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DELAY-DEPENDENT STABILITY OF DISCRETE-TIME SYSTEMS WITH MULTIPLE DELAYS AND NONLINEARITIES

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Abstract. *A discrete-time system under finite wordlength implementation may demonstrate instability in its behavior. This instability is enhanced further in the presence of delays and parameter uncertainties. This paper, therefore, considers the problem of global asymptotic stability of a class of discrete-time systems in the presence of finite wordlength nonlinearities (combination of quantization and overflow nonlinearities), multiple delays and parameter uncertainties. Two linear matrix inequality (LMI) based delay-dependent stability criteria are proposed. The first criterion deals with the systems involving multiple delays of interval-like time-varying in nature and the second criterion is applicable to systems with multiple constant delays. The parametric uncertainties are time-varying and unknown but are assumed to be norm-bounded. The forward difference of the Lyapunov functional is treated using the reciprocally convex approach. Numerical examples highlighting the usefulness of the proposed criteria are provided.*
Keywords: Discrete-time system, Finite wordlength nonlinearity, Lyapunov stability, Reciprocal convexity, Time-varying delay

1. **Introduction.** During the implementation of discrete-time dynamical systems, the signals or the data associated are represented using finite precision arithmetic (i.e., finite wordlength hardware) [1, 2]. It may happen that due to the arithmetic manipulations, the size of the data may exceed the maximum value that can be stored or processed in the given hardware. In order to limit the data, and represent with finite wordlength hardware, quantization (such as magnitude truncation, rounding and value truncation) and overflow correction (such as saturation, zeroing, triangular and two's complement) techniques are used. Thus, finite word-length implementation introduces quantization and overflow nonlinearities in the system [3]. In the presence of these nonlinearities, the system may become unstable and exhibit zero-input limit cycles. Several results have been reported on the stability of discrete-time systems in the presence of finite wordlength nonlinearities (see, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the references cited therein). Although, in the literature, researchers have studied the effects of quantization [3, 4, 5] and overflow [1, 2, 3, 6, 7, 8, 9, 10] nonlinearities separately under the decoupling approximation [19] but it is more realistic to consider the combined influence of quantization and overflow nonlinearities [11, 12, 13, 14, 16, 17, 18].

The presence of uncertainties due to the modeling errors, external disturbances or variation in parameters during implementation may also introduce instability into the system [20, 21, 22]. Parameter uncertainties are generally modeled as norm-bounded [13, 14, 16, 17, 18] or polytopic [22]. In this paper, the norm-bounded parametric uncertainties have been used for modeling the uncertainties.

Delays are another source of instability in discrete-time systems. As a result, many researchers have paid attention to developing stability criteria for discrete-time delayed systems (see, [7, 8, 9, 14, 16, 17, 18, 23, 24, 25, 26, 27, 28, 29, 30, 31, 20, 22] and the references cited therein). The stability criteria for systems with delays in the state are classified into delay-independent [8, 13, 20] and delay-dependent [7, 9, 14, 23, 24, 25, 26, 27, 28, 29, 30, 31, 20, 22]. Many systems are stable in the absence of delay but become unstable in the presence of delays; on the contrary there are systems which are unstable under zero delay conditions but are found to be stable over a delay range or interval [24]. The delay-independent stability criterion is unable to determine the stability of systems which are stable in the presence of delay (interval-like) having a nonzero lower delay bound [24]. Thus, delay-dependent stability criteria are less conservative than delay-independent criteria. Some results on discrete-time systems with multiple delays (constant and interval-like time-varying) have been reported in [32, 33, 34, 35].

A practical discrete-time system is generally under the combined influence of finite wordlength nonlinearities, multiple delays and parameter uncertainties. Examples of such systems include digital control systems with finite wordlength nonlinearities (for example memoryless state observer for linear discrete-time systems with multiple delays [36]), digital filters implemented in finite register length [10, 12, 13, 14] networked control system [27], where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying and can be assumed to have minimum and maximum delay bounds without loss of generality. While implementing networked control systems using computer or special purpose hardware with fixed-point arithmetic for data processing in the network transmission, the nonlinearities (quantization/overflow) due to finite wordlength are generated. Networked filtering, where multiple sensors send data to the filter over a common network [32], chemical reactor recycle system [34] are some other examples of practical systems with multiple interval-like time-varying delays or multiple constant delays. So, obtaining the global asymptotic stability criteria for systems with finite wordlength nonlinearities, uncertainties and multiple delays is an important problem. In [37], a delay-independent stability criterion for discrete-time systems with generalized overflow nonlinearities, uncertainties and multiple constant delays is established. A delay-independent stability criterion for discrete-time systems with multiple constant delays, quantization and overflow nonlinearities and uncertainties have been studied in [13].

To the best of authors' knowledge, delay-dependent stability criteria for uncertain discrete-time systems with multiple delays (time-varying and constant) and finite wordlength nonlinearities have not been established previously. The following are the main contributions of the paper:

(a) To obtain a delay-dependent stability criterion for uncertain discrete-time systems in the presence of quantization and overflow nonlinearities and multiple interval-like time-varying delays;

(b) To provide a delay-dependent stability criterion for a class of systems under the influence of finite wordlength nonlinearities and multiple constant delays;

(c) Using examples, we provide the applicability and advantage of the presented criteria as compared with previously reported results.

This paper is organized as follows. In Section 2, the system under consideration is described. Main results of the paper are presented in Section 3. Examples are provided in Section 4 for illustrating the usefulness of the criteria presented in this paper. Finally, Section 5 concludes this paper.

2. **System Description.** In this paper, we consider a class of discrete-time uncertain systems with interval-like time-varying delays under various combinations of quantization and overflow nonlinearities and for the situation where quantization occurs after summation only. Specifically, the system under consideration is described by

$$\begin{aligned}
\boldsymbol{x}(k+1) &= \boldsymbol{O}\{\boldsymbol{Q}(\boldsymbol{y}(k))\} = \boldsymbol{f}(\boldsymbol{y}(k)) && \text{(1a)}\\
&= [f_1(y_1(k)) \ \ f_2(y_2(k)) \ \ \cdots \ \ f_n(y_n(k))]^T \\
\boldsymbol{y}(k) &= (\boldsymbol{A} + \Delta\boldsymbol{A}(k))\boldsymbol{x}(k) + \sum_{i=1}^{m}(\boldsymbol{A}_{di} + \Delta\boldsymbol{A}_{di}(k))\boldsymbol{x}(k - d_i(k)) \\
&= [y_1(k) \ \ y_2(k) \ \ \cdots \ \ y_n(k)]^T && \text{(1b)}\\
d_{1i} &\le d_i(k) \le d_{2i}, \ \ i = 1, 2, \ldots, m && \text{(1c)}\\
\boldsymbol{x}(k) &= \boldsymbol{\varphi}(k), \quad \forall k = -d, -d+1, \ldots, 0 && \text{(1d)}\\
d &= \max\{d_{21}, d_{22}, \ldots, d_{2m}\} && \text{(1e)}
\end{aligned}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state vector; $\boldsymbol{A}$, $\boldsymbol{A}_{di}$ $(i = 1, 2, \ldots, m) \in \mathbb{R}^{n\times n}$ are the known constant matrices; $\Delta\boldsymbol{A}(k)$, $\Delta\boldsymbol{A}_{di}(k)$ $(i = 1, 2, \ldots, m) \in \mathbb{R}^{n\times n}$ are the unknown matrices representing parametric uncertainties in the state matrices; $\boldsymbol{\varphi}(k) \in \mathbb{R}^n$ is the initial state value at time k; $\boldsymbol{Q}(\cdot)$ represents the quantization nonlinearities; $\boldsymbol{O}(\cdot)$ denotes the overflow nonlinearities; $\boldsymbol{f}(\cdot)$ characterizes the composite nonlinear functions; $d_i(k)$ $(i = 1, 2, \ldots, m)$ is the time-varying delay satisfying (1c) in which d_{1i} and d_{2i} are known non-negative integers representing the lower and upper delay bounds, respectively.

In the event of $\boldsymbol{Q}(\cdot)$ being either magnitude truncation or roundoff, $\boldsymbol{f}(\cdot)$ is confined to the sector $[k_o, k_q]$, i.e.,

$$k_o \le \frac{f_j(y_j(k))}{y_j(k)} \le k_q, \qquad j = 1, 2, \ldots, n \tag{2a}$$

under the assumption that $f_j(0) = 0$, where

$$k_q = \begin{cases} 1, & \text{for magnitude truncation} \\ 2, & \text{for roundoff} \end{cases} \tag{2b}$$

$$k_o = \begin{cases} 0, & \text{for zeroing or saturation} \\ -\dfrac{1}{3}, & \text{for triangular} \\ -1, & \text{for two's complement} \end{cases} \tag{2c}$$

The uncertainties in the state matrices are assumed to be of the form

$$\Delta\boldsymbol{A}(k) = \boldsymbol{H}_0\boldsymbol{F}_0(k)\boldsymbol{E}_0 \tag{3a}$$

$$\Delta\boldsymbol{A}_{di}(k) = \boldsymbol{H}_i\boldsymbol{F}_i(k)\boldsymbol{E}_i, \quad i = 1, 2, \ldots, m \tag{3b}$$

where $\boldsymbol{H}_i \in \mathbb{R}^{n\times p_i}$, $\boldsymbol{E}_i \in \mathbb{R}^{q_i\times n}$ $(i = 0, 1, 2, \ldots, m)$ are known constant matrices and $\boldsymbol{F}_i(k) \in \mathbb{R}^{p_i\times q_i}$ $(i = 0, 1, \ldots, m)$ is a discrete time-varying unknown matrix which satisfies

$$\boldsymbol{F}_i^T(k)\boldsymbol{F}_i(k) \le \boldsymbol{I}, \qquad i = 0, 1, \ldots, m \tag{3c}$$

Equations (1)-(3) can be used to describe a broader class of uncertain discrete-time state-delayed dynamical systems involving both quantization and overflow nonlinearities. Examples of such systems are ubiquitous in engineering and include digital control systems with finite wordlength nonlinearities (for example memoryless state observer for linear discrete-time systems with multiple delays [36]), digital filters implemented in finite register length [10, 12, 13, 14], cold rolling mills [20, 22], etc. A typical example containing time delays that can be represented by (1c) can be found in networked control system [27], where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying and can be assumed to have minimum and maximum delay bounds without loss of generality. While implementing networked control systems using computer or special purpose hardware with fixed-point arithmetic for data processing in the network transmission, the nonlinearities (quantization/overflow) due to finite wordlength are generated. Networked filtering, where multiple sensors send data to the filter over a common network [32], chemical reactor recycle system [34] are some other examples of practical systems with multiple interval-like time-varying delays or multiple constant delays.

The following definition and lemmas are needed in the proof of our main results.

Definition 2.1. *The zero solution of the system given by* (1)-(3) *is globally asymptotically stable if the following holds:*
(i) it is stable in the sense of Lyapunov, i.e., for every $\mu > 0$ *there exists a* $\delta = \delta(\mu)$ *such that* $\|\boldsymbol{x}(k)\| < \mu$ *for all* $k = 0, 1, 2, \ldots,$ *whenever* $\|\boldsymbol{x}(0)\| < \mu$*;*
(ii) it is attractive, i.e., $\boldsymbol{x}(k) \longrightarrow \mathbf{0}$ *as* $k \longrightarrow \infty$.

Lemma 2.1. [31, 38] *For any vectors* $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, *matrices* $\boldsymbol{R}$, $\boldsymbol{S}$ *and real numbers* $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ *satisfying*

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{S} \\ * & \boldsymbol{R} \end{bmatrix} \geq 0, \quad \alpha_1 + \alpha_2 = 1 \tag{4}$$

$\boldsymbol{\xi}_i = \mathbf{0}$, *if* $\alpha_i = 0$, $i = 1, 2$ *then*

$$-\frac{1}{\alpha_1}\boldsymbol{\xi}_1{}^T\boldsymbol{R}\boldsymbol{\xi}_1 - \frac{1}{\alpha_2}\boldsymbol{\xi}_2{}^T\boldsymbol{R}\boldsymbol{\xi}_2 \leq -\begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{R} & \boldsymbol{S} \\ * & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} \tag{5}$$

Lemma 2.2. [39] *Let* $\boldsymbol{\Sigma}$, $\boldsymbol{\Gamma}$, $\boldsymbol{F}$, *and* $\boldsymbol{\Lambda}$ *be real matrices of appropriate dimensions with* $\boldsymbol{\Lambda}$ *satisfying* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T$, *and then*

$$\boldsymbol{\Lambda} + \boldsymbol{\Sigma}\boldsymbol{F}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^T\boldsymbol{F}^T\boldsymbol{\Sigma}^T < \mathbf{0} \tag{6}$$

$\forall \boldsymbol{F}^T\boldsymbol{F} \leq \boldsymbol{I}$, *if and only if there exists a scalar* $\epsilon > 0$ *such that*

$$\boldsymbol{\Lambda} + \epsilon^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T + \epsilon\boldsymbol{\Gamma}^T\boldsymbol{\Gamma} < \mathbf{0} \tag{7}$$

Lemma 2.3. [24] *For any positive definite matrix* $\boldsymbol{J} \in \mathbb{R}^{n\times n}$, *two positive integers* r *and* r_0 *satisfying* $r \geq r_0 \geq 1$, *and vector function* $\boldsymbol{x}(i) \in \mathbb{R}^n$, *one has*

$$\left(\sum_{i=r_0}^{r}\boldsymbol{x}(i)\right)^T \boldsymbol{J} \left(\sum_{i=r_0}^{r}\boldsymbol{x}(i)\right) \leq (r - r_0 + 1)\sum_{i=r_0}^{r}\boldsymbol{x}^T(i)\boldsymbol{J}\boldsymbol{x}(i) \tag{8}$$

3. **Main Results.** Inspired by the work of [28, 31], we now present the main results of the paper.

Theorem 3.1. *The zero solution of the system described by* (1)-(3) *is globally asymptotically stable if there exist* $n \times n$ *positive definite symmetric matrices* $\boldsymbol{P}$, $\boldsymbol{Q}_{1i}$, $\boldsymbol{Q}_{2i}$, $\boldsymbol{Q}_{3i}$, $\boldsymbol{Z}_{1i}$, $\boldsymbol{Z}_{2i}$, *matrices* $\boldsymbol{S}_i$ $(i = 1, 2, \ldots, m)$ *with appropriate dimensions, a positive definite diagonal matrix* $\boldsymbol{G} = diag(g_1, g_2, \ldots, g_n)$, *and positive scalars* ϵ_i $(i = 0, 1, \ldots, m)$ *satisfying* (9) *and* (10)

$$\boldsymbol{\Theta}_i = \begin{bmatrix} \boldsymbol{Z}_{2i} & \boldsymbol{S}_i \\ * & \boldsymbol{Z}_{2i} \end{bmatrix} > \mathbf{0}, \quad i = 1, 2, \ldots, m \tag{9}$$

$$\begin{bmatrix} \Xi_1 & \mathbf{0} & \Xi_2 & \mathbf{0} & \Xi_3 & \Xi_4 & \mathbf{0} \\ * & \Xi_5 & \Xi_6 & \Xi_7 & \Xi_8 & \Xi_9 & \mathbf{0} \\ * & * & \Xi_{10} & \Xi_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \Xi_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & \Xi_{13} & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} & \Xi_{14} \\ * & * & * & * & * & -k_q\boldsymbol{G} & \Xi_{15} \\ * & * & * & * & * & * & \Xi_{16} \end{bmatrix} < \mathbf{0} \tag{10}$$

where

$$\Xi_1 = -\boldsymbol{P} + \sum_{i=1}^{m}\Big[\boldsymbol{Q_{1i}} + \boldsymbol{Q_{2i}} + (d_{12i}+1)\boldsymbol{Q}_{3i} - \boldsymbol{Z}_{1i}\Big] + \sum_{i=1}^{m}\Big[d_{1i}^2\boldsymbol{Z}_{1i} + d_{12i}^2\boldsymbol{Z}_{2i}\Big] + \epsilon_0\boldsymbol{E}_0^T\boldsymbol{E}_0 \tag{11a}$$

$$\Xi_2 = \begin{bmatrix} \boldsymbol{Z}_{11} & \boldsymbol{Z}_{12} & \dots & \boldsymbol{Z}_{1m} \end{bmatrix} \tag{11b}$$

$$\Xi_3 = -\sum_{i=1}^{m}\Big[d_{1i}^2\boldsymbol{Z}_{1i} + d_{12i}^2\boldsymbol{Z}_{2i}\Big] + k_q\boldsymbol{A}^T\boldsymbol{G} \tag{11c}$$

$$\Xi_4 = -k_q\sqrt{-2k_o}\boldsymbol{A}^T\boldsymbol{G} \tag{11d}$$

$$\Xi_5 = diag\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_m\} \tag{11e}$$

$$\boldsymbol{\rho}_i = -\boldsymbol{Q}_{3i} - 2\boldsymbol{Z}_{2i} + \boldsymbol{S}_i + \boldsymbol{S}_i^T + \epsilon_i\boldsymbol{E}_i^T\boldsymbol{E}_i, \quad i = 1, 2, \dots, m \tag{11f}$$

$$\Xi_6 = diag\left\{\boldsymbol{Z}_{21} - \boldsymbol{S}_1^T, \boldsymbol{Z}_{22} - \boldsymbol{S}_2^T, \dots, \boldsymbol{Z}_{2m} - \boldsymbol{S}_m^T\right\} \tag{11g}$$

$$\Xi_7 = diag\{\boldsymbol{Z}_{21} - \boldsymbol{S}_1, \boldsymbol{Z}_{22} - \boldsymbol{S}_2, \dots, \boldsymbol{Z}_{2m} - \boldsymbol{S}_m\} \tag{11h}$$

$$\Xi_8 = \begin{bmatrix} k_q\boldsymbol{G}\boldsymbol{A}_{d1} & k_q\boldsymbol{G}\boldsymbol{A}_{d2} & \cdots & k_q\boldsymbol{G}\boldsymbol{A}_{dm} \end{bmatrix}^T \tag{11i}$$

$$\Xi_9 = \begin{bmatrix} -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{A}_{d1} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{A}_{d2} & \cdots & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{A}_{dm} \end{bmatrix}^T \tag{11j}$$

$$\Xi_{10} = diag\{-\boldsymbol{Q}_{11} - \boldsymbol{Z}_{11} - \boldsymbol{Z}_{21}, -\boldsymbol{Q}_{12} - \boldsymbol{Z}_{12} - \boldsymbol{Z}_{22}, \dots, -\boldsymbol{Q}_{1m} - \boldsymbol{Z}_{1m} - \boldsymbol{Z}_{2m}\} \tag{11k}$$

$$\Xi_{11} = diag\{\boldsymbol{S}_1, \boldsymbol{S}_2, \dots, \boldsymbol{S}_m\} \tag{11l}$$

$$\Xi_{12} = diag\{-\boldsymbol{Q}_{21} - \boldsymbol{Z}_{21}, -\boldsymbol{Q}_{22} - \boldsymbol{Z}_{22}, \dots, -\boldsymbol{Q}_{2m} - \boldsymbol{Z}_{2m}\} \tag{11m}$$

$$\Xi_{13} = \boldsymbol{P} - 2\boldsymbol{G} + \sum_{i=1}^{m}\Big[d_{1i}^2\boldsymbol{Z}_{1i} + d_{12i}^2\boldsymbol{Z}_{2i}\Big] + \Big(\frac{k_o}{2k_q}\Big)\boldsymbol{G} \tag{11n}$$

$$\Xi_{14} = \begin{bmatrix} k_q\boldsymbol{G}\boldsymbol{H}_0 & k_q\boldsymbol{G}\boldsymbol{H}_1 & \dots & k_q\boldsymbol{G}\boldsymbol{H}_m \end{bmatrix} \tag{11o}$$

$$\Xi_{15} = \begin{bmatrix} -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_0 & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_1 & \dots & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_m \end{bmatrix} \tag{11p}$$

$$\Xi_{16} = diag\{-\epsilon_0\boldsymbol{I}, -\epsilon_1\boldsymbol{I}, \dots, -\epsilon_m\boldsymbol{I}\} \tag{11q}$$

$$d_{12i} = d_{2i} - d_{1i}, \ i = 1, 2, \dots, m \tag{11r}$$

Proof: Define

$$\boldsymbol{\eta}(k) = \boldsymbol{x}(k+1) - \boldsymbol{x}(k) = \boldsymbol{f}(\boldsymbol{y}(k)) - \boldsymbol{x}(k) \tag{12}$$

Consider the following Lyapunov functional

$$V(\boldsymbol{x}(k)) = V_1(\boldsymbol{x}(k)) + V_2(\boldsymbol{x}(k)) + V_3(\boldsymbol{x}(k)) + V_4(\boldsymbol{x}(k)) + V_5(\boldsymbol{x}(k))$$

where

$$V_1(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k)$$

$$V_2(\boldsymbol{x}(k)) = \sum_{i=1}^{m}\left[\sum_{j=k-d_{1i}}^{k-1}\boldsymbol{x}^T(j)\boldsymbol{Q}_{1i}\boldsymbol{x}(j) + \sum_{j=k-d_{2i}}^{k-1}\boldsymbol{x}^T(j)\boldsymbol{Q}_{2i}\boldsymbol{x}(j)\right]$$
$$V_3(\boldsymbol{x}(k)) = \sum_{i=1}^{m}\sum_{j=-d_{2i}}^{-d_{1i}}\sum_{r=k+j}^{k-1}\boldsymbol{x}^T(r)\boldsymbol{Q}_{3i}\boldsymbol{x}(r)$$
$$V_4(\boldsymbol{x}(k)) = \sum_{i=1}^{m}\left[\sum_{\theta=-d_{1i}+1}^{0}\sum_{j=k-1+\theta}^{k-1}d_{1i}\boldsymbol{\eta}^T(j)\boldsymbol{Z}_{1i}\boldsymbol{\eta}(j)\right]$$
$$V_5(\boldsymbol{x}(k)) = \sum_{i=1}^{m}\left[\sum_{\theta=-d_{2i}+1}^{-d_{1i}}\sum_{j=k-1+\theta}^{k-1}d_{12i}\boldsymbol{\eta}^T(j)\boldsymbol{Z}_{2i}\boldsymbol{\eta}(j)\right] \tag{13}$$

Taking the forward difference of the Lyapunov functional along the trajectories of the system (1) yields

$$\Delta V(\boldsymbol{x}(k)) = \sum_{r=1}^{5}[V_r(\boldsymbol{x}(k+1)) - V_r(\boldsymbol{x}(k))] = \sum_{r=1}^{5}\Delta V_r(\boldsymbol{x}(k)) \tag{14}$$

where

$$\Delta V_1(\boldsymbol{x}(k)) = \boldsymbol{f}^T(\boldsymbol{y}(k))\boldsymbol{P}\boldsymbol{f}(\boldsymbol{y}(k)) - \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{15}$$
$$\Delta V_2(\boldsymbol{x}(k)) = \sum_{i=1}^{m}\boldsymbol{x}^T(k)[\boldsymbol{Q}_{1i}+\boldsymbol{Q}_{2i}]\boldsymbol{x}(k) - \sum_{i=1}^{m}\boldsymbol{x}^T(k-d_{1i})\boldsymbol{Q}_{1i}$$
$$\times\,\boldsymbol{x}(k-d_{1i}) - \sum_{i=1}^{m}\boldsymbol{x}^T(k-d_{2i})\boldsymbol{Q}_{2i}\boldsymbol{x}(k-d_{2i}) \tag{16}$$
$$\Delta V_3(\boldsymbol{x}(k)) = \sum_{i=1}^{m}(d_{12i}+1)\boldsymbol{x}^T(k)\boldsymbol{Q}_{3i}\boldsymbol{x}(k) - \sum_{i=1}^{m}\sum_{j=k-d_{2i}}^{k-d_{1i}}\boldsymbol{x}^T(j)\boldsymbol{Q}_{3i}\boldsymbol{x}(j)$$
$$\le \sum_{i=1}^{m}(d_{12i}+1)\boldsymbol{x}^T(k)\boldsymbol{Q}_{3i}\boldsymbol{x}(k) - \boldsymbol{x}^T(k-d_i(k))\boldsymbol{Q}_{3i}\boldsymbol{x}(k-d_i(k)) \tag{17}$$
$$\Delta V_4(\boldsymbol{x}(k)) = \sum_{i=1}^{m}d_{1i}^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_{1i}\boldsymbol{\eta}(k) - \sum_{i=1}^{m}\sum_{j=k-d_{1i}}^{k-1}d_{1i}\boldsymbol{\eta}^T(j)\boldsymbol{Z}_{1i}\boldsymbol{\eta}(j) \tag{18}$$
$$\Delta V_5(\boldsymbol{x}(k)) = \sum_{i=1}^{m}d_{12i}^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_{2i}\boldsymbol{\eta}(k) - \sum_{i=1}^{m}\sum_{j=k-d_{2i}}^{k-d_{1i}-1}d_{12i}\boldsymbol{\eta}^T(j)\boldsymbol{Z}_{2i}\boldsymbol{\eta}(j) \tag{19}$$

By employing Lemma 2.3 (i.e., the discrete Jensen inequality), we have

$$\Delta V_4(\boldsymbol{x}(k)) \le \sum_{i=1}^{m}d_{1i}^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_{1i}\boldsymbol{\eta}(k) - [\boldsymbol{x}(k)-\boldsymbol{x}(k-d_{1i})]^T\boldsymbol{Z}_{1i}[\boldsymbol{x}(k)-\boldsymbol{x}(k-d_{1i})] \tag{20}$$

and

$$\Delta V_5(\boldsymbol{x}(k)) \le \sum_{i=1}^{m}d_{12i}^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_{2i}\boldsymbol{\eta}(k) - \sum_{i=1}^{m}\frac{1}{\frac{(d_i(k)-d_{1i})}{d_{12i}}}\boldsymbol{\gamma}_{1i}^T(k)\boldsymbol{Z}_{2i}\boldsymbol{\gamma}_{1i}(k)$$

$$-\sum_{i=1}^{m} \frac{1}{\frac{(d_{2i}-d_i(k))}{d_{12i}}} \boldsymbol{\gamma}_{2i}^T(k)\boldsymbol{Z}_{2i}\boldsymbol{\gamma}_{2i}(k) \tag{21}$$

where

$$\boldsymbol{\gamma}_{1i}(k) = \boldsymbol{x}(k-d_{1i}) - \boldsymbol{x}(k-d_i(k)) \tag{22a}$$

$$\boldsymbol{\gamma}_{2i}(k) = \boldsymbol{x}(k-d_i(k)) - \boldsymbol{x}(k-d_{2i}), \quad i=1,2,\ldots,m \tag{22b}$$

It may be observed that $\boldsymbol{\gamma}_{1i}(k) = \boldsymbol{0}$, if $\dfrac{(d_i(k)-d_{1i})}{d_{12i}} = 0$ and $\boldsymbol{\gamma}_{2i}(k) = \boldsymbol{0}$, if $\dfrac{(d_{2i}-d_i(k))}{d_{12i}} = 0$. In the light of Lemma 2.1 (i.e., the reciprocally convex approach) and (21), the following relation is obtained if there exist matrices $\boldsymbol{S}_i$ $(i=1,2,\ldots,m)$ such that (9) is satisfied

$$\Delta V_5(\boldsymbol{x}(k)) \le \sum_{i=1}^{m}\left\{ d_{12i}^2 \boldsymbol{\eta}^T(k)\boldsymbol{Z}_{2i}\boldsymbol{\eta}(k) - \begin{bmatrix}\boldsymbol{\gamma}_{1i}(k)\\ \boldsymbol{\gamma}_{2i}(k)\end{bmatrix}^T \boldsymbol{\Theta}_i \begin{bmatrix}\boldsymbol{\gamma}_{1i}(k)\\ \boldsymbol{\gamma}_{2i}(k)\end{bmatrix} \right\} \tag{23}$$

Employing the terms $\Delta V_r(\boldsymbol{x}(k))$ $(r=1,2,\ldots,5)$, we have the following inequality

$$\Delta V(\boldsymbol{x}(k)) \le \boldsymbol{\xi}^T(k)\boldsymbol{\Psi}\boldsymbol{\xi}(k) - 2\beta \tag{24}$$

where

$$\begin{aligned}\beta &= \sum_{j=1}^{n} g_j[k_q y_j(k) - f_j(y_j(k))][f_j(y_j(k)) - k_o y_j(k)] \\ &= [k_q \boldsymbol{y}(k) - \boldsymbol{f}(\boldsymbol{y}(k))]^T \boldsymbol{G}[\boldsymbol{f}(\boldsymbol{y}(k)) - k_o \boldsymbol{y}(k)]\end{aligned} \tag{25}$$

is a nonnegative quantity [12, 13, 14] for the nonlinearities given by (2),

$$\begin{aligned}\boldsymbol{\xi}^T(k) = \Big[\; &\boldsymbol{x}^T(k)\ \boldsymbol{x}^T(k-d_1(k))\ \cdots\ \boldsymbol{x}^T(k-d_m(k))\ \boldsymbol{x}^T(k-d_{11}) \\ &\cdots\ \boldsymbol{x}^T(k-d_{1m})\ \boldsymbol{x}^T(k-d_{21})\ \cdots\ \boldsymbol{x}^T(k-d_{2m}) \quad \boldsymbol{f}^T(\boldsymbol{y}(k)) \;\Big]\end{aligned} \tag{26}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_1 & \boldsymbol{\Psi}_2 & \boldsymbol{\Xi}_2 & \boldsymbol{0} & \boldsymbol{\Psi}_3 \\ * & \boldsymbol{\Psi}_4 & \boldsymbol{\Xi}_6 & \boldsymbol{\Xi}_7 & \boldsymbol{\Psi}_5 \\ * & * & \boldsymbol{\Xi}_{10} & \boldsymbol{\Xi}_{11} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\Xi}_{12} & \boldsymbol{0} \\ * & * & * & * & \boldsymbol{\Psi}_6 \end{bmatrix} \tag{27}$$

$$\begin{aligned}\boldsymbol{\Psi}_1 = -\boldsymbol{P} &+ \sum_{i=1}^{m}\left[\boldsymbol{Q}_{\boldsymbol{1i}} + \boldsymbol{Q}_{\boldsymbol{2i}} + (d_{12i}+1)\boldsymbol{Q}_{3i} - \boldsymbol{Z}_{1i}\right] \\ &+ \sum_{i=1}^{m}\left[d_{1i}^2\boldsymbol{Z}_{1i} + d_{12i}^2\boldsymbol{Z}_{2i}\right] - 2k_q k_o \bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}\end{aligned} \tag{28a}$$

$$\boldsymbol{\Psi}_2 = \begin{bmatrix} -2k_q k_o \bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}_{d1} & -2k_q k_o \bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}_{d2} & \cdots & -2k_q k_o \bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}_{dm} \end{bmatrix} \tag{28b}$$

$$\boldsymbol{\Psi}_3 = -\sum_{i=1}^{m}\left[d_{1i}^2\boldsymbol{Z}_{1i} + d_{12i}^2\boldsymbol{Z}_{2i}\right] + (k_q+k_o)\bar{\boldsymbol{A}}^T\boldsymbol{G} \tag{28c}$$

$$\boldsymbol{\Psi}_4 = \begin{bmatrix} \bar{\boldsymbol{\rho}}_1 - 2k_q k_o \bar{\boldsymbol{A}}_{d1}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d1} & -2k_q k_o \bar{\boldsymbol{A}}_{d1}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d2} & \cdots & -2k_q k_o \bar{\boldsymbol{A}}_{d1}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{dm} \\ -2k_q k_o \bar{\boldsymbol{A}}_{d2}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d1} & \bar{\boldsymbol{\rho}}_2 - 2k_q k_o \bar{\boldsymbol{A}}_{d2}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d2} & \cdots & -2k_q k_o \bar{\boldsymbol{A}}_{d2}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{dm} \\ \vdots & \vdots & \ddots & \vdots \\ -2k_q k_o \bar{\boldsymbol{A}}_{dm}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d1} & -2k_q k_o \bar{\boldsymbol{A}}_{dm}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{d2} & \cdots & \bar{\boldsymbol{\rho}}_m - 2k_q k_o \bar{\boldsymbol{A}}_{dm}^T \boldsymbol{G} \bar{\boldsymbol{A}}_{dm} \end{bmatrix} \tag{28d}$$

$$\bar{\boldsymbol{\rho}}_i = -\boldsymbol{Q}_{3i} - 2\boldsymbol{Z}_{2i} + \boldsymbol{S}_i + \boldsymbol{S}_i^T, \quad i = 1, 2, \ldots, m \tag{28e}$$

$$\boldsymbol{\Psi}_5 = \begin{bmatrix} (k_q + k_o)\boldsymbol{G}\bar{\boldsymbol{A}}_{d1} & (k_q + k_o)\boldsymbol{G}\bar{\boldsymbol{A}}_{d2} & \cdots & (k_q + k_o)\boldsymbol{G}\bar{\boldsymbol{A}}_{dm} \end{bmatrix}^T \tag{28f}$$

$$\boldsymbol{\Psi}_6 = \boldsymbol{P} + \sum_{i=1}^{m} \left[d_{1i}^2 \boldsymbol{Z}_{1i} + d_{12i}^2 \boldsymbol{Z}_{2i} \right] - 2\boldsymbol{G} \tag{28g}$$

and

$$\bar{\boldsymbol{A}} = \boldsymbol{A} + \Delta\boldsymbol{A}, \ \ \bar{\boldsymbol{A}}_{di} = \boldsymbol{A}_{di} + \Delta\boldsymbol{A}_{di}, \ \ i = 1, 2, \ldots, m \tag{29}$$

Due to the non-negativeness of β (see (25)), one can infer from (24) that $\Delta V(\boldsymbol{x}(k)) < 0$ if $\boldsymbol{\Psi} < \boldsymbol{0}$. Thus, $\boldsymbol{\Psi} < \boldsymbol{0}$ along with (9) are sufficient conditions for the zero solution of the system (1)-(3) to be globally asymptotically stable. The condition $\boldsymbol{\Psi} < \boldsymbol{0}$ can be rewritten as

$$\hat{\boldsymbol{\Psi}} - \boldsymbol{\Upsilon}[-k_q\boldsymbol{G}]^{-1}\boldsymbol{\Upsilon}^T < \boldsymbol{0} \tag{30}$$

where

$$\boldsymbol{\Upsilon}^T = \left[-k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{A}} \ \ \boldsymbol{\Pi} \ \ \boldsymbol{0} \ \ \boldsymbol{0} \ \ \sqrt{\frac{-k_o}{2}}\boldsymbol{G} \right] \tag{31}$$

$$\boldsymbol{\Pi} = \left[-k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{A}}_{d1} \ \ -k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{A}}_{d2} \ \ \ldots - k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{A}}_{dm} \right] \tag{32}$$

$$\hat{\boldsymbol{\Psi}} = \begin{bmatrix} \hat{\boldsymbol{\Psi}}_1 & \boldsymbol{0} & \boldsymbol{\Xi}_2 & \boldsymbol{0} & \hat{\boldsymbol{\Psi}}_2 \\ * & \hat{\boldsymbol{\Psi}}_3 & \boldsymbol{\Xi}_6 & \boldsymbol{\Xi}_7 & \hat{\boldsymbol{\Psi}}_4 \\ * & * & \boldsymbol{\Xi}_{10} & \boldsymbol{\Xi}_{11} & \boldsymbol{0} \\ * & * & * & \boldsymbol{\Xi}_{12} & \boldsymbol{0} \\ * & * & * & * & \boldsymbol{\Xi}_{13} \end{bmatrix} < \boldsymbol{0} \tag{33}$$

$$\hat{\boldsymbol{\Psi}}_1 = -\boldsymbol{P} + \sum_{i=1}^{m} \left[\boldsymbol{Q_{1i}} + \boldsymbol{Q_{2i}} + (d_{12i} + 1)\boldsymbol{Q}_{3i} - \boldsymbol{Z}_{1i} \right] + \sum_{i=1}^{m} \left[d_{1i}^2 \boldsymbol{Z}_{1i} + d_{12i}^2 \boldsymbol{Z}_{2i} \right] \tag{34a}$$

$$\hat{\boldsymbol{\Psi}}_2 = -\sum_{i=1}^{m} \left[d_{1i}^2 \boldsymbol{Z}_{1i} + d_{12i}^2 \boldsymbol{Z}_{2i} \right] + k_q \bar{\boldsymbol{A}}^T \boldsymbol{G} \tag{34b}$$

$$\hat{\boldsymbol{\Psi}}_3 = diag\left\{ \bar{\boldsymbol{\rho}}_1, \bar{\boldsymbol{\rho}}_2, \ldots, \bar{\boldsymbol{\rho}}_m \right\} \tag{34c}$$

$$\hat{\boldsymbol{\Psi}}_4 = \begin{bmatrix} k_q\boldsymbol{G}\bar{\boldsymbol{A}}_{d1} & k_q\boldsymbol{G}\bar{\boldsymbol{A}}_{d2} & \cdots & k_q\boldsymbol{G}\bar{\boldsymbol{A}}_{dm} \end{bmatrix}^T \tag{34d}$$

Using the well-known Schur's complement, (30) yields

$$\left[\begin{array}{c|c} \hat{\boldsymbol{\Psi}} & \boldsymbol{\Upsilon} \\ \hline * & -k_q\boldsymbol{G} \end{array} \right] < \boldsymbol{0} \tag{35}$$

Employing (3a), (35) can be rewritten as

$$\boldsymbol{M}_0 + \bar{\boldsymbol{H}}_0 \boldsymbol{F}_0(k) \bar{\boldsymbol{E}}_0 + \bar{\boldsymbol{E}}_0^T \boldsymbol{F}_0^T(k) \bar{\boldsymbol{H}}_0^T < \boldsymbol{0} \tag{36}$$

where

$$\bar{\boldsymbol{H}}_0^T = \Big[\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ k_q\boldsymbol{H}_0^T\boldsymbol{G}\ \ -k_q\sqrt{-2k_o}\boldsymbol{H}_0^T\boldsymbol{G}\Big] \tag{37}$$

$$\bar{\boldsymbol{E}} = \Big[\boldsymbol{E}_0\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\Big] \tag{38}$$

$$\boldsymbol{M}_0 = \begin{bmatrix} \hat{\boldsymbol{\Psi}}_1 & \mathbf{0} & \boldsymbol{\Xi}_2 & \mathbf{0} & \boldsymbol{\Xi}_3 & \boldsymbol{\Xi}_4 \\ * & \hat{\boldsymbol{\Psi}}_3 & \boldsymbol{\Xi}_6 & \boldsymbol{\Xi}_7 & \hat{\boldsymbol{\Psi}}_4 & \boldsymbol{\Pi}^T \\ * & * & \boldsymbol{\Xi}_{10} & \boldsymbol{\Xi}_{11} & \mathbf{0} & \mathbf{0} \\ * & * & * & \boldsymbol{\Xi}_{12} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & \boldsymbol{\Xi}_{13} & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} \\ * & * & * & * & * & -k_q\boldsymbol{G} \end{bmatrix} < \mathbf{0} \tag{39}$$

In the light of Lemma 2.2, (36) is equivalent to

$$\boldsymbol{M}_0 + \epsilon_0^{-1}\bar{\boldsymbol{H}}_0\bar{\boldsymbol{H}}_0^T + \epsilon_0\bar{\boldsymbol{E}}_0^T\bar{\boldsymbol{E}}_0 < \mathbf{0} \tag{40}$$

where $\epsilon_0 > 0$. Using Schur's complement, (40) can be rewritten as

$$\begin{bmatrix} \boldsymbol{\Xi}_1 & \mathbf{0} & \boldsymbol{\Xi}_2 & \mathbf{0} & \boldsymbol{\Xi}_3 & \boldsymbol{\Xi}_4 & \mathbf{0} \\ * & \hat{\boldsymbol{\Psi}}_3 & \boldsymbol{\Xi}_6 & \boldsymbol{\Xi}_7 & \hat{\boldsymbol{\Psi}}_4 & \boldsymbol{\Pi}^T & \mathbf{0} \\ * & * & \boldsymbol{\Xi}_{10} & \boldsymbol{\Xi}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \boldsymbol{\Xi}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & \boldsymbol{\Xi}_{13} & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} & k_q\boldsymbol{G}\boldsymbol{H}_0 \\ * & * & * & * & * & -k_q\boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_0 \\ * & * & * & * & * & * & -\epsilon_0\boldsymbol{I} \end{bmatrix} < \mathbf{0} \tag{41}$$

Repeating the steps similar to (36)-(41), it is easy to show that (41) is equivalent to (10). This completes the proof of Theorem 3.1.

It can be observed that, under the condition where $d_{1i} = d_{2i} = d_i$ $(i = 1, 2, \ldots, m)$, Equations (1)-(3) can be used to represent a class of uncertain discrete-time systems with multiple constant delays subject to quantization and overflow nonlinearities.

Pertaining to the above, we have the following result.

Theorem 3.2. *The zero solution of the system described by* (1)-(3) *with* $0 < d_{1i} = d_{2i} = d_i$ $(i = 1, 2, \ldots, m)$ *is globally asymptotically stable if there exist* $n \times n$ *positive definite symmetric matrices* $\boldsymbol{P}$*,* $\boldsymbol{Q}_i$*,* $\boldsymbol{Z}_i$ $(i = 1, 2, \ldots, m)$*, a positive definite diagonal matrix* $\boldsymbol{G} = diag(g_1, g_2, \ldots, g_n)$*, and positive scalars* ϵ_i $(i = 0, 1, \ldots, m)$ *satisfying*

$$\begin{bmatrix} \boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2 & \boldsymbol{\Lambda}_3 & \boldsymbol{\Xi}_4 & \mathbf{0} \\ * & \boldsymbol{\Lambda}_4 & \boldsymbol{\Xi}_8 & \boldsymbol{\Xi}_9 & \mathbf{0} \\ * & * & \boldsymbol{\Lambda}_5 & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} & \boldsymbol{\Xi}_{14} \\ * & * & * & -k_q\boldsymbol{G} & \boldsymbol{\Xi}_{15} \\ * & * & * & * & \boldsymbol{\Xi}_{16} \end{bmatrix} < \mathbf{0} \tag{42}$$

where

$$\boldsymbol{\Lambda}_1 = -\boldsymbol{P} + \sum_{i=1}^{m}(\boldsymbol{Q}_i + d_i^2\boldsymbol{Z}_i - \boldsymbol{Z}_i) + \epsilon_0\boldsymbol{E}_0^T\boldsymbol{E}_0 \tag{43}$$

$$\boldsymbol{\Lambda}_2 = \begin{bmatrix} \boldsymbol{Z}_1 & \boldsymbol{Z}_2 & \cdots & \boldsymbol{Z}_m \end{bmatrix} \tag{44}$$

$$\mathbf{\Lambda}_3 = -\sum_{i=1}^{m} d_i^2 \boldsymbol{Z}_i + k_q \boldsymbol{A}^T \boldsymbol{G} \tag{45}$$

$$\mathbf{\Lambda}_4 = diag\left\{\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \ldots, \mathbf{\Gamma}_m\right\} \tag{46}$$

$$\mathbf{\Gamma}_1 = -\boldsymbol{Q}_1 - \boldsymbol{Z}_1 + \epsilon_1 \boldsymbol{E}_1^T \boldsymbol{E}_1 \tag{47}$$

$$\mathbf{\Gamma}_2 = -\boldsymbol{Q}_2 - \boldsymbol{Z}_2 + \epsilon_2 \boldsymbol{E}_2^T \boldsymbol{E}_2 \tag{48}$$

$$\mathbf{\Gamma}_m = -\boldsymbol{Q}_m - \boldsymbol{Z}_m + \epsilon_m \boldsymbol{E}_m^T \boldsymbol{E}_m \tag{49}$$

$$\mathbf{\Lambda}_5 = \boldsymbol{P} + \left(\frac{k_o}{2k_q} - 2\right) \boldsymbol{G} + \sum_{i=1}^{m} d_i^2 \boldsymbol{Z}_i \tag{50}$$

Proof: Choosing the Lyapunov functional as

$$\begin{aligned} V(\boldsymbol{x}(k)) = {} & \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) + \sum_{i=1}^{m} \sum_{j=-d_i}^{-1} \boldsymbol{x}^T(k+j)\boldsymbol{Q}_i \boldsymbol{x}(k+j) \\ & + \sum_{i=1}^{m} \left[\sum_{\theta=-d_i+1}^{0} \sum_{j=-1+\theta}^{-1} d_i \boldsymbol{\eta}^T(k+j)\boldsymbol{Z}_i \boldsymbol{\eta}(k+j) \right] \end{aligned} \tag{51}$$

and following the similar steps as shown in the proof of Theorem 3.1, one can easily arrive at Theorem 3.2. The details of the proof of Theorem 3.2 are, therefore, omitted. This completes the proof of Theorem 3.2.

In the following, we present how a delay-independent stability criterion [[13], Theorem 1] for systems with multiple constant delays is recovered from Theorem 3.2 as a special case.

Corollary 3.1. *The zero solution of the system described by* (1)-(3) *with* $0 < d_{1i} = d_{2i} = d_i$ $(i = 1, 2, \ldots, m)$ *is globally asymptotically stable if there exist* $n \times n$ *positive definite symmetric matrices* $\boldsymbol{P}$, $\boldsymbol{Q}_i$, $\boldsymbol{Z}_i$ $(i = 1, 2, \ldots, m)$, *a positive definite diagonal matrix* $\boldsymbol{G} = diag(g_1, g_2, \ldots, g_n)$, *and positive scalars* ϵ_i $(i = 0, 1, \ldots, m)$ *satisfying*

$$\begin{bmatrix} \mathbf{\Omega}_1 & \mathbf{0} & k_q \boldsymbol{A}^T \boldsymbol{G} & \mathbf{\Xi}_4 & \mathbf{0} \\ * & \mathbf{\Omega}_2 & \mathbf{\Xi}_8 & \mathbf{\Xi}_9 & \mathbf{0} \\ * & * & \mathbf{\Omega}_3 & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} & \mathbf{\Xi}_{14} \\ * & * & * & -k_q \boldsymbol{G} & \mathbf{\Xi}_{15} \\ * & * & * & * & \mathbf{\Xi}_{16} \end{bmatrix} < \mathbf{0} \tag{52}$$

where

$$\mathbf{\Omega}_1 = -\boldsymbol{P} + \sum_{i=1}^{m} \boldsymbol{Q}_i + \epsilon_0 \boldsymbol{E}_0^T \boldsymbol{E}_0 \tag{53}$$

$$\mathbf{\Omega}_2 = diag\left\{-\boldsymbol{Q}_1 + \epsilon_1 \boldsymbol{E}_1^T \boldsymbol{E}_1, -\boldsymbol{Q}_2 + \epsilon_2 \boldsymbol{E}_2^T \boldsymbol{E}_2, \ldots, -\boldsymbol{Q}_m + \epsilon_m \boldsymbol{E}_m^T \boldsymbol{E}_m\right\} \tag{54}$$

$$\mathbf{\Omega}_3 = \boldsymbol{P} + \left(\frac{k_o}{2k_q} - 2\right) \boldsymbol{G} \tag{55}$$

Proof: By choosing the parameters $\boldsymbol{Z}_i = \dfrac{\lambda_i \boldsymbol{I}}{d_i^2}$ $(i = 1, 2, \ldots, m)$, for sufficiently small scalars λ_i $(i = 1, 2, \ldots, m)$, the LMI (42) of Theorem 3.2 leads to Corollary 3.1. This completes the proof of Corollary 3.1.

Remark 3.1. *Relative to the methods presented in previous works* [13, 14], *the criteria presented in this paper utilize a tighter bounding technique based on the reciprocal convexity* *method* [31] *to deal with the sum terms, which may reduce the computational burden and simplify system analysis/synthesis procedure. It is also important to mention that the present criteria may provide better solution to check the global asymptotic stability of the system under consideration of which involves multiple time-varying delays/constant delays.*

Remark 3.2. *Condition* (52) *has been reported in* [13] *in the context of the global asymptotic stability of discrete-time state-delayed systems using quantization and overflow nonlinearities.*

Remark 3.3. *The delay-independent stability criteria do not depend on the size of the delays which makes them unsuitable for determining the global asymptotic stability of systems that are stable under non-zero delay conditions.*

Remark 3.4. *The global asymptotic stability criteria obtained in Theorems 3.1 and 3.2 are dependent on the values of k_o and k_q. For a given system described by* (1)-(3)*, it may happen that the system is globally asymptotically stable for a set of values of k_o and k_q, while the system may show unstable behavior for another set of values of k_o and k_q. Thus, Theorems 3.1 and 3.2 may also be helpful to identify the combinations of quantization and overflow that would ensure the global asymptotic stability of the systems under consideration.*

Remark 3.5. *The presented approach is based on exploiting the sector information of the nonlinearities and, therefore, not applicable for the discrete-time systems involving value truncation quantization. The value truncation quantization, owing to its peculiar characteristics is not fit for sector-based analysis.*

Remark 3.6. *The conditions given in Theorems 3.1 and 3.2 are in the form of LMIs and can be conveniently solved using MATLAB environment along with YALMIP 3.0 parser* [40] *and SeDuMi 1.21 solver* [41].

4. Examples.

Example 4.1. *To demonstrate the applicability of the presented results, we now consider the following example.*

Consider a discrete-time system described by Equations (1)-(3) *with*

$$m = 2,\ \boldsymbol{A} = \begin{bmatrix} 1 & 0.05 \\ 0.1 & 0.4 \end{bmatrix},\ \boldsymbol{A}_{d1} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix},\ \boldsymbol{A}_{d2} = \begin{bmatrix} 0 & 0.1 \\ 0.01 & -0.2 \end{bmatrix}, \tag{56}$$

$$\begin{aligned} &\boldsymbol{H}_0 = \boldsymbol{H}_1 = \begin{bmatrix} 0 \\ 0.12 \end{bmatrix},\ \boldsymbol{H}_2 = \begin{bmatrix} 0 \\ 0.14 \end{bmatrix},\ \boldsymbol{E}_0 = [0.01 \quad 0], \\ &\boldsymbol{E}_1 = [0 \quad 0.01],\ \boldsymbol{E}_2 = [0.02 \quad 0],\ d_{11} = 2,\ d_{12} = 4 \end{aligned} \tag{57}$$

and the sector $[k_o, k_q] = [0, 1]$ which includes saturation, zeroing, magnitude truncation, combination of saturation and magnitude truncation, combination of zeroing and magnitude truncation, etc. Now, our objective is to determine the upper delay bounds d_{21} and d_{22} for given lower delay bounds d_{11} and d_{12}, respectively such that the present system is globally asymptotically stable. Using the criterion given in Theorem 3.1, it is observed that the present system is found to be globally asymptotically stable over the delay ranges $2 \le d_1(k) \le 16$ and $4 \le d_2(k) \le 12$.

It may be noted that the example under consideration falls outside the application scope of [13, 14, 16].

In the next example, we present the advantages of our criterion as compared to a previously reported criterion.

Example 4.2. *Consider the following autonomous delay-difference system* [36]

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \sum_{i=1}^{3} \boldsymbol{A}_{di}\boldsymbol{x}(k-d_i) \tag{58}$$

where d_i $(i = 1, 2, 3)$ *represent the constant delays in the state and the system matrices* $\boldsymbol{A}$ *and* $\boldsymbol{A}_{di}$ $(i = 1, 2, 3)$ *are given by*

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & 0.15 \\ 0 & 0.45 \end{bmatrix}, \quad \boldsymbol{A}_{d1} = \begin{bmatrix} 0.05 & -0.2 \\ 0.05 & 0.05 \end{bmatrix},$$
$$\boldsymbol{A}_{d2} = \begin{bmatrix} 0 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}, \quad \boldsymbol{A}_{d3} = \begin{bmatrix} -0.2 & 0.2 \\ -0.1 & 0.22 \end{bmatrix} \tag{59}$$

Under finite wordlength implementation and in the presence of parameter uncertainties, the system described by (58) *can be written in the following form*

$$\boldsymbol{x}(k+1) = \boldsymbol{O}\{\boldsymbol{Q}(\boldsymbol{y}(k))\} = \boldsymbol{f}(\boldsymbol{y}(k)) \tag{60a}$$

$$\boldsymbol{y}(k) = (\boldsymbol{A} + \Delta\boldsymbol{A}(k))\boldsymbol{x}(k) + \sum_{i=1}^{m}(\boldsymbol{A}_{di} + \Delta\boldsymbol{A}_{di}(k))\boldsymbol{x}(k-d_i) \tag{60b}$$

with $m = 3$. *A delay-independent criterion for the stability of the system* (60) *is provided in the form of Theorem 1 in* [13] *which has been reproduced in the form of Corollary 3.1 in this paper. Now, with*

$$\boldsymbol{H}_0 = \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix}, \ \boldsymbol{H}_1 = \begin{bmatrix} 0 \\ 0.12 \end{bmatrix}, \ \boldsymbol{H}_2 = \begin{bmatrix} 0 \\ 0.14 \end{bmatrix}, \ \boldsymbol{H}_3 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \ \boldsymbol{E}_0 = [0.01 \quad 0.18],$$
$$\boldsymbol{E}_1 = [0 \quad 0.01], \ \boldsymbol{E}_2 = [0.02 \quad 0], \ \boldsymbol{E}_3 = [0.02 \quad 0], \ k_o = 0, \ k_q = 1 \tag{61}$$

it is observed that the delay-independent criterion Theorem 1 [13] *is unable to determine the feasibility of the system under consideration.*

Next, using Theorem 3.2 the present system is found to be stable for $d_1 = 8$, $d_2 = 5$ and $d_3 = 6$. Thus, the proposed Theorem 3.2 demonstrates the advantage in terms of conservativeness as compared to the previously reported criterion [13].

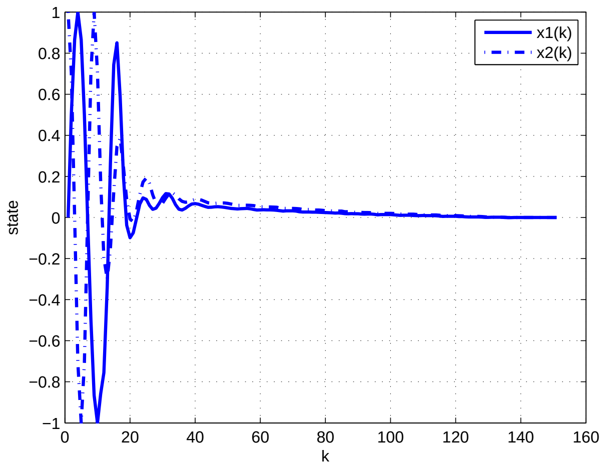

FIGURE 1. State trajectories of the system with quantization and overflow nonlinearities considered in Example 4.2

With arbitrary initial conditions, the results obtained in Example 4.2 via Theorem 3.2 have also been verified by the simulation result shown in Figure 1. The simulation result is obtained by assuming the system to be under the influence of magnitude truncation quantization and saturation overflow nonlinearities.

5. **Conclusion.** Two new delay-dependent stability criteria have been presented for uncertain discrete-time systems with multiple delays subject to various combinations of quantization and overflow nonlinearities. The examples show the applicability of the presented results for the system under consideration. The presented results may be extended to multidimensional systems.

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