{11} = P\tilde{A}_{\mu\hat{\mu}} + \tilde{A}^T_{\mu\hat{\mu}}P + Q_1 - W_1 + S_0 + S_1,\ \Psi_{nn} = -Q_{n-1} + Q_n - W_{n-1} - W_n$$
$$\Psi_{(N1,1)} = S_2 - R_1 - Q_N - W_N,\ \Psi_{(N2,2)} = -(1-d_1)S_0 - 2R_1$$
$$\Psi_{(N3,3)} = S_3 - S_2 - R_1 - R_2,\ \Psi_{(N4,4)} = -(1-d_2)S_1 - 2R_2$$
$$\Psi_{(N5,5)} = -S_3 - R_2,\ \Psi_{(N6,6)} = \sum_{n=1}^{N} h^2 W_n + (\rho\delta)^2 R_1 + ((1-\rho)\delta)^2 R_2$$

and

$$\Omega_1 = \begin{bmatrix}\tilde{E}^T P & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$
$$\Omega_{2\mu\hat{\mu}} = \begin{bmatrix}\tilde{H}_{c\mu\hat{\mu}} & 0 & \cdots & 0 & \delta(t)\tilde{H}_{\tau c\mu\hat{\mu}} & 0 & (1-\delta(t))\tilde{H}_{\tau c\mu\hat{\mu}} & 0 & 0 & \tilde{H}_{d\mu}\end{bmatrix}$$

By separating uncertain part of inequality (45) and using Lemma 2.1, the condition (45) is equivalent to the existence of $\varepsilon > 0$ such that

$$\begin{bmatrix}\Psi_{\mu\hat{\mu}} & \Gamma^T_{3\mu} & \varepsilon\Omega_1^T & \Omega^T_{2\mu\hat{\mu}} \\ * & -I & 0 & 0 \\ * & * & -\varepsilon I & 0 \\ * & * & * & -\varepsilon I\end{bmatrix} < 0 \tag{46}$$

Since matrices P and $A_{c\mu\hat{\mu}}$, $A_{\tau c\mu\hat{\mu}}$, $B_{c\hat{\mu}}$ are multiplied in equality (46), it is non-convex. Therefore, to derive LMI conditions, let symmetric positive definite matrix P and its inverse matrix P^{-1} be partitioned as

$$P = \begin{bmatrix}Y & N \\ * & W\end{bmatrix},\ P^{-1} = \begin{bmatrix}X & M \\ * & Z\end{bmatrix}$$

Since $PP^{-1} = I$, where $N^T X + W M^T = 0$, $YM + NZ = 0$, we denote

$$F_1 = \begin{bmatrix}X & I \\ M^T & 0\end{bmatrix},\ F_2 = \begin{bmatrix}I & Y \\ 0 & N^T\end{bmatrix}$$

and then it follows that $PF_1 = F_2$.

Let $\Upsilon_1^T = \text{diag}\{\overbrace{F_1^T, F_1^T, \ldots, F_1^T, F_1^T, F_1^T, F_1^T, I}^{(N+7)}\}$, pre- and post-multiplying (46) by the diag $\left\{\Upsilon_1^T, I, I, I\right\}$ and its transpose, respectively, and denoting

$$\hat{A}_{\mu\hat{\mu}} = Y^T(A_\mu + BC_{\hat{\mu}})X + NB_{c\hat{\mu}}C_\mu X + Y^T BC_{c\hat{\mu}}M^T + N\left(A_{c\mu\hat{\mu}}M^T\right)$$
$$\hat{A}_{\tau\mu\hat{\mu}} = Y^T\left(A_{\tau\mu} + BC_{\tau\hat{\mu}}\right)X + NB_{c\hat{\mu}}C_{\tau\mu}X + Y^T BC_{\tau c\hat{\mu}}M^T + N\left(A_{\tau c\mu\hat{\mu}}M^T\right)$$
$$\hat{B}_{\hat{\mu}} = NB_{c\hat{\mu}},\ \hat{C}_{\hat{\mu}} = C_{\hat{\mu}}X + C_{c\hat{\mu}}M^T,\ \hat{C}_{\tau\hat{\mu}} = C_{\tau\hat{\mu}}X + C_{\tau c\hat{\mu}}M^T$$

we have

$$\Xi_A^{\mu\hat{\mu}} = F_1^T P\tilde{A}_{\mu\hat{\mu}}F_1 = \begin{bmatrix}A_\mu X + B\hat{C}_{\hat{\mu}} & A_\mu + BC_{\hat{\mu}} \\ \hat{A}_{\mu\hat{\mu}} & Y^T\left(A_\mu + BC_{\hat{\mu}}\right) + \hat{B}_{\hat{\mu}}C_\mu\end{bmatrix} \tag{47}$$

The equal conditions to satisfy (47) when $i, j = 1, 2, \ldots, r$ are as follows:

$$\hat{\Xi}_A^{ij} = F_1^T P\tilde{A}_{ij}F_1 = \begin{bmatrix}A_i X + B\hat{C}_j & A_i + BC_j \\ \hat{A}_{ij} & Y^T(A_i + BC_j) + \hat{B}_j C_i\end{bmatrix} \tag{48}$$

Similarly,

$$\hat{\Xi}_{A_\tau}^{ij} = F_1^T P\tilde{A}_{\tau ij}F_1 = \begin{bmatrix}A_{\tau i}X + B\hat{C}_{\tau j} & A_{\tau i} + BC_{\tau j} \\ \hat{A}_{\tau ij} & Y^T(A_{\tau i} + BC_{\tau j}) + \hat{B}_j C_{\tau i}\end{bmatrix} \tag{49}$$

$$\hat{\Xi}_{B_\omega}^{ij} = F_1^T P \tilde{B}_{\omega ij} = \begin{bmatrix} B_{di} + BD_{dj} & -B \\ Y^T(B_{di} + BD_{dj}) + \hat{B}_j D_{di} & -Y^T B \end{bmatrix} \tag{50}$$

$$\hat{\Omega}_1 = \begin{bmatrix} \begin{pmatrix} \lambda E & 0 \\ \lambda Y^T E & \lambda N E \end{pmatrix} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

$$\hat{\Omega}_{2ij} = \begin{bmatrix} \hat{\Omega}_{2,1}^{ij} & 0 & \cdots & 0 & \delta_0 \hat{\Omega}_{2,N2}^{ij} & 0 & (1-\delta_0)\hat{\Omega}_{2,N4}^{ij} & 0 & 0 & \hat{\Omega}_{2,N7}^{ij} \end{bmatrix} \tag{52}$$

where

$$\hat{\Omega}_{2,1}^{ij} = \begin{bmatrix} XH_{ci}^T & -MH_{Aj}^T \\ H_{ci}^T & 0 \end{bmatrix},\ \hat{\Omega}_{2,N2}^{ij} = \hat{\Omega}_{2,N4}^{ij} = \begin{bmatrix} XH_{\tau ci}^T & -MH_{A\tau j}^T \\ H_{\tau ci}^T & 0 \end{bmatrix},\ \hat{\Omega}_{2,N7}^{ij} = \begin{bmatrix} H_{di}^T & 0 \\ 0 & 0 \end{bmatrix}$$

Then, it is easy to show that (45) is equivalent to the following inequalities:

$$\sum_{i=1}^{r} \mu_i(x(t))\mu_i\left(\hat{x}(t)\right)\hat{\Psi}_{ii} + \sum_{i=1}^{r}\sum_{i<j}^{r} \mu_i(x(t))\mu_j\left(\hat{x}(t)\right)\left(\hat{\Psi}_{ij} + \hat{\Psi}_{ji}\right) < 0 \tag{53}$$

Thus, if (42) and (43) are satisfied for all $i, j = 1, 2, \ldots, r$, then the closed-loop fuzzy system (40) with random time-varying delay is asymptotically stable (with $\tilde{\omega}(t) = 0$) while satisfying the prescribed H_∞ performance (41) when $\tilde{\omega}(t) \neq 0$. This completes the proof.