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DETERMINATION OF ALLOCATION RATE OF PRODUCTION PROJECTS UTILIZING RISK-SENSITIVE CONTROL THEORY

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Abstract. *This study is part of an ongoing report on an analysis of production processes using a lead-time function. Two types of production demand are classified in a production business. One is custom-type production (asynchronous type), which has a stochastic element. The other is mass-produced production (synchronous type), which has almost no stochastic element. We report an optimal production allocation to maximize the rate of increase in cash flow by these two types of production requests (complex type). Our approach is to take advantage of the risk-sensitive control method, which is a powerful technique that takes robustness into account. We present numerical results, which are optimal allocation and estimation calculation of our mathematical model. We apply optimal allocation to a production flow system as an example.*

Keywords: Optimal allocation, Risk-sensitive control, Log-normal distribution, Riccati equation, Financial theory

1. **Introduction.** Based on mathematical and physical understanding of production engineering, we are conducting research aimed at establishing an academic area called mathematical production engineering. As our business size is a small-to-medium-sized enterprise, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, when considering human intervention from outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of such intervention.

With respect to mathematical modeling of deterministic systems, a physical model of the production process was constructed using a one-dimensional diffusion equation in 2012 [1]. However, many concerns that occur in the supply chain are major problems facing production efficiency and business profitability. A stochastic partial bilinear differential equation with time delay was derived for outlet processes. The supply chain was modeled by considering as time delay [2]. With respect to the analysis of production processes in stochastic systems based on financial engineering, we have proposed that a production throughput rate can be estimated utilizing a Kalman filter based on a stochastic differential equation [3]. We have also proposed a stochastic differential equation (SDE) for the mathematical model describing production processes from the input of materials to the end. We utilized a risk-neutral principal in stochastic calculus based on the SDE [4].

With respect to the analysis of production processes based on physics, we have clarified that phenomena such as power-law distributions, self-similarity, phase transitions, and on-off intermittency can occur in production processes [5, 7, 8, 9, 10]. On the other hand, there is the famous theory of constraints (TOC) that describes the importance of avoiding bottlenecks in production processes [11]. Small fluctuations in an upstream subsystem appear as large fluctuations in the downstream (the so-called bullwhip effect) [12]. The bullwhip effect generates a large gap between the demand forecasts of the market and suppliers. Large fluctuations can be suppressed by the following mechanisms.

(1) Reducing the lead time, improving the throughput, and synchronizing the production process by the TOC.
(2) Sharing the demand information and performing mathematical evaluations.
(3) Analyzing the reduction and fluctuating demands of the subsystem (using nonlinear vibration theory).
(4) Basing the inventory management approach on stochastic demand.

In our previous studies, when using manufacturing equipment, delays in one production step are propagated to the next. Hence, the use of manufacturing equipment itself may lead to delays. The improvement of production processes was presented that the "synchronization with preprocess" method was the most desirable in practice using the actual data in production flow process based on the cash flow model by using the SDE of log-normal type [13]. In essence, we have proposed the best way, which is a synchronous method using the Vasicek model for mathematical finance [18]. Then, the supply chain theme, which was a time delay in the production processes, was proposed for the throughput improvement based on a stochastic differential equation of log-normal type [12].

Moreover, the analysis of the synchronized state indicated that this state was a much better method from the viewpoint of potential energy [12, 15]. We have also shown that the phase difference between stages in a process corresponded to the standard deviation of the working time [17]. When the phase difference was constant, the total throughput could be minimized. We showed that a synchronous process could be realized by the gradient system. The above problem is not limited to small- and medium-sized companies; in all cases, human interventions that directly affect the production process present a major challenge.

In general, we may reasonably consider that human interventions within and outside of the production system (internal and external forces, respectively) introduce uncertainties into the system's progress [4, 18]. The production system is formed by connecting both elements. When human intervention from outside companies involves an uncertainty, the noise element is frequently overlooked; instead, researchers have focused on efficient production or manufacturing the best system. Moreover, by including the noise element, we can recognize the unique advantage of the system.

In this study, we present the construction of a mathematical model of production rate increase in production business. The difference between a production order amount and production demand is derived from the probability density function of the log-normal type [5]. To this end, the allocation ratio of each production volume must be determined. This research addresses the maximization problem of expectation value of production rate increase as a composite system of each production system. Therefore, our objective is to decide the optimum value of the production allocation factor. We calculate the Bellman equation obtained from the maximization problem. As the solution of the Bellman equation satisfies a Riccati-type differential equation, we calculate the optimal portfolio function and the expected value maximization function (the expected optimum increase rate) after obtaining the solution of the Bellman equation. As a result, we present the approach used in this study as a method to evaluate a strategy for optimizing the structure of a production business. To the best of our knowledge, a determination of an optimal capacity has not been undertaken in previous studies.

2. Production Process Analysis for Lead-Time Function.

2.1. Production flow process. In Figure 1, the production methods used in manufacturing equipment are briefly covered in this paper. More information is provided in our report [5]. This system is considered to be a "make-to-order system with version control", which enables manufacturing after orders are received from clients, resulting in "volatility" according to its delivery date and lead time. In addition, there is volatility in the lead time, depending on the content of the make-to-order products (production equipment).

A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the production process. In Figure 2, the process consists of six stages. In each step S1-S6 of the manufacturing process, materials are being produced.

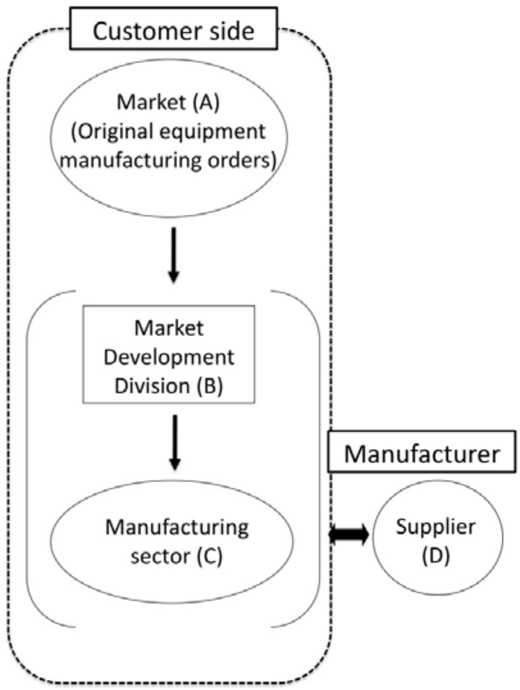

FIGURE 1. Business structure of company of research target

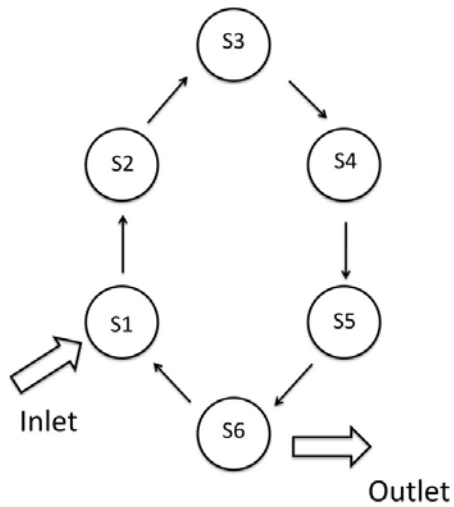

FIGURE 2. Production flow process

The direction of the arrows represents the direction of the production flow. Production materials are supplied through the inlet and the end-product is shipped from the outlet [13].

2.2. Lead time function. We define an expected production volume for a demand x as follows.

Definition 2.1. *Expected production volume $S_d(x)$ for a demand x.*

$$S_d(x) = \int_{-\infty}^{\infty} f(x) \cdot (px + q)dx \tag{1}$$

Figure 3 shows that a throughput is proportional to a rate of return in production processes. Then, we introduce the lead-time function so that we can analyze a production process [6]. The lead-time function $f(y)$ is assumed as a log-normal probability density function so that we can calculate the lead time using a continuous expected value calculation and $px + q$ denotes an expected cash flow function with a constant parameter p (> 0) and q (> 0) as shown in Figure 4.

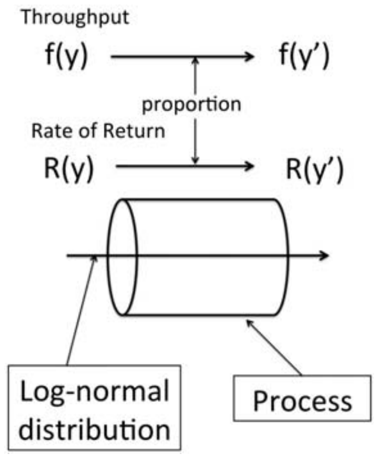

FIGURE 3. Throughput fluctuation in a process distribution amount

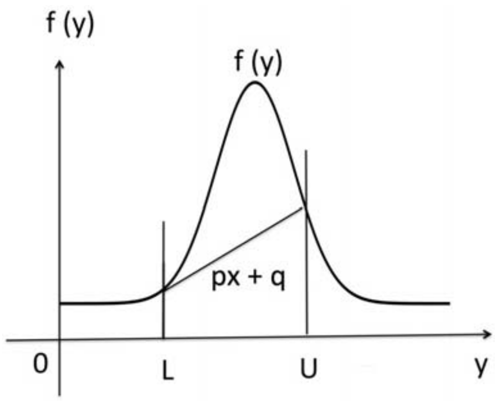

FIGURE 4. Lead time function $f(y)$ and loss function $(px + q)$

Assumption 2.1. *Lead time function of a probability density function with log-normal type* [13].

$$f(y) = \frac{1}{\sqrt{2\pi}(y - k_p)\sigma_p} \exp\left\{-\frac{1}{2}\left(\frac{(\ln y - k_p) - m}{\sigma}\right)^2\right\} \tag{2}$$

where k_p is a displacement of y, σ is a volatility and m is an average.

3. Production Model of Custom-Production Type and Mass-Production Type.

3.1. Description of complex production system.

We calculate the expectation value and volatility as follows [14]:

$$E_d(x) = \int_{-\infty}^{\infty} df(x)dx = \mu_d \tag{3}$$

$$V_d(x) = \int_{-\infty}^{\infty} (x - \mu_d)^2 df(x)dx = \sigma_d^2 \tag{4}$$

Definition 3.1. *External stochastic factor $C(x)$*

$$C(x) = S_d(x) - R_d + n_d \tag{5}$$

where $S_d(x)$ denotes the expected production considering demand distribution, x_d denotes the inventory volume and n_d denotes the amount of shortage.

At this time, we think about the positive and negative of expression $\{S_d(x_d) - x_d\}$.

- $S_d(x_d) - x_d > 0$: Excess production
- $S_d(x_d) - x_d < 0$: Under-production

where $x_d \equiv R_d - n_d$.

The probability density function of $\{S_d(x_d) - x_d\}$ is described as follows [14]:

$$g[S_d(x_d) - x_d] = f[-(S_d(x) - x_d) - \mu_d] \tag{6}$$

where let $S_d(x_d) \equiv S_d$ and $x = x_d$.

$$\begin{aligned} g[S_d - x_d] &= f[-(S_d - x_d) - \mu_s] \\ &= \frac{1}{\sqrt{2\pi}(S_d - x_d)\sigma_d} \exp\left\{-\frac{1}{2}\left(\frac{(\ln(S_d - x_d)) - (\mu_s - \mu_d)}{\sigma_d}\right)^2\right\} \end{aligned} \tag{7}$$

Then, the expectation value and volatility are derived as follows [14]:

$$\begin{aligned} E(S_d) &= \int_{-\infty}^{\infty} (S_d - x_d) g(S_d - x_d) d(S_d - x_d) \\ &= \frac{1}{\sqrt{2\pi}(S_d - x_d)\sigma_d} \int_{-\infty}^{\infty} (S_d - x_d) \exp\left\{-\frac{1}{2}\left(\frac{(\ln(S_d - x_d)) - (\mu_s - \mu_d)}{\sigma_d}\right)^2\right\} \\ &\quad \times d(S_d - x_d) \end{aligned} \tag{8}$$

$$\begin{aligned} V(S_d) &= \int_{-\infty}^{\infty} \left\{((S_d - x_d) - (\mu_s - \mu_d))^2 g(S_d - x_d)\right\} d(S_d - x_d) \\ &= \frac{1}{\sqrt{2\pi}(S_d - x_d)\sigma_d} \int_{-\infty}^{\infty} (S_d - x_d) \exp\left\{ -\frac{1}{2}\left(\frac{(\ln(S_d - x_d)) - (\mu_s - \mu_d)}{\sigma_d}\right)^2\right\} dx \\ &= \frac{1}{\sqrt{2\pi} x \sigma_d} \int_{-\infty}^{\infty} (x - \mu)^2 \exp\left\{ -\frac{1}{2}\left(\frac{(\ln(x) - \mu}{\sigma_d}\right)^2\right\} dx \end{aligned} \tag{9}$$

where $S(x) = S_d(x) - x_d$ and $\mu = \mu_s - \mu_d$.

At this time, according to Prof. Kohmura, a production adjustment is proposed as follows [19]:

$$S_i(x) = S_d(x) - \sum_{j=1}^{i-1} [S_j(x) - x_j] \tag{10}$$

where $S_i(x)$ denotes a stochastic production order amount, x_j denotes a demand amount and $i \neq j$.

The difference between a stochastic production order amount and production demand is derived from the probability density function of the log-normal type [5]. Therefore, we introduce the following equation.

Assumption 3.1.

$$S(x) = S_d(x) - x \tag{11}$$

where $S(x)$ denotes a stochastic production volume.

From our previous study [13], we define as follows [20].

Definition 3.2. *Stochastic production model $S(x)$ (Production volume) and $C(x)$ (External stochastic factors).*

$$dS(x) = S(x)\{(a + AC(x))dt\} + \sigma_s S(x) dW(x) \tag{12}$$

$$dC(x) = (b + BC(x))dt + \sigma_c dW(x) \tag{13}$$

We replace Equations (12) and (13) for simplicity as follows:

$$\mu_s(x) \equiv a + AC(x) \tag{14}$$

$$\mu_c C(x) \equiv b + BC(x) \tag{15}$$

where a, b, A, B are the market parameters. According to Equation (12), we define the mathematical model of a stochastic production system as a time variable t as follows:

$$dS(t) = \mu_s(t)S(t)dt + \sigma_s S(t)dW(t) \tag{16}$$

$$dC(t) = \mu_c C(t)dt + \sigma_c dW(t) \tag{17}$$

where $C(t)$ denotes the external stochastic factors, which represent a supply chain delay and logistic trouble delay, etc., $\mu_s(t)$ and μ_c denote an average value of trend respectively, and σ_s and σ_c denote a volatility respectively. $W(t)$ denotes the Wiener process.

4. **Business Allocation in a Production Business.** With respect to mathematical modeling of a no-risk system, a physical model of the production process was constructed using a one-dimensional diffusion process. In a real production business, business with no risk and business with risk are mixed. Determining the ratio between these businesses is important.

When a production business receives an order, production requests with uncertain specifications are included as well as established production projects. The most important objective of the business is to complete the uncertain project within the estimated manpower. Therefore, the ability to determine the portfolio ratio of uncertain production businesses becomes important.

We confirm the definition of stochastic and deterministic systems as follows.

- Deterministic system: Mass-production system (synchronous type) $\Rightarrow$ Production system of relatively small fluctuation factors.
- Stochastic system: Custom-type production system (asynchronous type) $\Rightarrow$ Depending on the stochastic fluctuations in demand, this is constrained by irregular fluctuation factors.

Then, we define the mathematical model of the deterministic production system as follows.

Definition 4.1. *Production volume $S^0(t)$ of the deterministic model*

$$dS^0(t) = r^0 S^0(t)dt \tag{18}$$

where r^0 denotes a risk free rate.

Thus, we define the complex system model as follows [24].

Definition 4.2. *Complex system model $G(t)$ of both the stochastic system and the deterministic system.*

$$\frac{dG(t)}{G(t)} = \frac{dS(t)}{S(t)} + \frac{dS^0(t)}{S^0(t)} = h^0(t)r(t)dt + h^s(t)\Big[\mu_s(t)S(t)dt + \sigma_s dW(t)\Big] \tag{19}$$

where h^0 and h^s denote the allocation factor and the portfolio of production business is derived as follows [24]:

$$h^0(t) + h^s(t) = 1 \tag{20}$$

From Equation (20), we obtain the following equation.

$$h^0(t) = 1 - h^s(t) \tag{21}$$

From Equation (21), Equation (19) can be modified as follows:

$$\begin{aligned}\frac{dG(t)}{G(t)} &= (1-h^s(t))r(t)dt + h^s(t)\Big[\mu_s(t)S(t)dt + \sigma_s dW(t)\Big] \\ &= r(t)dt + h^s(t)\Big[(\mu_s(t)S(t) - r(t))dt\Big] + h^s(t)\sigma_s dW(t)\end{aligned} \tag{22}$$

Thus, Equation (22) denotes a cash flow for both of a custom-type production (asynchronous type) and mass-production system (synchronous type).

4.1. **Optimal production ratio utilizing risk-sensitive control theory.** The purpose of this study is to maximize the rate of increase in cash flow (production rate) in a complex production system. We have two types of production systems: custom-produced and mass-production systems. Therefore, we define the evaluation function of a complex system as follows.

Definition 4.3. *Evaluation function of a complex system*

$$J(g,\theta,h,T) = -\frac{2}{\theta}\ln E\Big[e^{-\frac{\theta}{2}\ln G_T(h)}\Big] \tag{23}$$

Here, we consider the problem of finding the combination ratio. Nagai proposed several approaches to such a problem. We, in turn, propose applying Nagai's approaches to our complex production system.

$\ln G_T(h)$ in Equation (23) denotes a rate of increase in cash flow until time T, when we select the combination ratio h. θ is a risk sensitive parameter. For example, when $\theta \to 0$, Equation (23) is derived by asymptotic expansion as follows [20, 25]:

$$J(g,\theta,h,T) \simeq E\Big[e^{-\frac{\theta}{2}\ln G_T(h)}\Big] - \frac{\theta}{4}V_{ar}\Big[G_T(h)\Big] + O(\theta^2) \tag{24}$$

Thus, when θ $(> -2, \neq 0)$ and θ takes a value close to zero, the maximization problem of Equation (23) is equivalent to maximize the expectation of $\ln[G_T(h)]$ and to minimize the volatility [20, 25]. Here, $h(t)$ is measured function with respect to $\sigma(S(t),\theta,s\le t)$.

According to Nagai's approach, we put the following equation to y:

$$e^{-\frac{\theta}{2}\ln G_T(h)} = y \tag{25}$$

Then, when taking the logarithm of both sides in Equation (25), we obtain as follows:

$$-\frac{\theta}{2}\ln G_T(h) = \ln y \tag{26}$$

Consequently, we obtain as follows:

$$\ln G_T(h)^{-\frac{\theta}{2}} = \ln y \tag{27}$$

Hence, we apply Ito's theorem to $y = G_T(h)^{-\frac{\theta}{2}}$ and obtain as follows [25]:

$$\begin{aligned}\partial G_T(h)^{-\frac{\theta}{2}} = g^{-\frac{\theta}{2}}\exp\Bigg[&\int_0^T \frac{\theta}{2}\left\{\frac{1}{2}\left(\frac{\theta}{2}+1\right)h\sigma_s^2 h - r - h(\mu_s(t)-r)\right\}ds \\ &- \frac{\theta}{2}\int_0^T h(t)\sigma_s dW(t) - \frac{1}{2}\left(\frac{\theta}{2}\right)^2\int_0^T h\sigma_s^2 h ds\Bigg]\end{aligned} \tag{28}$$

From Nagai's approach, we obtain as follows [25]:

$$P(\xi) = E\left[\exp\left(-\frac{\theta}{2}\int_0^T h(t)\sigma_s dW(t) - \frac{1}{2}\left(\frac{\theta}{2}\right)^2\int_0^T h(s)\sigma_s^2 h(s)ds : \xi\right)\right] \tag{29}$$

We obtain as follows after stochastic measure translation:

$$\hat{W}(t) = W(t) + \frac{\theta}{2}\int_0^T h(s)\sigma_s ds \tag{30}$$

Equation (30) denotes a Brownian motion.

Let T be $(T-t)$. To maximize $J(\bullet)$ is equivalent to consider the following equation [25]:

$$\begin{aligned} & g(t,C) \\ &= \sup_{h(\cdot)}\left[-\frac{\theta}{2}\ln E\left[g^{-\frac{\theta}{2}}\cdot\exp\left(\int_0^T \frac{\theta}{2}\left\{\left(\frac{\theta}{2}+1\right)h(s)\sigma_s^2 h(s) - r\right\} - h(\mu_s(t)-r)\right)ds\right]\right] \end{aligned} \tag{31}$$

$$\Phi(C,h,r:\theta) \equiv \left(\frac{\theta}{2}+1\right)h(s)\sigma_s^2 h(s) - r - h(\mu_s(t)-r) \tag{32}$$

$C(t)$ in Equation (31) is derived as follows:

$$dC(t) = \left(\mu_c C(t) - \frac{\theta}{2}\sigma_s\sigma_c h(t)\right)dt + \sigma_c dW(t) \tag{33}$$

Consequently, Bellman's equation is derived as follows [25]:

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma_c^2\frac{\partial^2 g}{\partial C^2} + \inf_h\left[\left\{\left(\mu_c C - \frac{\theta}{2}\sigma_c\sigma_s h\right)\frac{\partial g}{\partial C} + \frac{\theta}{2}\Phi(C,h,r:\theta)g\right\}\right] = 0 \tag{34}$$

where $g(T,C) = g^{-\frac{\theta}{2}}$.

At this time, $g(t,C)$ is derived as follows:

$$g(t,C) = \frac{1}{2}P(t)C^2(t) + k(t)C(t) \tag{35}$$

Moreover, $P(t)$ is obtained by solving Riccati type differential equation as follows [25]:

$$\frac{dP(t)}{dt} - P^2(t)K_0 + 2K_1P(t) + \frac{2}{\theta+2}\mu_s(t)^2\sigma_s^2 = 0 \tag{36}$$

$$P(T) = 0 \tag{37}$$

Then, $k(t)$ is obtained by solving the following equation:

$$\frac{dk(t)}{dt} + P^2(t)\sigma_s^2 = 0 \tag{38}$$

where $k(T) = 0$.

Therefore, $h^s(t)$ is derived as follows:

$$h^s(t) = \frac{2\sigma_s^{-2}}{\theta+2}\{\mu_s(t) - r\} - \frac{\theta}{2}\sigma_c\sigma_s\{P(t)C(t) + k(t)\} \tag{39}$$

Further, the expected optimal rate of increase can be calculated as follows:

$$J(g,C;h,T) = \frac{1}{2}P(\bullet)C^2 + k(\bullet)C \tag{40}$$

K_0 and K_1 in Equation (36) denote as follows:

$$\begin{aligned} K_0 &= \frac{\theta}{2}\sigma_c\left(1 - \frac{\theta}{\theta+2}\right)\sigma_c \\ K_1 &= B - \frac{\theta}{\theta+2}(\sigma_s\sigma_c)\left(\sigma_c^2\right)^{-1}\cdot\mu_s(t) \end{aligned} \tag{41}$$

Hence, with respect to a solution method of Equation (36), we present again as follows:

$$\frac{dP(t)}{dt} - P^2(t)K_0 + 2K_1P(t) + \frac{2}{\theta+2}\mu_s(t)^2\sigma_s^2 = 0 \tag{42}$$

$$P(T) = 0 \tag{43}$$

In Equation (42), let $t = T - \tau$ and $\tau \in [0, T]$ under $P(0) = 0$. From Equation (42), we obtain as follows:

$$\frac{dP(\tau)}{d\tau} = 2K_1P(\tau) - K_0P^2(\tau) + b^2 \tag{44}$$

where b^2 is as follows:

$$b^2 = \left(\frac{2}{\theta+2}\right)\mu_s(t)^2\sigma_s^2 \tag{45}$$

With respect to the solution of Equation (44), please refer to Appendix A for the derivation process.

4.2. **Estimation of $\boldsymbol{E_t[C(t)]}$.** The average and volatility of $C(t)$ are defined respectively as follows.

Definition 4.4. *Average $\mu(t)$ and volatility $\nu(t)$ of $C(t)$.*

$$\mu(t) = E[C(t)|\mathscr{F}_t^w] \tag{46}$$

$$\nu(t) = V_{ar}[C(t)|\mathscr{F}_t^w] \tag{47}$$

where $w(0) = 0$ and $\mathscr{F}_t^w$ denotes filtration. We can obtain a theoretical solution of Riccati type equation describing $\nu(t)$ in Equations (50) and (51). It is assumed that the estimation values of original equation depend on the estimation values of average $\mu(t)$ in case of volatility fixed.

Generally, we can represent by using an estimation factor g as follows:

$$dC(t) = \mu_c C(t)dt + \sigma_C C(t)dW(t) \tag{48}$$

$$dw(t) = gC(t)dt + \sigma_w dW(t) \tag{49}$$

where $g \neq 0$ and σ_w denotes a volatility.

Though the information obtained from the original mathematical model is uncertain, it is assumed that the initial distribution $N(\mu_0, \nu_0)$ is known. From Kalman-Bucy's filter theory, we obtain as follows [26]:

$$d\mu(t) = \left[\mu_c - \frac{\nu(t)}{\sigma_w^2}\right]\mu(t)dt + \frac{\nu(t)}{\sigma_w^2}dZ(t) \tag{50}$$

$$\frac{d\nu}{dt} = 2\mu_c\nu(t) - \frac{g^2}{\sigma_w^2}\nu^2(t) + \sigma_C^2 \tag{51}$$

$$\nu(0) = a_0^2 \tag{52}$$

where a_0 denotes the initial value.

Equation (51) can be solved as follows [26]:

$$\nu(t) = \frac{\alpha_1 + K\alpha_2 \exp\left\{\frac{(\alpha_2-\alpha_1)g^2}{\sigma_w^2}t\right\}}{1 + K\exp\left\{\frac{(\alpha_2-\alpha_1)g^2}{\sigma_w^2}t\right\}} \tag{53}$$

where α_1, α_2 and K in Equation (54) are derived as follows:

$$\alpha_1 = g^{-2}\left(-\mu_c\sigma_w^2 + \sigma_w\sqrt{\mu_c^2\sigma_w^2 + g^2\sigma_C^2}\right) \tag{54}$$

$$\alpha_2 = g^{-2}\left(-\mu_c\sigma_w^2 + \sigma_w\sqrt{\mu_c^2\sigma_w^2 + g^2\sigma_C^2}\right) \tag{55}$$

$$K = \frac{\alpha_1 - a_0^2}{a_0^2 - \alpha_2} \tag{56}$$

With respect to $h(t)$, we put $h(t)$ as follows:

$$h(t) = \mu_c - \frac{\nu(t)}{\sigma_w^2} \tag{57}$$

From Equations (50) and (57), we can obtain as follows:

$$d\mu(t) = h(t)\mu(t)dt + \frac{\nu(t)}{\sigma_w^2}dZ(t) \tag{58}$$

where $\mu(0) = E[C(0)]$.

We define $f(t,\mu)$ as follows.

Definition 4.5.

$$f(t,\mu) = \exp\left\{-\int_0^t h(u)du\right\}\mu(t) \tag{59}$$

We obtain by applying Itoh's theorem to Equation (59) as follows:

$$\begin{aligned}
&\exp\left\{-\int_0^t h(u)du\right\}\mu(t) - \mu(0) \\
&= \int_0^t \left(-h(s)\exp\left\{-\int_0^t h(u)du\right\}\mu(s)\right)ds + \int_0^t \exp\left\{\int_0^t h(u)du\right\}d\mu(s) \\
&= \int_0^t \exp\left\{\int_0^t h(u)du\right\}\cdot\frac{g}{\sigma_w^2}\nu(s)dw(s)
\end{aligned} \tag{60}$$

From Equation (60), we obtain $\mu(t)$ as follows:

$$\mu(t) = \mu(0)\exp\left\{\int_0^t h(u)du\right\} + \frac{g}{\sigma_w^2}\int_0^t \exp\left\{\int_0^t h(u)du\right\}\cdot\nu(s)dw(s) \tag{61}$$

Then, when $t\to\infty$, $\alpha_1 > 0$ and $\alpha_2 < 0$ from Equation (54). Thus, $\nu(t)\simeq\alpha_1$. Here, Equation (61) is derived as follows:

$$\mu(t) \simeq \mu(0)\exp\left\{\left(\mu_c - \frac{g^2\alpha_1}{\sigma_w^2}\right)t\right\} + \frac{g\alpha_1}{\sigma_w^2}\int_0^t \exp\left\{\left(\mu_c - \frac{g^2\alpha_1}{\sigma_w^2}\right)(t-s)\right\}dw(s) \tag{62}$$

Moreover, let $\mu_c - \frac{g^2\alpha_1}{\sigma_w^2} = -\beta$, and we obtain as follows:

$$\mu(t) \simeq \exp(-\beta t)\cdot\left(\mu(0) + \frac{g^2\alpha_1}{\sigma_w^2}\int_0^t \exp(\beta s)dw(s)\right) \tag{63}$$

Assuming that $g = 1$, Equation (63) becomes the solution of system's Equations (46), (47) and (48). Therefore, from Equation (63), the estimated value $\mu(t)$ can be obtained as Equation (63). Therefore, the explicit solution of portfolio function is derived as follows:

$$h^s(t) \simeq \frac{2\sigma_s^{-2}}{\theta+2}\{\mu_s(t)\cdot C(t) - r\} - \frac{\theta}{2}\sigma_c\sigma_s P(t)C(t) \tag{64}$$

Thus, according to Equation (64), the expected value $h(t)$ can be calculated as follows:

$$E_t[h(t)] \simeq \frac{2\sigma_s^{-2}}{\theta+2}\{\mu_s(t)E_t[C(t)] - r\} - \frac{\theta}{2}\sigma_c\sigma_s\{P(0)E_t[C(t)]\}$$

$$\simeq \frac{2\sigma_s^{-2}}{\theta+2}\{\mu_s(t)\cdot\mu(t)-r\}-\frac{\theta}{2}\sigma_c\sigma_s\{P(0)\cdot\mu(t)\} \tag{65}$$

As described above, represent a model for the increasing rate of production volume in the production system as Equations (19) and (20). In addition, we discussed the problem of determining the allocation factor of the production volume of each production system. Our objective is to calculate the optimum value of the allocation factor, which involves defining the maximization problem of expected value of the increasing rate in a production system at that time. Thus, we calculate the Bellman equation obtained from the maximization problem. Knowing for a fact that the explicit solution of the demonstrating function satisfies the Riccati-type differential equation, we compute the solution of this equation, optimal allocation factor, and expected value of the maximization function (i.e., the expected value of the optimal increasing rate). Thus, our novel approach in this study is a useful method to evaluate strategies for optimizing the structure of a production system.

4.3. **Calculation example of risk free rate r.** We set all factors to 1 except θ for simplicity and we obtain as follows:

$$\frac{dP}{d\tau}=2\lambda P(\tau)-\frac{\theta}{2}\lambda P^2(\tau)+b^2 \tag{66}$$

where $\lambda = 1-\frac{\theta}{\theta+2}$, $P(0)=0$, $\tau = T-\tau$.

Thus, we obtain $P(\tau)$ as follows:

$$P(\tau)=\frac{1}{K_0}\cdot\beta_1\cdot\frac{1-\exp(\beta_2-\beta_1)\tau}{1-\left(\frac{\beta_1}{\beta_2}\right)\exp(\beta_2-\beta_1)\tau} \tag{67}$$

Therefore, as $\tau\to\infty$, P is derived as follows:

$$P\simeq K_0^{-1}\cdot\beta_1=\frac{2}{\theta\lambda}\beta_1,\quad K_0=\frac{\theta}{2}\lambda \tag{68}$$

Consequently, $h(\tau)$ is derived approximately as follows:

$$h(\tau)\cong\frac{2\sigma_s^{-2}}{\theta+2}(\mu_s(t)C(\tau)-r)-\frac{\theta}{2}\sigma_c\sigma_s\left\{P(\tau)\bar{C}(\tau)\right\} \tag{69}$$

In Equation (69), as $\bar{C}(t)$ denotes an expectation of $C(t)$, we replace $\bar{C}(t)$ as follows:

$$\bar{C}(t)=E_t[C(t)] \tag{70}$$

where σ_s and σ_c denote a risk value, and let $a=0$ and $A=1$, the steady-state value $\bar{h}$ is derived as follows:

$$\bar{h}\cong\frac{2\sigma_s^{-2}}{\theta+2}(E_t[C(t)]-r)-\frac{\theta}{2}\sigma_c\sigma_s P(t)E_t[C(t)] \tag{71}$$

$\bar{h}$ in Equation (71) is called as a risk distributed factor.

Since all of the above condition can be set to 1 except for θ, Equation (71) can be written as follows:

$$\bar{h}\cong\frac{2\sigma_s^{-2}}{\theta+2}(E_t[C(t)]-r)-\frac{\theta}{2}\left[\frac{2\beta_1}{\theta\lambda}\right]\bar{C}(t)=\frac{2\sigma_s^{-2}}{\theta+2}(E_t[C(t)]-r)-\frac{\beta_1}{\lambda}\bar{C}(t) \tag{72}$$

Therefore, as $\bar{h}<1$, an establishment condition of this system is obtained as follows:

$$\frac{2\sigma_s^{-2}}{\theta+2}(E_t[C(t)]-r)-\frac{\beta_1}{\lambda}E_t[C(t)]<1 \tag{73}$$

Then, we try to calculate a numerical example as follows.

$$\beta_1 = -\lambda + \sqrt{\lambda^2 + \left(\frac{\theta}{\theta+2}\right) \times \frac{\theta\lambda}{2}} = -\frac{2}{3} + \sqrt{\left(\frac{2}{3}\right)^2 + \frac{4}{9}} \approx 0.08 \tag{74}$$

$$P(\infty) \cong \frac{2\beta_1}{\theta\lambda} = \frac{2 \times 0.08}{\frac{2}{3}} \approx 0.24 \tag{75}$$

$$h = \frac{2}{3}\left\{-E_t[C(t)] - r\right\} - \frac{0.24}{2} E_t[C(t)] \approx 0.546 \cdot E_t[C(t)] - 0.666r \tag{76}$$

According to $h < 1$, we set the risk free rate as follows from Equation (76).

$$\begin{aligned} 0.546E_t[C(t)] &< 0.666r \\ r &> \frac{0.546}{0.666} \approx 0.82E_t[C(t)] \end{aligned} \tag{77}$$

5. Numerical Results.

5.1. Numerical simulation.

It is concluded as follows from the above results.

- For the larger trend coefficient $\mu_s(t)$ indeed, $h^s(t)$ is increased.
- For the smaller volatility σ_s indeed, $h^s(t)$ is increased.
- For a change in θ, $h^s(t)$ is not affected.

In the case of the target company, a closest value is the data in substantially the table data from Equation (77) as follows.

Deterministic system : Stochastic system

0.8 : 0.2

In the case of a high ratio of the deterministic system, the production system forms a synchronization process and is smoothed by the introduction of the production flow system. As detailed in the aforementioned results, we obtained similar results in the case of the sales ratio. Figure 5 shows a solution process of stochastic differential equation with respect to the external factors. The initial value $\sigma = 0.1$, the average value $\mu = 0.73$ and the volatility $\sigma = 0.29$. Figure 6 shows an origin data and observation data. The average value $\mu = 0.73$, the volatility $\sigma = 0.29$ and the volatility of observation process is 0.1. Figure 7 shows a comparison of the original data and the observation data. The average value $\mu = 0.73$, the volatility $\sigma = 0.29$ in original process, the volatility of observation process is 0.1, the average of estimation process is 0.12 and the volatility of estimation process is 0.286. The data related to the observation process are obtained using Kalman filter theory. Figure 8 shows the original data, observation data, and estimation data. The average value $\mu = 0.73$ of a stochastic systematization of the original process, the volatility in the observation process is 0.1, average of the estimation process is 0.12, and volatility of the estimation process is 0.286. We summarize the numerical results, i.e., Figure 5 through Figure 8, as follows:

- Stochastic process (asynchronous process). Many processes involve lead time and because the completion date is irregular, extensive idle time occurs in the overall process. Thus, structuring a production flow process (synchronous production process) is difficult. Fluctuations in the process are relatively more likely to occur, including logistic risks with delays, production risks with specification changes, and manpower risks with the differences in workers' abilities.

- Deterministic process (synchronous process). This process points to a mass-production system such as the production flow process. In this method, the volatility of throughput is assumed to be suppressed through the control of the production flow. Please refer Appendix B with respect to test runs 2 and 3.

In Figure 9, the left edge of the "P" is the project number, the value of the brackets on the line represents the lead time, and "output" is complete ship respectively. Here, we obtained the mean value ($\mu_s(t) \approx 0.46$) and volatility ($\sigma_s \approx 0.27$) after calculation respectively based on normalization by using maximum lead time.

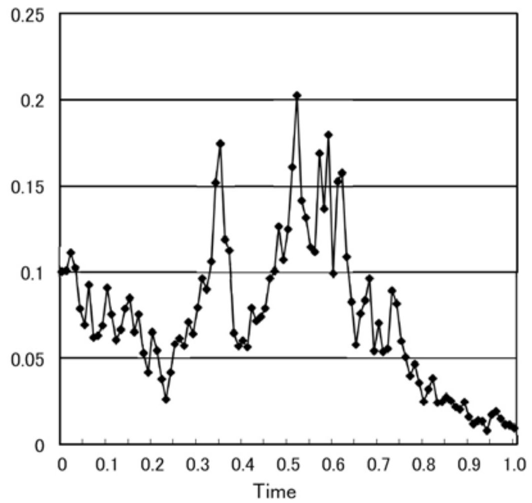

FIGURE 5. Solution process of stochastic differential equation with respect to the external factors

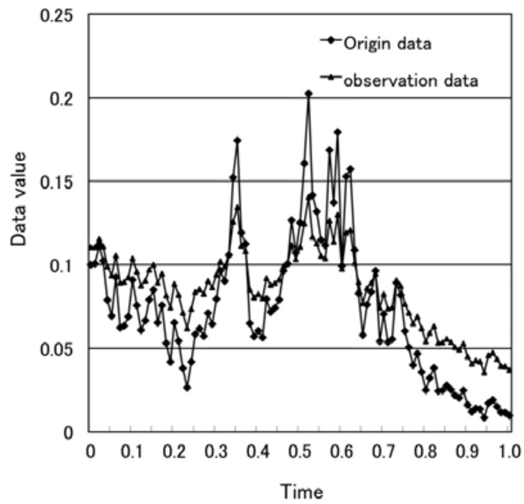

FIGURE 6. Origin data and observation data

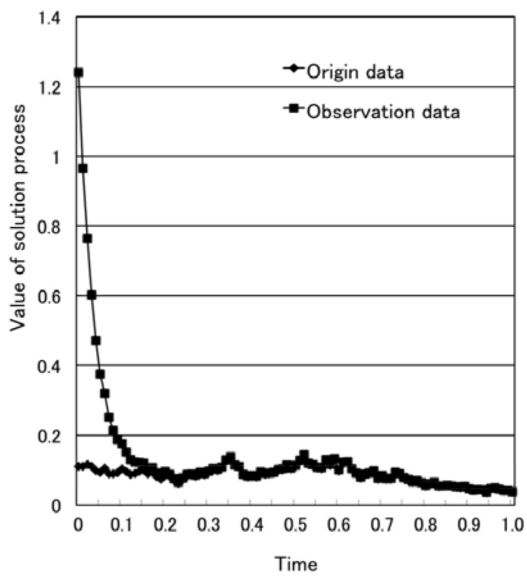

FIGURE 7. Comparison of the original system and the observation process

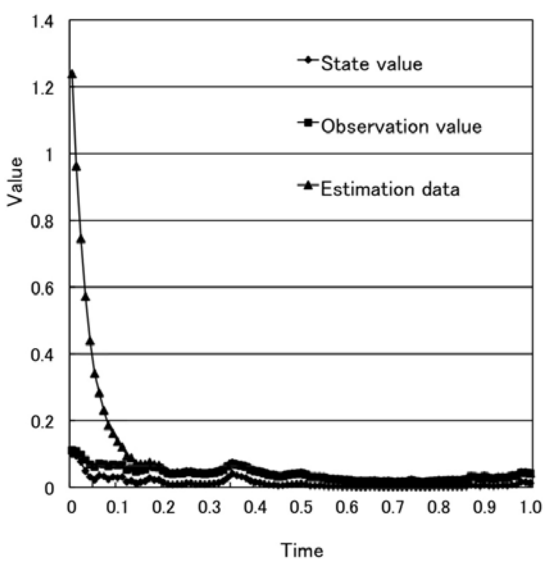

FIGURE 8. Original data, observation data and estimation data

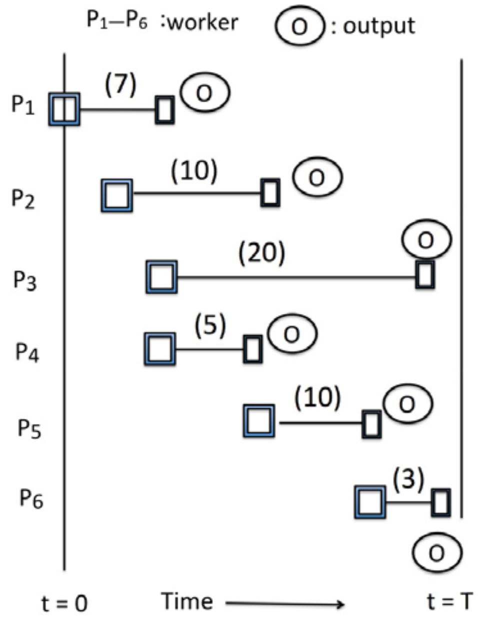

FIGURE 9. Typical process diagram of stochastic production system

The throughput (lead time) model in the actual data of test run 1 (an asynchronous process in Table 5 at Appendix B) is represented as follows [13]:

$$dS(t) = 0.73S(t)dt + 0.27S(t)dW(t) \tag{78}$$

where 0.73 denotes an average value of trend, 0.27 denotes a volatility and $W(t)$ denotes the Wiener process.

Similarly, the synchronization model is obtained as follows [13]:

$$dS(t) = 0.92S(t)dt + 0.06S(t)dW(t) \tag{79}$$

where 0.92 denotes an average value of trend, and 0.06 denotes a volatility.

Equation (79) denotes the test run 2 of the actual data referred in Appendix B. Alternatively, the synchronization model with improved processes is obtained as follows [13]:

$$dS(t) = 0.95S(t)dt + 0.03S(t)dW(t) \tag{80}$$

where 0.95 denotes an average value of trend, and 0.03 denotes a volatility.

Equation (80) denotes the test run 3 of the actual data referred in Appendix B. Thus, we represent an asynchronous model as follows, because it contains a risk factor.

$$\begin{aligned} dS(t) &= \mu_s(t)S(t)dt + \sigma_s S(t)dW(t) \\ dC(t) &= \mu_c C(t)dt + \sigma_c dW(t) \end{aligned} \tag{81}$$

where $S(t)$ denotes the production volume, $C(t)$ denotes the external stochastic factors, the mean value $\mu_s(t) = 0.46$ and volatility $\sigma_s = 0.27$ in real processes, the mean value $\mu_c = 0.73$ and volatility $\sigma_c = 0.29$ in an asynchronous throughput model.

5.2. **Determination production processes.** In a deterministic production process, we can get a large mean value and a small volatility as follows [13]:

$$dS(t) = 0.95S(t)dt + 0.06S(t)dW(t) \tag{82}$$

Basically, as we try to apply the financial engineering, we use the risk free rate r as follows.

Definition 5.1.

$$r = 1 - \mu_r = 1 - 0.95 = 0.05 \tag{83}$$

We call $r = 0.05$ as a subjective risk rate. However, we use $r = 0.1$ for the sake of convenience in this paper. Then, we obtain as follows:

$$dS^0(t) = r_0 S^0(t)dt \tag{84}$$

Figures 10-17 are calculated by the following equation. Table 7 denotes the parameters of Figures 10-17 respectively.

$$E_t[h_t] = \frac{2\sigma_s^{-2}}{\theta + 2}(\mu_s(t)E_t[C] - r) - \frac{\theta}{2}\sigma_c\sigma_s P_0 E_t[C] \tag{85}$$

We obtain the following results from the numerical simulation, which shows in Figures 10-17.

- If the trend coefficient $\mu_s(t)$ is large indeed, the portfolio factor $h_s(t)$ is increasing.
- If the volatility σ_s is small indeed, the portfolio factor $h_s(t)$ is increasing.
- If the risk sensitive parameter θ varies indeed, the portfolio factor $h_s(t)$ is not affected.

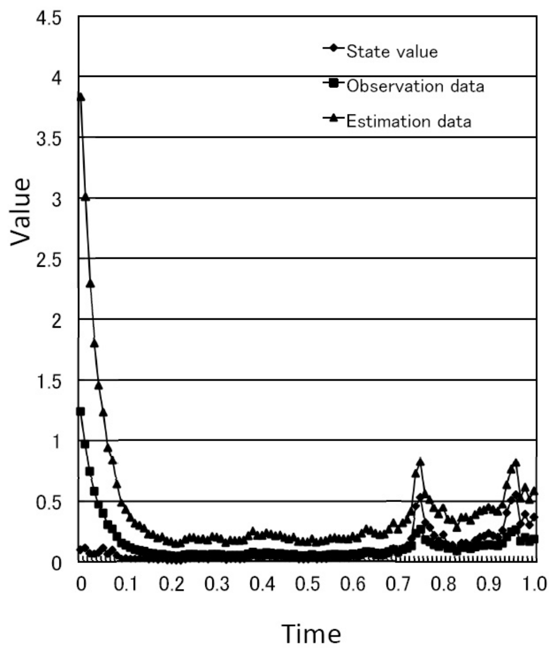

FIGURE 10. Risk sensitive optimal allocation function (Type 1)

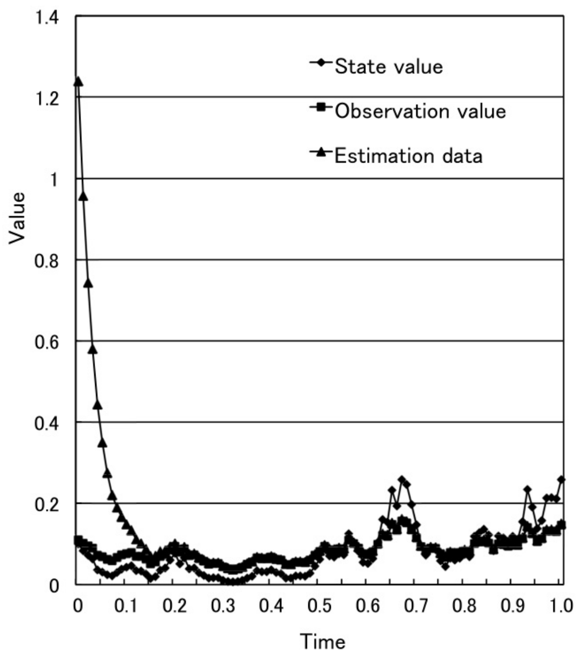

FIGURE 11. Risk sensitive optimal allocation function (Type 2)

6. **Dynamic Simulation of Production Processes.** We attempted to perform a dynamic simulation of the production process by utilizing the simulation system that NTT DATA Mathematical Systems Inc. (www.msi.co.jp) has developed. With respect to the meaning of the individual parts in Figure 24, we conducted a simulation of the following procedure. When the simulation began, it generated one of the products on a "start" part go to "finish".

- In each process, including the six workers in parallel, the slowest worker waited till the work was completed.
- When the work of each process was completed, it moved to the next process.

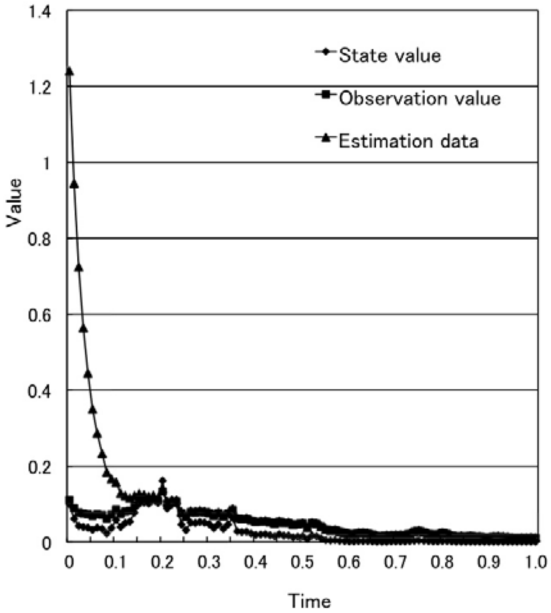

Figure 12. Risk sensitive optimal allocation function (Type 3)

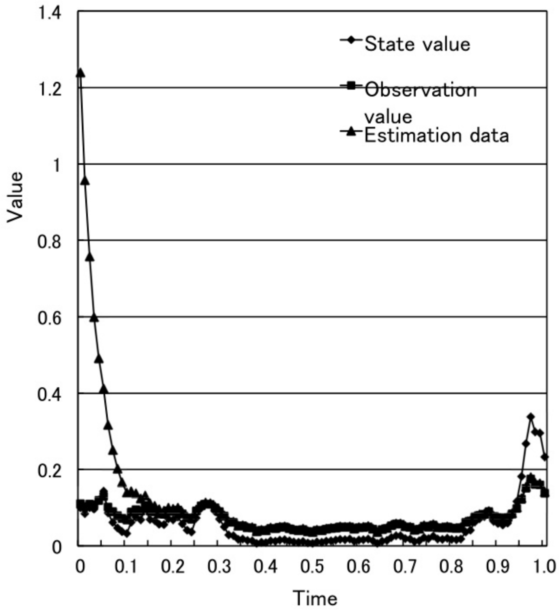

Figure 13. Risk sensitive optimal allocation function (Type 4)

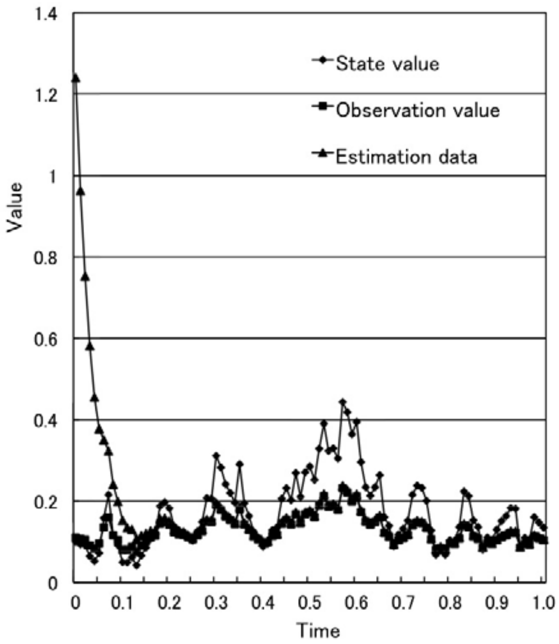

Figure 14. Risk sensitive optimal allocation function (Type 5)

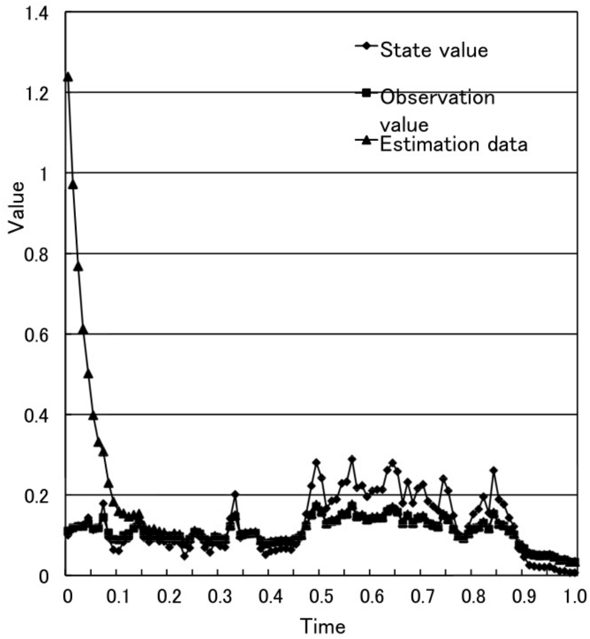

Figure 15. Risk sensitive optimal allocation function (Type 6)

- Simultaneously as each process was completed, it recorded the working time of each process.

With respect to Table 3 and Table 4,

- Process No. indicates each process (1-6).
- Average indicates the average time.
- STD indicates the standard deviation of process time (sec).

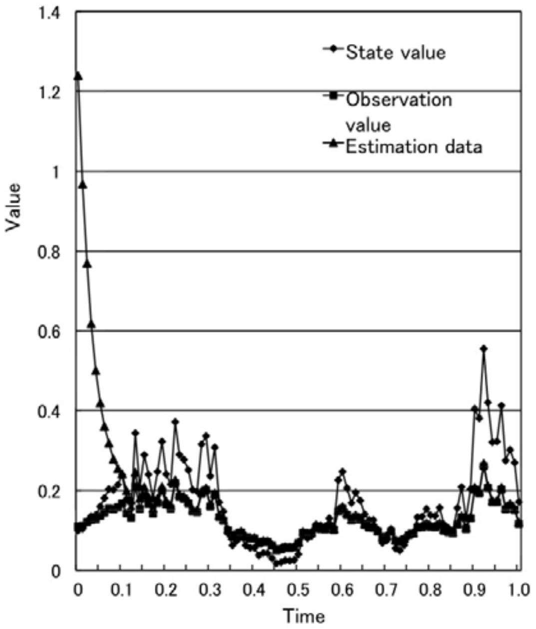

FIGURE 16. Risk sensitive optimal allocation function (Type 7)

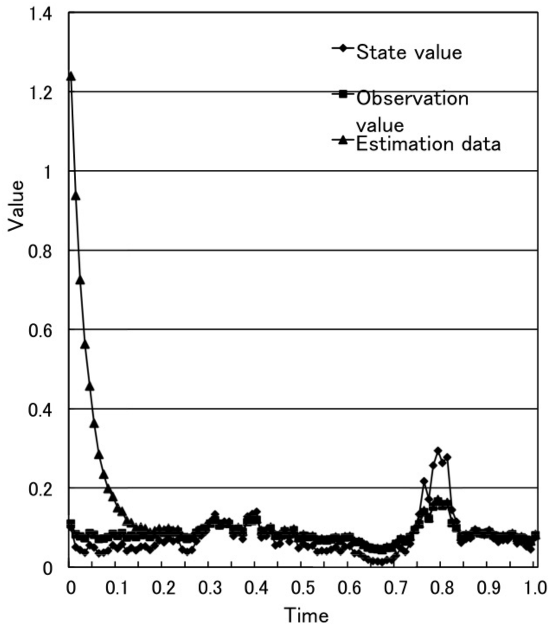

FIGURE 17. Risk sensitive optimal allocation function (Type 8)

TABLE 1. Total manufacturing time at each stage for each worker

Type\	$\mu_s(t)$	σ_s	σ_c	r	θ	h
1	0.46	0.27	0.29	0.1	2	0.20
2	0.6	0.27	0.29	0.1	2	0.51
3	0.3	0.27	0.29	0.1	2	0.10
4	0.46	0.4	0.29	0.1	2	0.09
5	0.46	0.6	0.29	0.1	2	0.001
6	0.46	0.2	0.29	0.1	2	0.35
7	0.46	0.27	0.29	0.1	1	0.37
8	0.46	0.27	0.29	0.1	0.5	0.40

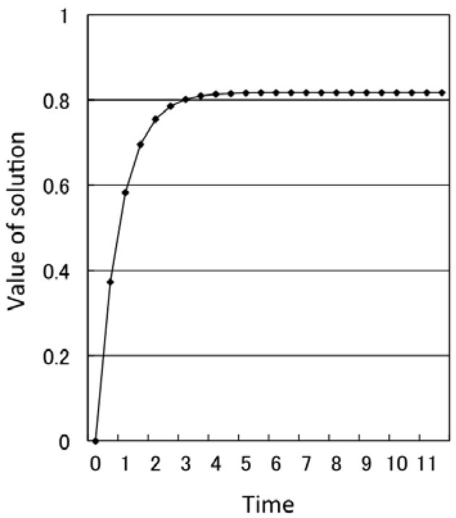

FIGURE 18. Solution of Riccati equation

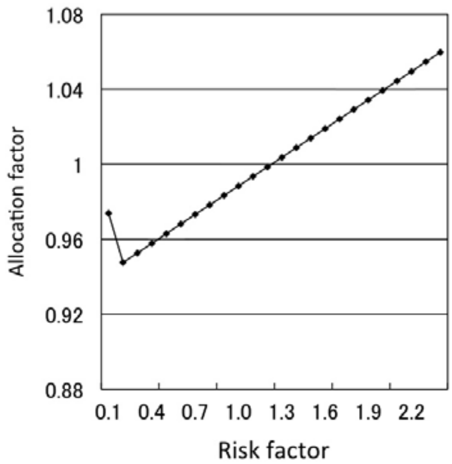

FIGURE 19. Allocation factor

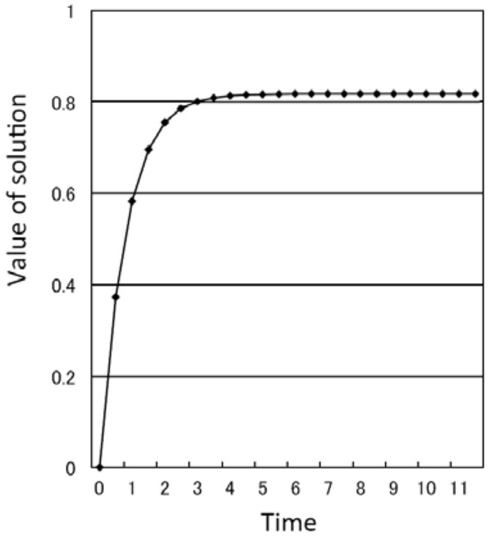

Figure 20. Solution of Riccati equation

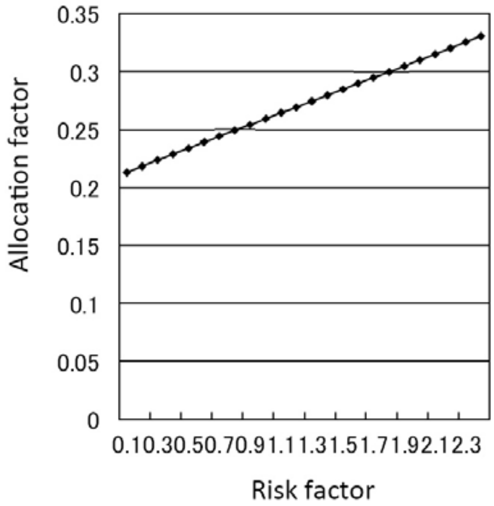

Figure 21. Allocation factor

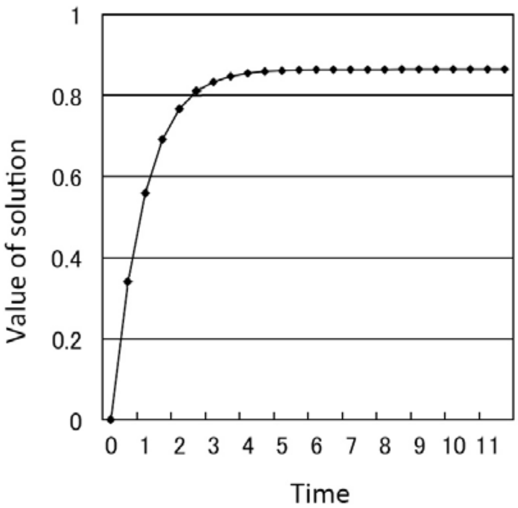

Figure 22. Solution of Riccati equation

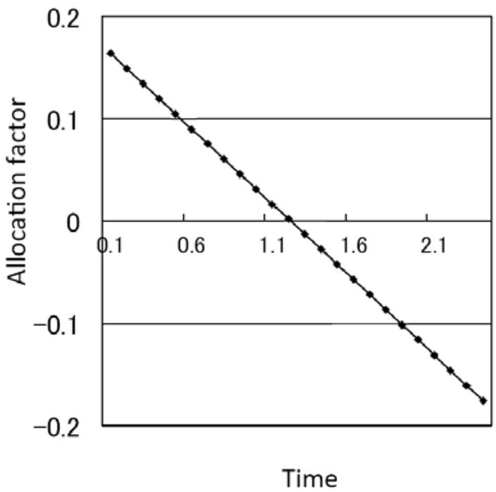

Figure 23. Allocation factor

- Worker efficiency (WE) indicates the efficiency of six workers.

"record" calculates the worker's operating time, which is obtained by multiplying the specified WE data for the log-normally distributed random numbers in Table 3.

Figure 25 shows the operating time of processes 1-6 (record1-record6). As the working time of the synchronous process is less volatile, the work efficiency became higher than the asynchronous process. In Figure 25, the total working time of asynchronous and synchronous processes are 1241.7(sec) and 586.4(sec) respectively. The synchronous process shows better production efficiency than the asynchronous process.

7. **Conclusion.** The target company example is equivalent to type 1 in Table 7 in this paper. We presented the ratio between the deterministic and stochastic systems. The proportion of the deterministic system is large because this business leads to profits. Through the synchronization process and standardization under the production flow system, we can achieve a higher throughput. As described in the aforementioned results, we also

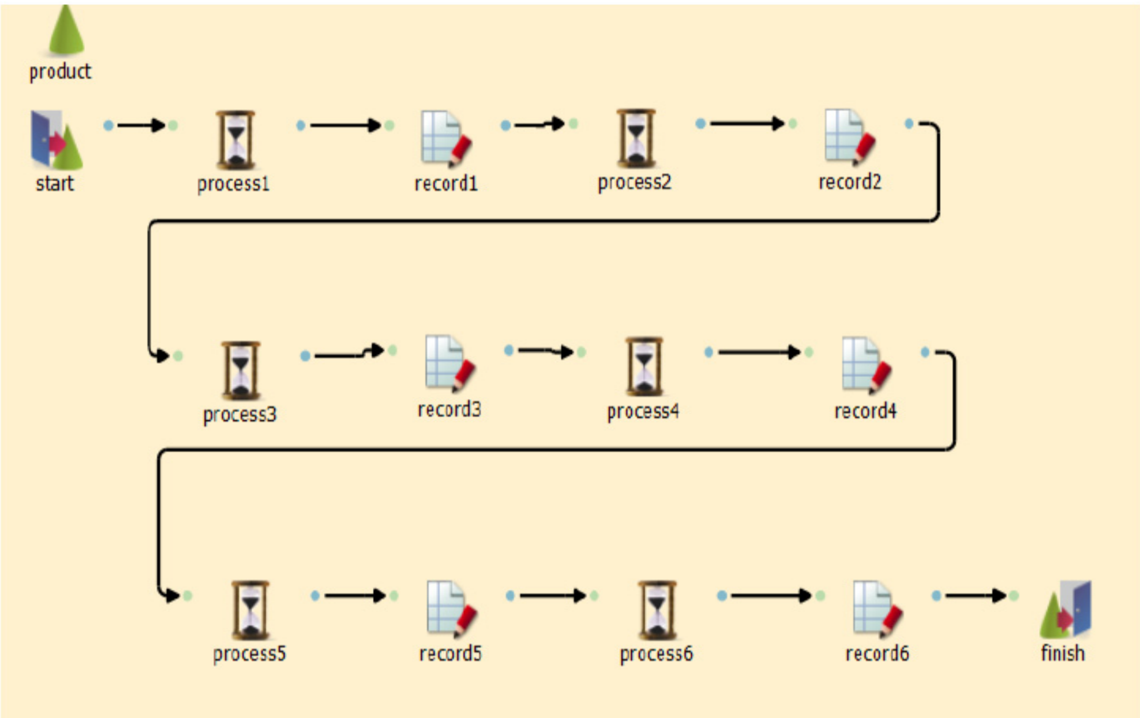

FIGURE 24. Simulation model of production flow system

TABLE 2. Parameter setting of Riccati equation and allocation factor

Figure	r	θ
18	0.1	1
19	0.8	1
20	0.8	1.5
21	0.1	1
22	0.8	1
23	0.8	1.5

TABLE 3. Working data for six production asynchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	22	25	22	25	21
STD	2.1	2.5	1.6	1.9	2.0	1.9
W.E 1	0.83	1.0	0.66	0.76	0.88	0.91
W.E 2	1.27	1.26	1.21	1.31	1.17	1.20
W.E 3	0.96	1.11	1.01	1.12	0.88	0.89
W.E 4	0.92	0.96	1.06	0.98	0.91	0.9
W.E 5	1.2	1.03	1.07	0.89	1.03	1.1
W.E 6	1.09	1.1	1.2	0.98	1.13	0.89

TABLE 4. Working data for six production synchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	20	20	20	20	20
STD	1.1	1.5	1.2	1.4	1.0	1.4
W.E 1	1.0	1.0	1.0	1.0	1.0	1.0
W.E 2	1.0	1.0	1.2	1.3	1.1	1.2
W.E 3	1.7	1.1	1.0	1.1	1.0	1.0
W.E 4	1.0	1.0	1.0	1.0	1.0	1.0
W.E 5	1.0	1.0	1.0	1.0	1.0	1.0
W.E 6	1.0	1.3	1.2	1.0	1.1	1.0

obtain the same ratio with respect to the sales ratio. The risk sense evaluation enables the determination of the ratio between the deterministic and stochastic systems, which

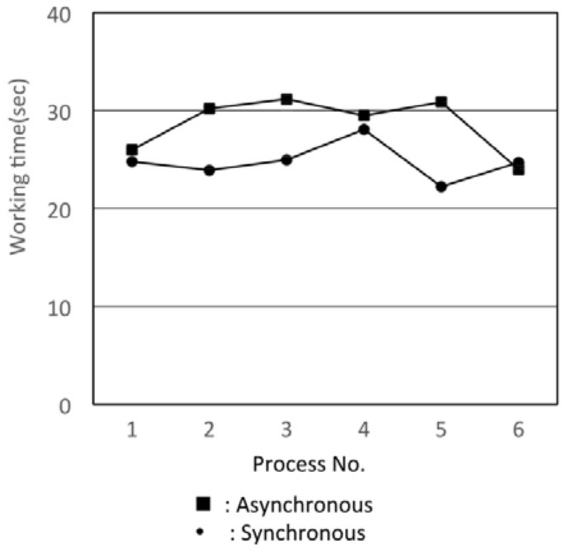

FIGURE 25. Working time for process number one through six

represents a remarkable achievement. Determining the risk free rate is often a problem in corporate finance as well. Although this time was set as a specific value, how to set the risk free rate is a future challenge.

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Appendix A. The Process of Obtaining the Solution of the Riccati Equation. We solve the Riccati equation from Equation (44).

$$-\frac{dP(\tau)}{d\tau} - P^2(\tau)K_0 + 2K_1P(\tau) + b^2 = 0 \tag{86}$$

$$b = \sqrt{\frac{2}{\theta+2}}\mu_s(t)\sigma_s \tag{87}$$

Definition A.1.

$$P(\tau) = Q(\tau) \cdot \frac{u'(\tau)}{u(\tau)} \tag{88}$$

We obtain the differential equation from Equation (88) as follows:

$$P'(\tau) = Q(\tau)\left[\frac{u''(\tau)}{u(\tau)} - \frac{(u')^2(\tau)}{u^2(\tau)}\right] + Q'(\tau)\frac{u'(\tau)}{u(\tau)} \tag{89}$$

Equation (86) is rewritten from Equation (89) as follows:

$$Q(\tau)\left[\frac{u''(\tau)}{u(\tau)} - \frac{(u')^2(\tau)}{u^2(\tau)}\right] + Q'(\tau)\frac{u'(\tau)}{u(\tau)}$$

$$= 2K_1 Q(\tau) \cdot \frac{u'(\tau)}{u(\tau)} - K_0 (Q(\tau))^2 \cdot \left(\frac{u'(\tau)}{u(\tau)}\right)^2 + b^2 \tag{90}$$

From Equation (90), we obtain as follows:

$$Q(\tau)\left[\frac{u''(\tau)}{u(\tau)}\right] + Q(\tau)[-K_0 Q(\tau) + 1]\left(\frac{u'(\tau)}{u(\tau)}\right)^2 + \frac{u'(\tau)}{u(\tau)}\left[Q'(\tau) + 2K_1 Q(\tau)\right] - b^2 = 0 \tag{91}$$

We obtain the following equation after $Q = 1/K_0$.

$$\frac{1}{K_0} \cdot \frac{u''(\tau)}{u(\tau)} - 2\frac{K_1}{K_0} \cdot \frac{u'(\tau)}{u(\tau)} - b^2 = 0 \tag{92}$$

From Equation (92), we obtain the following equation.

$$\frac{1}{K_0} \cdot u''(\tau) + 2\frac{K_1}{K_0} \cdot u'(\tau) - b^2 \cdot u(\tau) = 0 \tag{93}$$

The general solution of Equation (93) is as follows:

$$u(\tau) = A_1 \exp(\beta_1 \tau) + A_2 \exp(\beta_2 \tau) \tag{94}$$

Therefore, $P(\tau)$ is derived as follows:

$$P(\tau) = Q(\tau) \cdot \frac{u'(\tau)}{u(\tau)} = \frac{1}{K_0} \cdot \frac{\beta_1 A_1 \exp(\beta_1 \tau) + \beta_2 A_2 \exp(\beta_2 t)}{A_1 \exp(\beta_1 t) + A_2 \exp(\beta_2 \tau)} \tag{95}$$

Then, β_1 and β_2 are derived as follows:

$$a u''(\tau) + 2\tilde{b} u'(\tau) - b^2 u(\tau) = 0 \tag{96}$$

$$a = \frac{1}{K_0}, \quad \tilde{b} = \frac{K_1}{K_0} \tag{97}$$

The solution of β_1 and β_2 are derived from Equations (96) and (97) as follows:

$$\beta_{1,2} = \frac{-\tilde{b} \pm \sqrt{\tilde{b} + ab^2}}{a} = \frac{-K_1/K_0 \pm \sqrt{\left(K_1/K_0\right)^2 + \left(\tilde{b}\right)^2 / K_0}}{1/K_0}$$
$$= -K_1 \pm K_0 \sqrt{\left(K_1/K_0\right)^2 + \left(\tilde{b}\right)^2 / K_0} = -K_1 \pm \sqrt{K_1^2 + \left(\tilde{b}\right)^2 K_0} \tag{98}$$

Consequently, from Equations (95) and (98), $P(\tau)$ is derived as follows:

$$P(\tau) = \left(\frac{1}{K_0}\right) \cdot \frac{\beta_1 + \left(\frac{A_2}{A_1}\right)\beta_2 \exp\{(\beta_2 - \beta_1)\tau\}}{1 + \left(\frac{A_2}{A_1}\right) \exp\{(\beta_2 - \beta_1)\tau\}} \tag{99}$$

The initial value $P(0)$ is derived as follows at $t = 0$:

$$\left(\frac{1}{K_0}\right) \cdot \frac{\beta_1 + \left(\frac{A_2}{A_1}\right)\beta_2}{1 + \left(\frac{A_2}{A_1}\right)} = 0 \tag{100}$$

Namely, we obtain as follows from Equation (100):

$$\left(\frac{A_2}{A_1}\right) = -\left(\frac{\beta_1}{\beta_2}\right) \tag{101}$$

Appendix B. **Analysis of Actual Data in the Production Flow System.** Figure 2 represents a manufacturing process called a flow production system, which is a manufacturing method employed in the production of control equipment. The flow production system, which in this case has six stages, is commercialized by the production of material in steps S1-S6 of the manufacturing process.

The direction of the arrow represents the direction of the production flow. In this system, production materials are supplied from the inlet and the end product will be shipped from the outlet.

Assumption B.1. *The production structure is nonlinear.*

Assumption B.2. *The production structure is a closed structure; that is, the production is driven by a cyclic system (production flow system).*

Assumption B.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the throughput generation structure in a stochastic manufacturing process (hereafter called the manufacturing field). Because such a structure is at least dependent on the demand, it is considered to have a nonlinear structure.

Because the value of such a product depends on the throughput, its production structure is nonlinear. Therefore, Assumption B.1 reflects the realistic production structure and is somewhat valid. Assumption B.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption B.2 is reasonable because new production starts from S1.

Based on the control equipment, the product can be manufactured in one cycle. The production throughput required to maintain 6 pieces of equipment/day is as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25 \text{ (min)} \tag{102}$$

where the throughput of the previous process is set as 20 (min). In Equation (102), “28” represents the throughput of the previous process plus the idle time for synchronization. “8” is the number of processes and the total number of all processes is “8” plus the previous process. “60” is given by 20 (min) – 3 (cycles).

One process throughput (20min) in full synchronization is

$$T_s = 3 \times 120 + 40 = 400 \text{ (min)} \tag{103}$$

Therefore, a throughput reduction of about 10 can be achieved. However, the time between processes involves some asynchronous idle time.

As a result, the above test run is as follows.

- (test run 1) Each throughput in every process (S1-S6) is asynchronous, and its process throughput is asynchronous. Table 5 represents the manufacturing time (min) in each process. Table 6 represents the variance in each process performed by workers. Table 5 represents the target time, and the theoretical throughput is given by $3 \times 199 + 2 \times 15 = 627$(min).

  In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. Figure 26 is a graph illustrating the measurement data in Table 5, and it represents the total working time for each worker (K1-K9). The graph in Figure 27 represents the variance data for each working time in Table 5.
- (test run 2) Set to synchronously process the throughput.

  The target time in Table 7 is 500 (min), and the theoretical throughput (not including the synchronized idle time) is 400 (min). Table 8 represents the variance data of each working process (S1-S6) for each worker (K1-K9).

- (test run 3) The process throughput is performed synchronously with the reclassification of the process. The theoretical throughput (not including the synchronized idle time) is 400 (min) in Table 9.

  Table 10 represents the variance data of Table 9. "WS" in the measurement tables represents the standard working time. This is an empirical value obtained from long-term experiments.

TABLE 5. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

TABLE 6. Volatility of Table 5

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

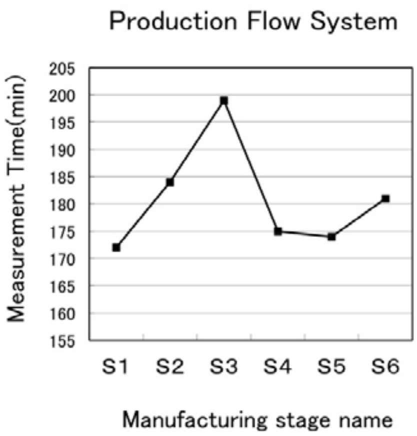

FIGURE 26. Total work time for each stage (S1-S6) in Table 5

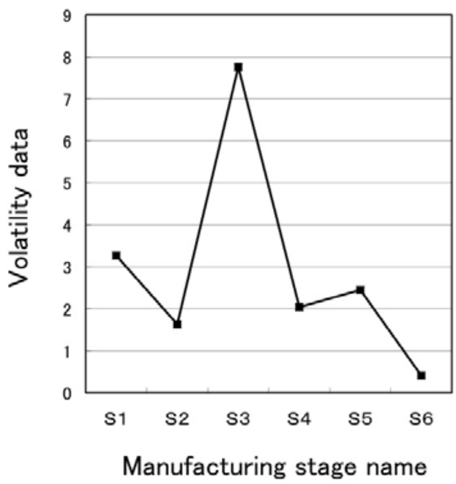

FIGURE 27. Volatility data for each stage (S1-S6) in Table 5

TABLE 7. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

TABLE 8. Volatility of Table 7

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 9. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	21	21	21	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	22	22	20	20	20	20
K9	20	25	25	25	20	20	20
Total	180	165	164	161	180	180	180

TABLE 10. Variance of Table 9

K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.33	3	3	0	0	0
K5	1.33	1.33	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.33	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0