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INVESTIGATION OF THE RELATION BETWEEN PRODUCTION DENSITY AND LEAD-TIME VIA STOCHASTIC ANALYSIS

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Abstract. *We modeled the mathematical modeling of the production process using the diffusion equation in the deterministic system in previous paper. We have also proposed some modeling and system evaluation by considering the production process as stochastic system. Based on these results, we report on the stochastic model of the production density distribution. The relationship between production density and lead time (throughput) has not yet been clearly established. As a result, we clarify the effect of fluctuations of lead time on production density by utilizing stochastic analysis.*

Keywords: Production density, Lead time, Diffusion process, Fokker-Plank equation, Stochastic analysis

1. Introduction.

We have working on mathematical modeling and system evaluation of production process targeting a small-to-medium-sized equipment manufacturing industry. A human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior in our business area.

Firstly, we had worked on a physical model of the production process using a one-dimensional diffusion equation with respect to mathematical modeling of deterministic systems [1, 2]. In our previous studies related to this topic, we reported a production propagation model as a deterministic system and subsequently proposed a lead-time analysis method [3].

With respect to a stochastic system of the production process, we have introduced an idea of a production level corresponding to an energy level being discussed in physics. A valence electron transits to a conducting state due to a rise in potential (transition of a manufacturing process), and lowers an energy level by radiating energy with time [4, 5, 6]. We have also proposed a stochastic differential equation (SDE) for the mathematical model describing production processes from the input of materials to the end. We utilized a risk-neutral principal in stochastic calculus based on the SDE [4, 7].

With respect to the analysis of production processes in stochastic systems based on financial engineering, we have proposed that a production throughput rate can be estimated utilizing a Kalman filter based on a stochastic differential equation [4, 8, 9].

However, the many concerns that occur in the supply chain are major problems facing production efficiency and business profitability. A stochastic partial bilinear differential equation with time delay was derived for outlet processes. The supply chain was modeled by considering as time delay [10].

On the other hand, when a delay occurs in a stage, the delay propagates to the successive stage in manufacturing. This delays the entire production process, which is equivalent to fluctuations in physical phenomena. A delay in the entire process is attributed to the propagation of fluctuations (volatility). We mathematically analyzed this phenomenon [7, 11, 12].

With respect to a lead-time analysis, we implemented a lead-time function and a loss function to calculate the expected loss value. In other words, it can be assumed that if the cash flow is critically required for lead-time, it can be obtained before the production process. Furthermore, it is possible to identify lead-time in advance as suitable targets, which is a very innovative approach [3, 13].

In this study, we present the stochastic model of the production density distribution. The relationship between production density and lead time (throughput) has not yet been clearly established. The changes in lead time cause fluctuations in production density. We build a production propagation model based on a stochastic theory by considering lead time as a medium. We propose two stochastic differential equations as a mathematical model. One is related to production density, and the other relates to lead time. A major feature of this paper is that the production density is a functional as a variable with time and a lead-time as a variable. We further consider through stochastic analysis that lead time is strongly related to production density. Then, on the basis of the concept of continuity approximation, we build a mathematical model that considers production density. This idea is based on the diffusion approximation of a production process, which was used as a deterministic model in our previous research.

2. Production Business of a Small-to-Midsize Firm.

2.1. Production systems in the production equipment industry. We refer to the production system in manufacturing equipment industry studied in this paper. This is not a special system but “Make-to-order system with version control”. Make-to-order system is a system which allows necessary manufacturing after taking orders from clients, resulting in “volatility” according to its delivery date and lead time. In addition, “volatility” occurs in lead time depending on the contents of make-to-order products (production equipment).

However, effective utilization of the production forecast information on the orders may suppress certain amount of “variation”, but the complete suppression of variation will be difficult. In other words, “volatility” in monthly cash flow occurs and of course influences a rate of return in these companies. Production management system, suitable for the separate make-to-order system which is managed by numbers assigned to each product upon order, is called as “product number management system” and is widely used.

All productions are controlled with numbered products and instructions are given for each numbered product.

Thus, ordering design, logistics and suppliers are conducted for each manufacturer’s serial numbers in most cases except for semifinished products (unit incorporated into the final product) and strategic stocks.

Therefore, careful management of the lead time or production date may not suppress “volatility” in manufacturing (production).

The company in this study is the “supplier” in Figure 1 and “factory” here. Companies are under the assumption that there are N (numbers of) suppliers; however, this study deals with one company because no data is published for the rest of the companies ($N-1$).

2.2. Production flow process. A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the

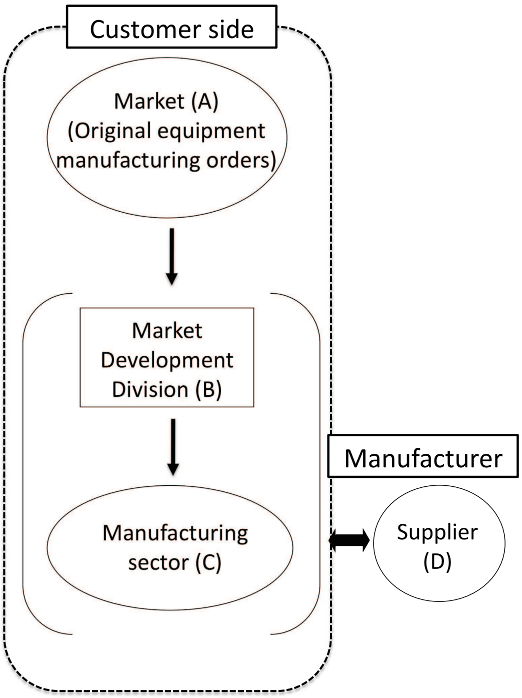

FIGURE 1. Business structure of company of research target

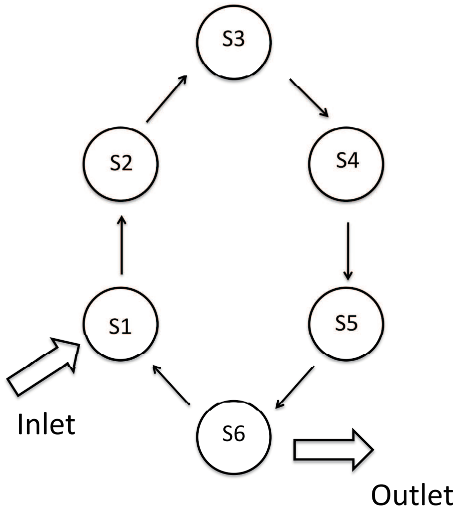

FIGURE 2. Production flow process

production process. In Figure 2, the process consists of six stages. In each step S1-S6 of the manufacturing process, materials are being produced.

Figure 2 represents a manufacturing process called a flow production system, which is a manufacturing method employed in the production of control equipment. The flow production system, which in this case has six stages, is commercialized by the production of material in steps S1-S6 of the manufacturing process.

The direction of the arrow represents the direction of the production flow. In this system, production materials are supplied from the inlet and the end product will be shipped from the outlet.

Assumption 2.1. *The production structure is nonlinear.*

Assumption 2.2. *The production structure is a closed structure; that is, the production is driven by a cyclic system (production flow system).*

Assumption 2.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the throughput generation structure in a stochastic manufacturing process (hereafter called the manufacturing field). Because such a structure is at least dependent on the demand, it is considered to have a nonlinear structure.

Because the value of such a product depends on the throughput, its production structure is nonlinear. Therefore, Assumption 2.1 reflects the realistic production structure and is somewhat valid. Assumption 2.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption 2.2 is reasonable because new production starts from S1.

Based on the control equipment, the product can be manufactured in one cycle. The production throughput required to maintain 6 pieces of equipment/day is as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25 \text{ (min)} \tag{1}$$

where the throughput of the previous process is set as 20 (min). In Equation (1), "28" represents the throughput of the previous process plus the idle time for synchronization. "8" is the number of processes and the total number of all processes is "8" plus the previous process. "60" is given by 20 (min) $\times$ 3 (cycles).

One process throughput (20 min) in full synchronization is

$$T_s = 3 \times 120 + 40 = 400 \text{ (min)} \tag{2}$$

Therefore, a throughput reduction of about 10% can be achieved. However, the time between processes involves some asynchronous idle time.

As a result, the above test run is as follows. Tables 4-8 are shown in Appendix A.

- (test run1): Each throughput in every process (S1-S6) is asynchronous, and its process throughput is asynchronous. Table 4 represents the manufacturing time (min) in each process. Table 5 represents the variance in each process performed by workers. Table 4 represents the target time, and the theoretical throughput is given by $3 \times 199 + 2 \times 15 = 627$ (min).

  In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. Figure 22 is a graph illustrating the measurement data in Table 4, and it represents the total working time for each worker (K1-K9). The graph in Figure 23 represents the variance data for each working time in Table 4.
- (test run2): Set to synchronously process the throughput.

  The target time in Table 6 is 500 (min), and the theoretical throughput (not including the synchronized idle time) is 400 (min). Table 7 represents the variance data of each working process (S1-S6) for each worker (K1-K9).
- (test run3): The process throughput is performed synchronously with the reclassification of the process. The theoretical throughput (not including the synchronized idle time) is 400 (min) in Table 8.

  Table 9 represents the variance data of Table 8. "WS" in the measurement tables represents the standard working time. This is an empirical value obtained from long-term experiments.

3. **Distribution System and Diffusion Equation of the Production Process.** From Figure 3, we refer to the network capacity (i.e., a statically acceptable amount of production) in an interprocess network (a production field) as R. An interprocess network indicates a sequential flow from one process to the other after the completion of the current process. Here assuming that the production density function for the i-th equipment is $S_i(x,t)$, $S_i(x,t)$ is expressed by

$$[J(x,t)dt - J(x+dx,t)dt]R = [S_i(x,t+dt) - S_i(x,t)]Rdx \tag{3}$$

where J is the production flow [1, 14].

We define production flow as the displacement of a production density function in the unit-production direction. The production density function is proportional to the cost necessary for production; thus, it can be considered as the production cost per unit production. Furthermore, as production leads to returns, the production density function can be considered as returns.

$$\frac{\partial S_i(x,t)}{\partial t} = D\frac{\partial^2 S_i(x,t)}{\partial x^2} \tag{4}$$

where D is the diffusion coefficient, t is the time variable, and x is the spatial variable.

A model of the production process, which is connected in one dimension, is described as follows. The process of production is indicated by the movement of production units from one process (node) to another. This production flow is equivalent to transmission

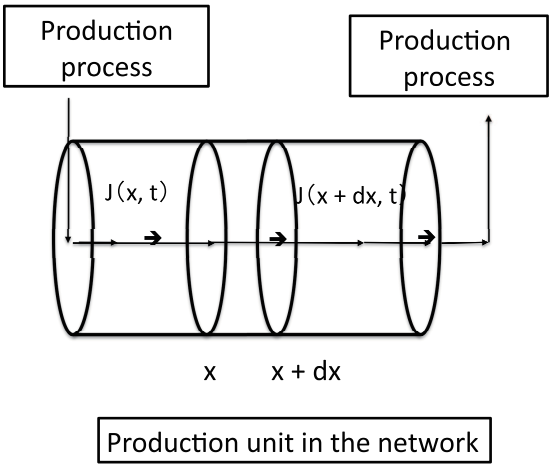

FIGURE 3. Network inter-process division of worker

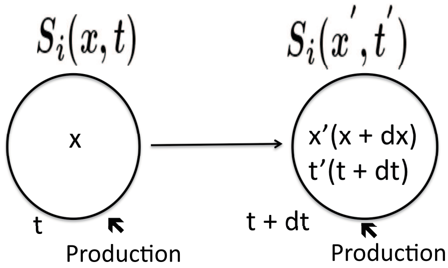

FIGURE 4. Unit of production by changing the excitation force

rate, which is defined as the rate of data flow between connected nodes in communication engineering. Accordingly, we formulate the production model in a manner similar to heat propagation in physics. Thus, the production process is modeled mathematically using a continuous diffusion type of partial differential equation consisting of time and spatial variables [1].

Setting the network capacity (the available static production volume) to R in an inter-process network (production field, equivalent to a stochastic field), we obtain the following:

$$[J(x)dt - J(X + dx)dt]R = [S(t + dt) - S(t)]Rdx \tag{5}$$

where J is the production flow and S is the production density.

In the present model, the production flow indicates the displacement of production processes in the direction related to the production density. In other words, the production cost per production is as follows.

Definition 1. *Production cost per unit production*

$$J = -D\frac{\partial S}{\partial x} \tag{6}$$

where D is a diffusion coefficient.

From Equation (5), we obtain

$$-\frac{\partial J}{\partial x} = \frac{\partial S}{\partial t} \tag{7}$$

From Equations (6) and (7), we obtain

$$\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} \tag{8}$$

where $t \in [0, T]$, $x \in [0, L] \equiv \Omega$, $S(0, x) = S_0(x)$, $B_x S(t, x)|_{x=\partial\Omega}$.

This equation is equivalent to the diffusion equation derived from the minimization condition of free energy in a production field [1, 2]. The connections between processes can be treated as a diffusive propagation of products (refer to Figure 3).

As shown in Figure 4, X represents the production elements that constitute a unit production and varies $X \to X'$ at $[t + dt]$. In other words, the unit production varies by exciting the external force and is the basis for revenue generation (an increase of potential energy). Therefore, in the transition $S_i(t, x) \to S_i(t, x')$, the production cost, which is the cumulated external force, increases. The connections between production processes are referred to as "joints".

In the general idea of production flow, we define the joint propagation model at multiple stages in the production process and the potential energy in the production field.

Thereafter, we can construct a control system, which increases the process throughput, by calculating the gradient function in the autonomous distributed system. The gradient function is described in the next opportunity.

$$\frac{\partial S}{\partial t} + \Delta(v \cdot S) = \frac{1}{2}\Delta(D^2 S) + \lambda \tag{9}$$

where λ denotes a forced external force function and v denotes a production propagation speed. Here, λ is omitted.

We assume that S defines as follows: S represents a production density with a fluctuation, and v also causes a fluctuation in throughput. As a result, a production is proportional to the slope of production density.

Definition 3.1. *Mathematical model of each stage*

$$dx(t) = \left\{a(t,x)dt + c(t,x)d\tilde{B}(t)\right\} + D(t,x)dB(t) \tag{10}$$

where $\tilde{B}$ and B denote an independent Brownian motion. c denotes a fluctuation term, which follows a stochastic differential equation.

The first term on the right-hand side of Equation (10) denotes the flow of the medium, and the second term represents the fluctuation of diffusion. Moreover, $a(t,\cdot)$ denotes an average lead-time and $c(t,x)d\tilde{B}(t)$ denotes a fluctuation around processes [10, 12].

We report a stochastic approach for a production process based on the production density equation [1], i.e., a fluctuation is induced by a stochastic characteristic of a lead-time function. In this case, we apply stochastic analysis to evaluating the manner in which the production density is constrained.

Generally, Equation (9) with constrained such as Equation (10) can be derived as follows:

$$\begin{aligned}\partial S(t,x) = &\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x) + c^2(t,x)\right\}S(t,x) - \frac{\partial}{\partial x}\left(a(t,x)S(t,x)\right)\right]dt \\ &+ \frac{\partial}{\partial x}\left\{c(t,x)S(t,x)\right\}\partial\tilde{B}(t)\end{aligned} \tag{11}$$

where $S(t,x)$ denotes a production density and is derived as follows [1]:

$$S(t, I_h^x) = \int_0^t P(\tau, x_0; t, I_h^x) S(\tau, x_0) d\tau \tag{12}$$

where $I_h^x \equiv [x, x+h]$.

From Equation (12), a production density distribution varies according to increasing a production density.

$S(t,x)$ satisfies a Fokker-Plank equation as follows [16, 17, 18, 19].

$$\frac{\partial S(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x)S(t,x)\right\} - \frac{\partial}{\partial x}\{a(t,x)S(t,x)\} \tag{13}$$

where $x(t)$ satisfies Equation (10).

According to Okazaki's analysis, we obtain as follows [15]:

$$\partial S(t,x) = \left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x) + c^2(t,x)\right\}S(t,x) - \frac{\partial}{\partial x}\left(a(t,x)S(t,x)\right)\right]dt$$

$$+\frac{\partial}{\partial x}\left\{c(t,x)S(t,x)\right\}\partial\tilde{B}(t) \tag{14}$$

where $D^2(t,x)+c^2(t,x)$ denotes a trend, $a(t,x)S(t,x)$ denotes a fluctuation of stages and $c(t,x)S(t,x)$ denotes also a fluctuation of lead-time.

Definition 3.2. *Trend function of a production density distribution*

$$m(t,x)=E[S(t,x)] \tag{15}$$

According to Equation (9), $m(t,x)$ is derived as follows:

$$\frac{\partial}{\partial t}m(t,x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left\{D^2(t,x)+c^2(t,x)\right\}m(t,x)\right]-\left\{a(t,x)m(t,x)\right\} \tag{16}$$

where the dispersion covariance of a production density $\chi(t,x,x^{'})$ is defined as follows.

Definition 3.3. *Dispersion covariance of a production density* $\chi(t,x,x^{'})$

$$\chi\left[t,x,x^{'}\right]=E\left[S(t,x)\cdot S\left(t,x^{'}\right)\right],\ t\in R,\ x^{'}\in R \tag{17}$$

where R *denotes Euclidean space.*

From Equation (15), we obtain as follows:

$$Cov.\left[S(t,x)\cdot S\left(t,x^{'}\right)\right]=\chi\left(t,x,x^{'}\right)-m(t,x)\cdot m\left(t,x^{'}\right) \tag{18}$$

According to a stochastic process theory, the following equation holds.

$$\begin{aligned}
d\left\{S(t,x)\cdot S\left(t,x^{'}\right)\right\} &= S(t,x)\cdot dS\left(t,x^{'}\right)+S\left(t,x^{'}\right)\cdot dS(t,x)\\
&\quad +\frac{1}{2}\cdot 2\cdot d<S(\bullet,x),S\left(\bullet,x^{'}\right)>_t\\
&= S(t,x)\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x)+c^2(t,x)\right\}S\left(t,x^{'}\right)\right.\\
&\quad \left.-\frac{\partial}{\partial x^{'}}\left\{a\left(t,x^{'}\right)S\left(t,x^{'}\right)\right\}\right]dt\\
&\quad +S\left(t,x^{'}\right)\left[\frac{1}{2}\frac{\partial^2}{\partial x^{'2}}\left\{D^2(t,x)+c^2(t,x)\right\}S(t,x)\right.\\
&\quad \left.-\frac{\partial}{\partial x}\left\{a(t,x)S(t,x)\right\}\right]dt\\
&\quad +\frac{\partial}{\partial x}\{c(t,x)S(t,x)\}\frac{\partial}{\partial x^{'}}\left\{c\left(t,x^{'}\right)S\left(t,x^{'}\right)\right\}\\
&\quad +S(t,x)\frac{\partial}{\partial x^{'}}\left\{c\left(t,x^{'}\right)S\left(t,x^{'}\right)\right\}d\tilde{B}(t)\\
&\quad +S\left(t,x^{'}\right)\frac{\partial}{\partial x}\{c(t,x)S(t,x)\}d\tilde{B}(t)
\end{aligned} \tag{19}$$

Then, we obtain the dispersion covariance of a production density between stages as follows by taking the average value.

$$\begin{aligned}
\frac{\partial}{\partial t}\chi\left[t,x,x^{'}\right] &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left(D^2(t,x)+c^2(t,x)\right)\chi\left(t,x,x^{'}\right)\right]\\
&\quad +\frac{1}{2}\frac{\partial^2}{\partial x^{'2}}\left[D^2\left(t,x^{'}\right)+c^2\left(t,x^{'}\right)\chi\left(t,x,x^{'}\right)\right]\\
&\quad -\frac{\partial}{\partial x}\left\{a(t,x)\chi\left(t,x,x^{'}\right)\right\}-\frac{\partial}{\partial x^{'}}\left\{a\left(t,x^{'}\right)\chi\left(t,x,x^{'}\right)\right\}
\end{aligned} \tag{20}$$

where $a(t,x) > 0$, $b(t,x) > 0$ and $c(t,x) > 0$.

Definition 3.4. *Correlation function of lead-time function between stages*

$$dx^{i+1}(t) = \left\{ a\left(t, x^{i+1}\right) dt + \int_R c\left(t, x^i, x^{i+1}\right) \tilde{B}\left(dt, dx^{i+1}\right) \right\} + b\left(t, x^{i+1}(t)\right) dB^i(t) \quad (21)$$

The production density distribution satisfies as follows based on Equation (21):

$$dS(t,x) = \left[\frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ b^2(t,x) + \int_R c^2(t,x,z) dz S(t,x) \right\} - \frac{\partial}{\partial x} \{a(t,x)S(t,x)\} \right] dt + \int_R \frac{\partial}{\partial x} \left\{ c\left(t,x,x'\right) S(t,x) \right\} \tilde{B}\left(dt, dx'\right) \quad (22)$$

4. **Preparation for Numerical Calculation.**

4.1. **Trend function of production density distribution.** We present an example for numerical parameters such as the following: $a > 0$, $b > 0$ and $c > 0$ are constant parameters. Let $S(0,x) = \delta(x)$, which denotes as follows:

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} \right), \; x \in R \quad (23)$$

The condition of these parameters $a > 0$, $b > 0$ and $c > 0$ represents that a production density exists between any stages.

Then, according to Equation (16), we obtain as follows:

$$\frac{\partial}{\partial t} m(t,x) = \frac{1}{2} \left(r \frac{\partial^2}{\partial x^2} \right) - am(t,x) \quad (24)$$

According to Equation (16), we obtain as follows:

$$\frac{\partial \chi\left(t,x,x'\right)}{\partial t} = \frac{1}{2} r \left\{ \frac{\partial^2 \chi\left(t,x,x'\right)}{\partial x^2} + \frac{\partial^2 \chi\left(t,x,x'\right)}{\partial x'^2} \right\} - a \left\{ \frac{\partial \chi\left(t,x,x'\right)}{\partial x} + \frac{\partial \chi\left(t,x,x'\right)}{\partial x'} \right\} \quad (25)$$

From Equation (24), we obtain as follows:

$$m(t,x) = \frac{1}{\sqrt{2\pi} rt} \exp\left(-\frac{(x-at)^2}{2r^2t^2} \right) \quad (26)$$

Similarly, according to Equation (25), we obtain as follows:

$$\chi\left(t,x,x'\right) = \frac{1}{2\pi(r^2 - c^4)t} \exp\left(-\frac{1}{2(r^2-c^4)t} \times \left\{ r(x-at)^2 - 2c^2(x-at)\left(x'-at\right) + r\left(x'-at\right)^2 \right\} \right) \quad (27)$$

where $r = D^2 + c^2$.

From Equation (27), the numerical data of correlation function can be calculated for x and x' of production density.

$$dS(t,x) = \frac{1}{2} \left[\left\{ D^2 + \int_R c^2\left(t,x,x'\right) \right\} \frac{\partial^2 S(t,x)}{\partial x^2} - a \frac{\partial S(t,x)}{\partial x} \right] dt + \frac{\partial}{\partial x} \left[\int_R c\left(t,x,x'\right) S(t,x) \tilde{B}\left(dt, dx'\right) \right] \quad (28)$$

where $\tilde{B}(dt, dx')$ denotes any of the k interval $F_1 = I_1 \times J_1, F_2 = I_2 \times J_2, \cdots, F_k = I_k \times J_k \subset R^2$. $(B(F_1), B(F_2), \cdots, B(F_k))'$ in $\tilde{B}(dt, dx')$ denotes a k-dimensional normal distribution with average zero. However, from Equation (11) in case of a single Brownian motion, we obtain as follows:

$$\partial S(t,x) = \frac{1}{2}\left[\left(D^2 + c^2\right)\frac{\partial^2 S(t,x)}{\partial x^2}\right]\partial t - a\frac{\partial}{\partial x}S(t,x)\partial t + c\frac{\partial}{\partial x}S(t,x)\tilde{B}(t) \tag{29}$$

The aforementioned calculation clarifies that the trend of a production density distribution fluctuation represents a normal distribution from Equation (26). In the case of a single Brownian motion, the trend denotes a stochastic diffusion partial differential equation from Equation (29). In other words, the motion of the trend is affected by the coefficient c, which is caused in a lead-time fluctuation.

With respect to a lead-time distribution, we obtain from Equation (10) as follows:

$$dx(t) = \left\{adt + cd\tilde{B}(t)\right\} + DdB(t) \tag{30}$$

When Equation (31) is derived, the stochastic model of a production density distribution is derived by Equation (29).

For simplicity, let $\tilde{B}(t) \approx B(t)$, we can rewrite Equation (31) as follows:

$$dx(t) = adt + (c + D)dB(t) \tag{31}$$

4.2. **Partial differential equation eigenvalue problem.** We define a partial differential equation eigenvalue problem under appropriate boundary conditions.

Definition 4.1. *Partial differential operator $\mathcal{L}$*

$$\mathcal{L} \equiv \left(D^2 + c^2\right)\frac{\partial^2}{\partial x^2} - a\frac{\partial}{\partial x} \tag{32}$$

Definition 4.2.

$$S_i(t) = \int_R S(t,x)\varphi_i(x)dx \tag{33}$$

$$S(t,x) = \sum_i S_i(t)\varphi_i(t),\ i = 1, 2, \cdots \tag{34}$$

$$\mathcal{L} \equiv \left(D^2 + c^2\right)\frac{\partial^2}{\partial x^2} - a\frac{\partial}{\partial x} \tag{35}$$

From Green's theorem, Equation (29) can be rewritten as follows:

$$dS_i(t) = \lambda_i S_i(t) + \sigma(x)S_i(t)dB(t),\ i = 1, 2, \cdots \tag{36}$$

where $\sigma(x) \equiv \frac{d\varphi_i(x)}{dx}$.

Equation (36) denotes a state-dependent stochastic differential equation (log-normal type). In Figure 5, lead-time fluctuations are strongly related to production density; they represent a phenomenon in which mutual fluctuation occurs. Characters "A" and "B" in Figure 5 represent a lead-time fluctuation and a production density fluctuation, respectively. The mutual fluctuation between a lead time and production density represents the fluctuations in the actual data of test runs 1 through 3 in Appendix A. The work time indicated by the circle in the table indicates that the reference time is over in Tables 4, 6 and 8 of Appendix A.

Definition 4.3. *Probability density function of production density distribution in stationary $S_p(x : \mu_x, \sigma_x)$, where μ_x and σ_x represent a trend coefficient and a volatility depending on x respectively.*

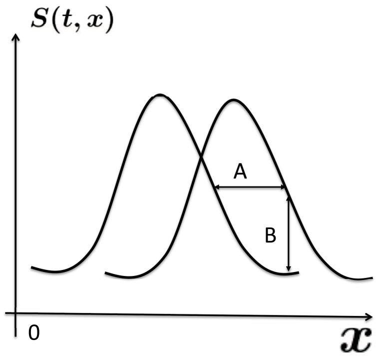

Figure 5. Influence of lead-time fluctuation

Thus, the expectation $F_E(x;\mu_x,\sigma_x)$ is derived as follows:

$$F_E(x;\mu_x,\sigma_x) = \int_R g(x)dF_p(x;\mu_x,\sigma_x),\ \frac{dF_p(x;\mu_x,\sigma_x)}{dx} = S_p(x;\mu_x,\sigma_x) \tag{37}$$

Similarly, the production density fluctuation affects the expected total production volume. Then, the stochastic model of production density distribution is denoted as follows in the case of a single Brownian motion:

$$dx(t) = a(t,x) + c(t,x)d\tilde{B}(t) \tag{38}$$

$$\partial S(t,x) = \frac{1}{2}\left\{\left(D^2+c^2\right)\frac{\partial^2 S(t,x)}{\partial x^2} - a(t,x)\frac{\partial S(t,x)}{\partial x}\right\} + c(t,x)\frac{\partial S(t,x)}{\partial x}d\tilde{B}(t) \tag{39}$$

4.3. **Numerical simulation of the trend function of a production density distribution.** A trend function, which denotes an expectation of a production density function, represents a lead time and is dependent on the trend function coefficient but not on the volatility. An autocorrelation function is also dependent on the trend function coefficient. With respect to the time trend of a production density function $S(t,x)$, the function is affected by the trend function and the effect of x is large, especially in the case of nonlinear terms such as a triangle function or δ function.

Figures 6 through 9 show the solution values of a stochastic differential equation, which denotes the constant data of Figure 6 and the nonlinear data of Figures 7 through 9 with respect to σ. In Figures 10 through 15, regarding parameter "a", symbol "▲" is set to three times of the symbol "■" and the symbol "■" is set to ten times of the symbol "♦".

In each Figures 10 through 15, "a" affects the trend function value greatly. However, with respect to Figures 10 through 15, it can be said that "a" has less influence than "c" and "D" on the trend function value. In other words, "c" denotes a fluctuation parameter and "D" denotes the parameter corresponding to the diffusion coefficient. Because the trend function values of Figures 14 and 15 are lower than those of the other Figures 10 through 13. It can be said that "D" is a parameter corresponding to the diffusion coefficient. It can be also said that "c" is a parameter corresponding to the production retention.

Table 1. Parameter setting of trend function of production density distribution

Figure number	Initial value S_0	Average μ	Volatility σ
Figure 6	0.1	0.6	0.3
Figure 7	0.1	0.6	$\sin(2\pi x)$
Figure 8	0.1	0.6	$\sin(2\pi x)$ + constant value
Figure 9	0.1	0.6	constant value + δ function

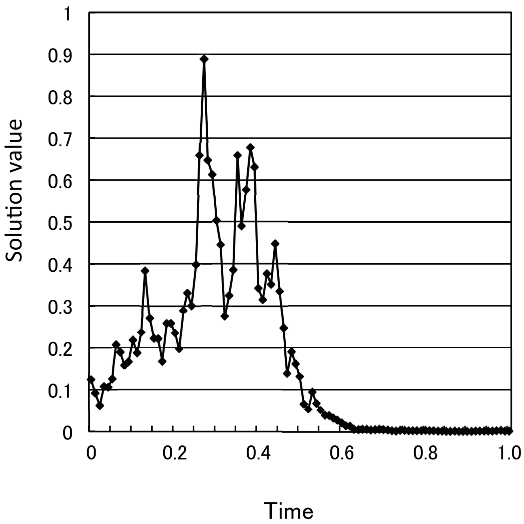

FIGURE 6. Solution process of stochastic differential Equation (38)

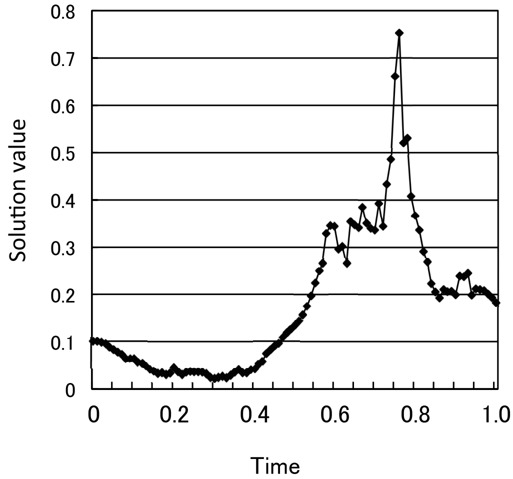

FIGURE 7. Solution process of stochastic differential Equation (38)

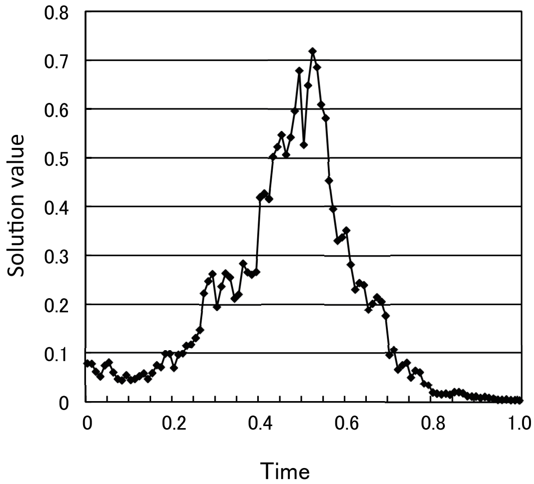

FIGURE 8. Solution process of stochastic differential Equation (38)

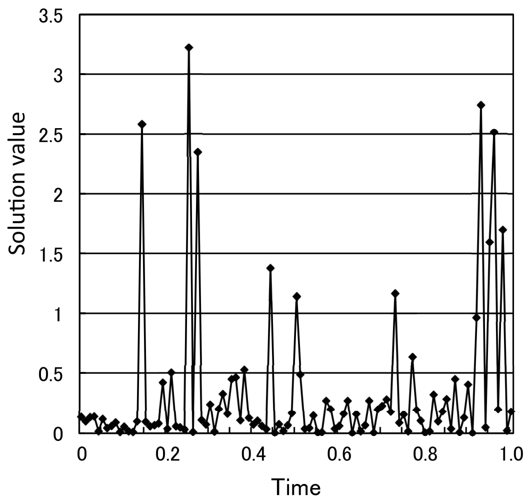

FIGURE 9. Solution process of stochastic differential Equation (38)

4.4. **Autocorrelation function of stages in a production process.** We can calculate an autocorrelation function of stages at $x = x'$ as follows:

$$\begin{aligned}\xi(t,x) &= \frac{1}{2\pi(r^2-c^4)t}\exp\left[-\frac{r(x-at)^2-c^2(x-at)^2}{(r^2-c^4)t}\right]\\ &= \frac{1}{2\pi(r^2-c^4)t}\exp\left[\frac{(r-c^2)(x-at)^2}{(r^2-c^4)t}\right]\end{aligned} \tag{40}$$

With respect to Figures 16 through 21, the influence of the stage on the autocorrelation function is the same as the trend function value. In each Figures 16 through 21, "a" affects the trend function value greatly. However, with respect to Figures 16 through 21, it can be said that "a" has less influence than "c" and "D" on the trend function value.

TABLE 2. Parameter setting of trend function of production density distribution

Figure number	a	r	c	D
Figure 10	0.3	0.5	0.5	1
Figure 11	1	0.5	0.5	1
Figure 12	0.5	0.5	0.5	1
Figure 13	0.1	0.5	0.5	1
Figure 14	1	0.5	0.2	0.7
Figure 15	1	0.5	0.1	0.6

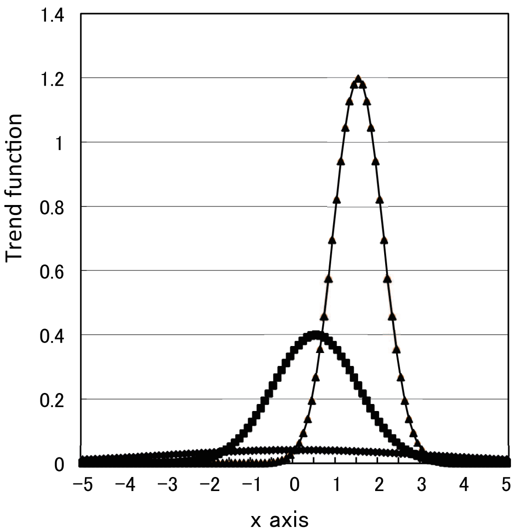

FIGURE 10. Trend function of production density distribution (Table 2)

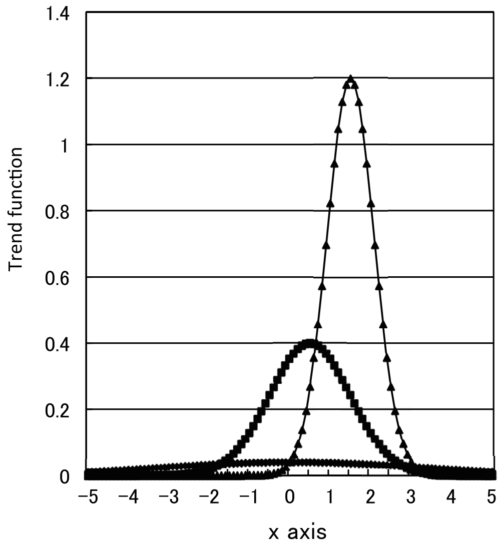

FIGURE 11. Trend function of production density distribution (Table 2)

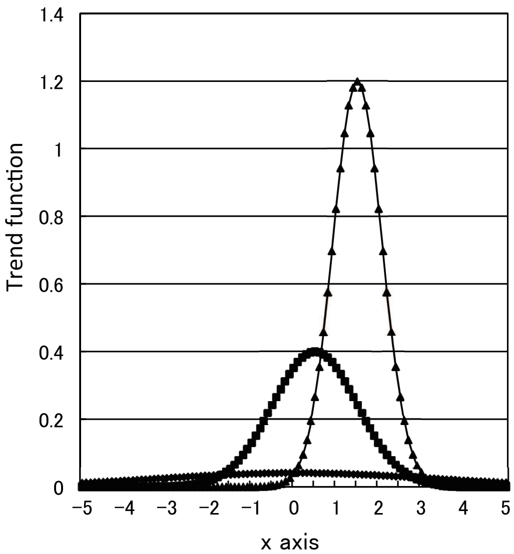

FIGURE 12. Trend function of production density distribution (Table 2)

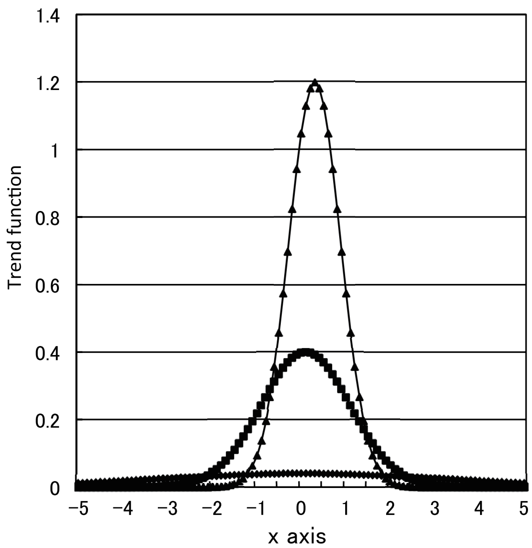

FIGURE 13. Trend function of production density distribution (Table 2)

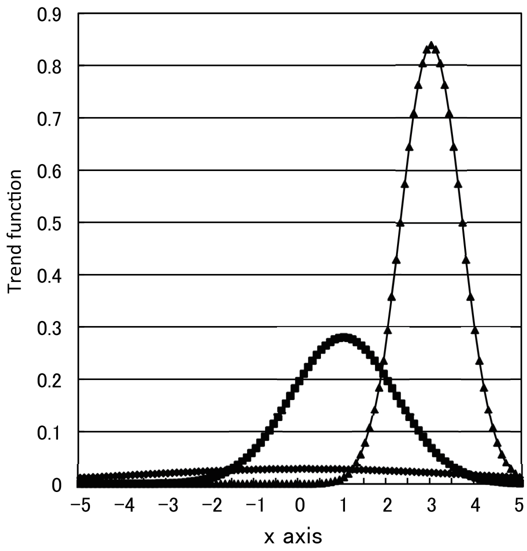

FIGURE 14. Trend function of production density distribution (Table 2)

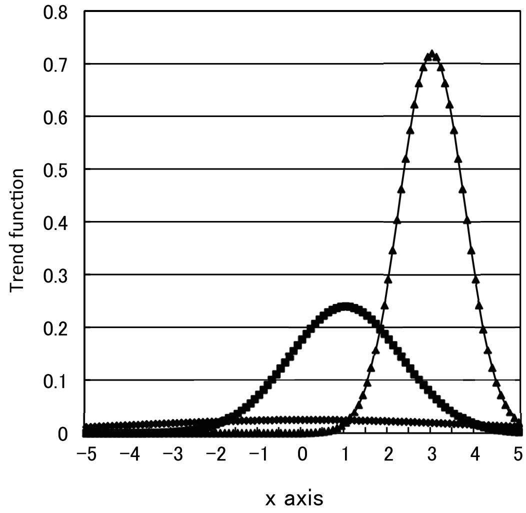

FIGURE 15. Trend function of production density distribution (Table 2)

TABLE 3. Parameter setting of trend function of production density distribution

Figure number	a	r	c	D
Figure 16	0.3	0.5	0.5	1
Figure 17	1	0.5	0.5	1
Figure 18	0.5	0.5	0.5	1
Figure 19	0.1	0.5	0.5	1
Figure 20	1	0.5	0.2	0.7
Figure 21	1	0.5	0.1	0.6

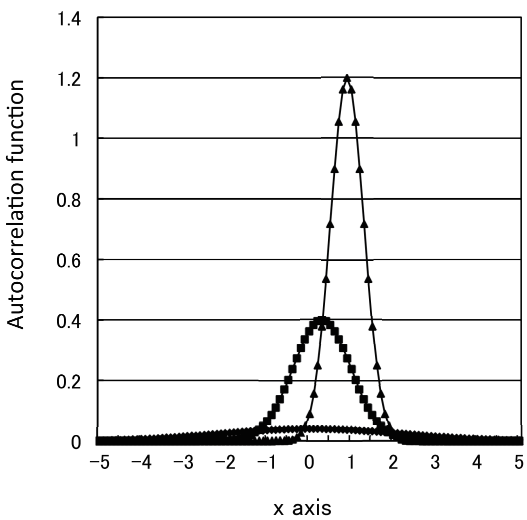

FIGURE 16. Autocorrelation function of production density distribution (Table 3)

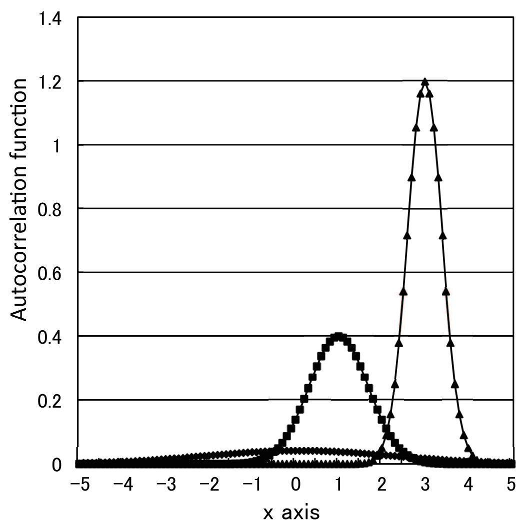

FIGURE 17. Autocorrelation function of production density distribution (Table 3)

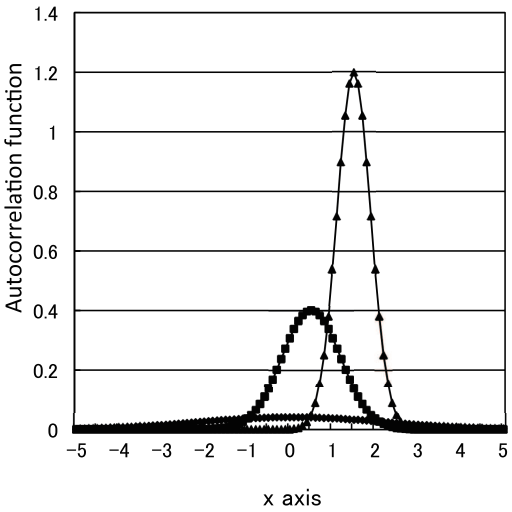

FIGURE 18. Autocorrelation function of production density distribution (Table 3)

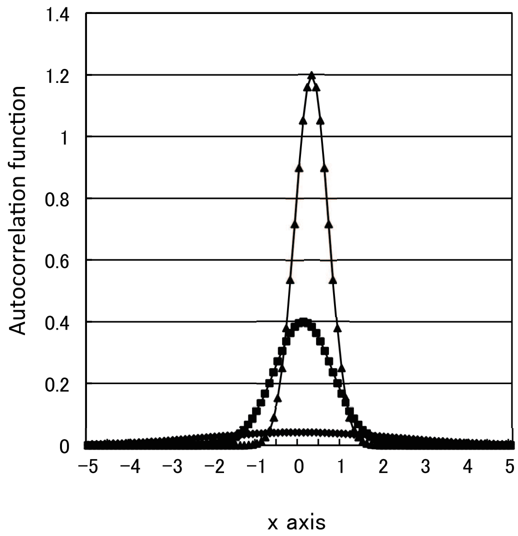

FIGURE 19. Autocorrelation function of production density distribution (Table 3)

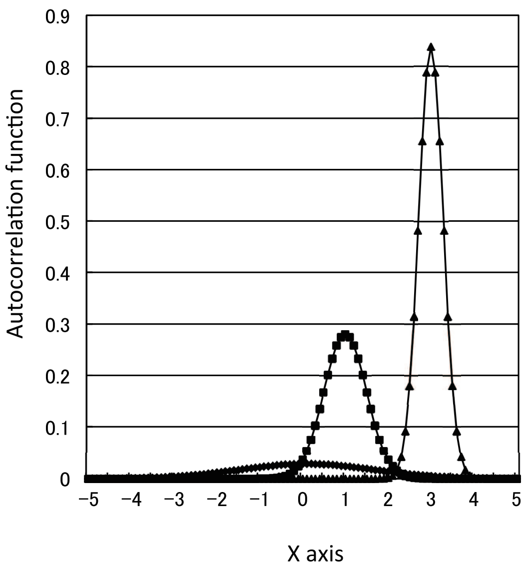

FIGURE 20. Autocorrelation function of production density distribution (Table 3)

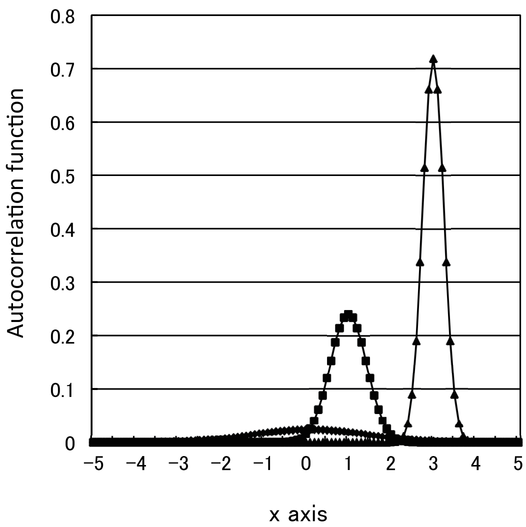

FIGURE 21. Autocorrelation function of production density distribution (Table 3)

5. **Conclusion.** We clarified the relation between lead time and production density by constructing two stochastic differential equations as a mathematical model. We also clarified that production density is greatly affected by a fluctuation in lead time. In other words, the production density distribution is highly dependent on the trend and volatility of the lead time (throughput) as a medium. The mathematical stochastic model of production density distribution presented in the present work has significant meaning, and its validity was recognized under the conditions mentioned in this paper. A model that considers external forces will be reported in due course.

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Appendix A. Analysis of Actual Data in the Production Flow System.

Table 4. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

Table 5. Volatility of Table 4

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

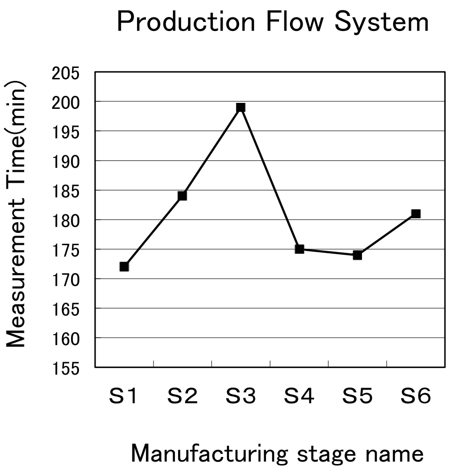

Figure 22. Total work time for each stage (S1-S6) in Table 4

Figure 23. Volatility data for each stage (S1-S6) in Table 4

Table 6. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

Table 7. Volatility of Table 6

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 8. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	$\boxed{21}$	$\boxed{21}$	$\boxed{21}$	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	$\boxed{22}$	$\boxed{22}$	20	20	20	20
K9	20	$\boxed{25}$	$\boxed{25}$	$\boxed{25}$	20	20	20
Total	180	165	164	161	180	180	180

TABLE 9. Variance of Table 8

K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.33	3	3	0	0	0
K5	1.33	1.33	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.33	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0