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ON FUZZY QUASI-PRIME AND WEAKLY FUZZY QUASI-PRIME IDEALS IN LEFT ALMOST RINGS

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Abstract. *The purpose of this paper is to introduce the notion of fuzzy quasi-prime and weakly fuzzy quasi-prime ideals in LA-rings, and to study fuzzy quasi-prime, weakly fuzzy quasi-prime, fuzzy completely prime and weakly fuzzy completely prime ideals in LA-rings. Some characterizations Cartesian product of weakly fuzzy completely prime and fuzzy completely prime ideals are obtained. Moreover, we investigated relationships between fuzzy completely prime and fuzzy quasi-prime ideals in LA-rings.*
Keywords: LA-ring, Fuzzy quasi-prime, Weakly fuzzy quasi-prime, Fuzzy completely prime, Weakly fuzzy completely prime

1. **Introduction.** Let R be a non-empty set. A fuzzy subset of R is, by definition, an arbitrary mapping $f : R \to [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. Zadeh [19] in 1965 introduced the notion of a fuzzy set to describe vagueness mathematically in its very abstractness and to solve such problems he gave a certain grade of membership to each member of a given set. Focusing on the structure of ring, the early paper of Liu [4], defining fuzzy ideals, initiated the investigation of rings by means of expanding the class of ideals with these fuzzy objects. Malik, Mordeson and Mukherjee, Sen [5, 6] have studied fuzzy ideals. In [7], Mukherjee and Sen have attempted to determine all nonnull fuzzy prime ideals in the special case where R is the ring of integers and L is the unit interval $[0, 1]$, which is a complete chain. In [1], Hur et al. introduced the notions of intuitionistic fuzzy (completely) prime ideals and intuitionistic fuzzy weak completely prime ideals in a ring.

In 2006, Yusuf in [18] introduces the concept of a left almost ring (LA-ring). That is, a non empty set R with two binary operations $+$ and $\cdot$ is called a left almost ring, if $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and distributive laws of $\cdot$ over $+$ holds. Further in [10] Shah et al. generalize the notions of commutative semigroup rings into LA-rings. However, Shah et al. in [14] generalize the notion of an LA-ring into a near left almost ring. A near left almost ring (nLA-ring) N is an LA-group under $+$, an LA-semigroup under $\cdot$ and left distributive property of $\cdot$ over $+$ holds.

In this study we followed lines as adopted in [1, 4, 5, 6, 7] and established the notion of fuzzy subsets of LA-rings. Specifically we characterize the fuzzy quasi-prime, fuzzy completely prime, weakly fuzzy quasi-prime and weakly fuzzy completely prime ideals in LA-rings. It is natural to extend the concept of a fuzzy prime ideal to a fuzzy quasi-prime of an LA-ring. Moreover, we investigated relationships between fuzzy completely prime and weakly fuzzy quasi-prime ideals in LA-rings.

2. **Preliminaries.** Let R be an LA-ring. If S is a nonempty subset of R and S is itself an LA-ring under the binary operation induced by R, then S is called an LA-subring of R. An LA-subring I of R is called a left ideal of R if $RI \subseteq I$ and I is called a right ideal of R if $IR \subseteq I$ and I is called an ideal of R if I is both a left and a right ideal of R [8]. A left ideal P of an LA-ring R is said to be quasi-prime (weakly quasi-prime) ideal of R if and only if $AB \subseteq P$ ($\{0\} \neq AB \subseteq P$) implies either $A \subseteq P$ or $B \subseteq P$, where A and B are ideals in R [8]. A left ideal P is called completely prime (weakly completely prime) if $ab \in P$ ($0 \neq ab \in P$) implies either $a \in P$ or $b \in P$. It can be easily seen that a completely prime (weakly completely prime) ideal of an LA-ring with left identity R is quasi-prime (weakly quasi-prime).

A function f from R to the unit interval $[0, 1]$ is a fuzzy subset of R [8]. The LA-ring R itself is a fuzzy subset of R such that $R(x) = 1$ for all $x \in R$, denoted also by R. Let f and g be two fuzzy subsets of R. Then the inclusion relation $f \subseteq g$ is defined $f(x) \leq g(x)$, for all $x \in R$. $f \cap g$ and $f \cup g$ are fuzzy subsets of R defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\}, \quad (f \cup g)(x) = \max\{f(x), g(x)\}$$

for all $x \in R$. More generally, if $\{f_\alpha : \alpha \in \beta\}$ is a family of fuzzy subsets of R, then $\bigcap\limits_{\alpha\in\beta} f_\alpha$ and $\bigcup\limits_{\alpha\in\beta} f_\alpha$ are defined as follows:

$$\left(\bigcap_{\alpha\in\beta} f_\alpha\right)(x) = \bigcap_{\alpha\in\beta} f_\alpha(x) = \inf\{f_\alpha(x) : \alpha \in \beta\},$$
$$\left(\bigcup_{\alpha\in\beta} f_\alpha\right)(x) = \bigcup_{\alpha\in\beta} f_\alpha(x) = \sup\{f_\alpha(x) : \alpha \in \beta\}$$

and will be the intersection and union of the family $\{f_\alpha : \alpha \in \beta\}$ of fuzzy subset of R. The product $f \circ g$ [8] is defined as follows:

$$(f \circ g)(x) = \begin{cases} \bigcup\limits_{x=yz} \min\{f(y), g(z)\}; & \exists y, z \in R, \text{ such that } x = yz \\ 0; & \text{otherwise.} \end{cases}$$

As is well known [8], a fuzzy subset f of R is called a fuzzy LA-subring of R if $f(x-y) \geq \min\{f(x), f(y)\}$ and $f(xy) \geq \min\{f(x), f(y)\}$, for all $x, y \in R$. A fuzzy LA-subring f of an LA-ring R is called a fuzzy left (right) ideal of R if $f(xy) \geq f(y)$ ($f(xy) \geq f(x)$) for all $x, y \in R$, if f is both fuzzy left and right ideal of R, then f is called a fuzzy ideal of R [8]. It is easy that f is a fuzzy ideal of R if and only if f is a fuzzy LA-subring of R such that $f(xy) \geq \max\{f(x), f(y)\}$ for all $x, y \in R$ [8].

Lemma 2.1. [8] *Let R be an LA-ring. If f, g, h are fuzzy subsets of R, then $(f \circ g) \circ h = (h \circ g) \circ f$.*

Proof: The proof is available in [8]. □

Remark 2.1. *If R is an LA-ring and $F(R)$ is the collection of all fuzzy subsets of R, then $(F(R), \circ)$ is an LA-semigroup.*

Lemma 2.2. [8] *Let R be an LA-ring with left identity. If f, g, h, k are fuzzy subsets of R, then*

1) $f \circ (g \circ h) = g \circ (f \circ h)$,
2) $(f \circ g) \circ (h \circ k) = (k \circ h) \circ (g \circ f)$.

Proof: The proof is available in [8]. □

Lemma 2.3. [8] *Let f be a fuzzy LA-subring of an LA-ring R. Then the following properties hold.*

1) $f \circ f \subseteq f$ and $f(x-y) \geq \min\{f(x), f(y)\}$, for all $x, y \in R$.
2) f is a fuzzy left ideal of R if and only if $R \circ f \subseteq f$.
3) f is a fuzzy right ideal of R if and only if $f \circ R \subseteq f$.
4) f is a fuzzy ideal of R if and only if $R \circ f \subseteq f$ and $f \circ R \subseteq f$.

Proof: The proof is available in [8]. □

Lemma 2.4. [8] *Let f be a fuzzy left ideal of an LA-ring with left identity R. Then $R \circ R = R$.*

Proof: The proof is available in [8]. □

Theorem 2.1. [8] *Let I be a non-empty subset of an LA-ring R and $f_I : R \to [0,1]$ be a fuzzy subset of R such that*

$$f_I(x) = \begin{cases} 1; & x \in I \\ 0; & \textit{otherwise}. \end{cases}$$

Then I is a left ideal (right ideal, ideal) of R if and only if f_I is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of R.

Proof: The proof is available in [8]. □

Theorem 2.2. *Let I be a non-empty subset of an LA-ring R, $m \in (0,1]$ and f_I be a fuzzy set of R such that*

$$f_I(x) = \begin{cases} m; & x \in I \\ 0; & \textit{otherwise}. \end{cases}$$

Then the following properties hold.

1) I is an LA-subring of R if and only if f_I is a fuzzy LA-subring of R.
2) I is a left ideal of R if and only if f_I is a fuzzy left ideal of R.
3) I is a right ideal of R if and only if f_I is a fuzzy right ideal of R.
4) I is an ideal of R if and only if f_I is a fuzzy ideal of R.

Proof: 1) ($\Rightarrow$) Suppose I is an LA-subring of R. Let $a, b \in R$. If $a \notin A$ or $b \notin A$, then $f_I(a) = 0$ or $f_I(b) = 0$ so $f_I(ab) \geq 0 = \min\{f_I(a), f_I(b)\}$ and $f_I(a-b) \geq 0 = \min\{f_I(a), f_I(b)\}$. If $a, b \in A$, then $f_I(a) = m$ and $f_I(b) = m$, so $f_I(ab) = m = \min\{f_I(a), f_I(b)\}$ and $f_I(a-b) = m = \min\{f_I(a), f_I(b)\}$. This implies f_I is a fuzzy LA-subring of R.

($\Leftarrow$) Assume that f_I is a fuzzy LA-subring of R. Let $a, b \in I$. Since $f_I(a-b) \geq \min\{f_I(a), f_I(b)\} = m$ and $f_I(ab) \geq \min\{f_I(a), f_I(b)\} = m$. This implies $f_I(a-b) = m$ and $f_I(ab) = m$. Thus, $a - b, ab \in A$. Hence, I is an LA-subring.

2) ($\Rightarrow$) Suppose I is a left ideal of R. Let $a, b \in R$. If $b \notin I$, then $f_I(b) = 0$, so $f_I(ab) \geq 0 = f_I(b)$. If $b \in I$, then $ab \in I$, since I is a left ideal. This implies $f_I(ab) = m = f_I(b)$. Thus, f_I is a fuzzy left ideal of R.

($\Leftarrow$) Assume that f_I is a fuzzy left ideal of R. Let $r \in R$ and $a \in I$. Since $f_I(ra) \geq f_I(a) = m$, we get $f_I(ra) = m$. This implies $ra \in I$. Thus, I is a left ideal of R.

3) ($\Rightarrow$) Suppose I is a right ideal of R. Let $a, b \in R$. If $a \notin I$, then $f_I(a) = 0$, so $f_I(ab) \geq 0 = f_I(a)$. If $a \in I$, then $ab \in I$, since I is a right ideal. This implies $f_I(ab) = m = f_I(a)$. Thus, f_I is a fuzzy right ideal of R.

($\Leftarrow$) Assume that f_I is a fuzzy right ideal of R. Let $r \in R$ and $a \in I$. Since $f_I(ar) \geq f_I(a) = m$, we get $f_I(ar) = m$. This implies $ar \in I$. Thus, I is a right ideal of R.

4) ($\Rightarrow$) Suppose I is an ideal of R. Let $a, b \in R$. If $a \notin I$ and $b \notin I$, then $f_I(a) = 0$ and $f_I(b) = 0$, so $f_I(ab) \geq 0 = \max\{0, 0\} = \max\{f_I(a), f_I(b)\}$. If $a \in I$ or $b \in I$, then $ab \in I$, since I is an ideal. This implies $f_I(ab) = m = \max\{f_I(a), f_I(b)\}$. Thus, f_I is a fuzzy ideal of R.

($\Leftarrow$) Assume that f_I is a fuzzy ideal of R. Let $r \in R$ and $a \in I$. Since $f_I(ra) \geq f_I(a) = m$ or $f_I(ar) \geq f_I(a) = m$, we get $f_I(ra) = m$ and $f_I(ar) = m$. This implies $ra, ar \in I$. Thus, I is an ideal of R. □

Definition 2.1. *Let R be an LA-ring, $x \in R$ and $t \in (0, 1]$. A fuzzy point x_t of R is defined by the rule that*

$$x_t(y) = \begin{cases} t; & x = y \\ 0; & \textit{otherwise.} \end{cases}$$

It is accepted that x_t is a mapping from R into $[0, 1]$, and then a fuzzy point of R is a fuzzy subset of R. For any fuzzy subset f of R, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel. Let tf_A be a fuzzy subset of R defined as follows:

$$tf_A(x) = \begin{cases} t \in (0, 1]; & x \in A \\ 0; & \textit{otherwise.} \end{cases}$$

3. **Fuzzy Quasi-Prime Ideals in LA-Rings.** The results of the following lemmas seem to play an important role to study fuzzy quasi-prime ideals in LA-rings; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 3.1. *A fuzzy subset f of an LA-ring of R is called fuzzy completely prime if* $\max\{f(x), f(y)\} \geq f(xy)$ *and* $\max\{f(x), f(y)\} \geq f(x - y)$, *for all $x, y \in R$.*

Example 3.1. *Let $R = \{0, 1, 2\}$ be a set under the binary operations defined as follows,*

+	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then R is an LA-ring. We define a fuzzy subset $f : R \to [0; 1]$ by $f(x) = 0$, for all $x \in R$. It is easy to show that f is a fuzzy completely prime subset of R.

Lemma 3.1. *A fuzzy ideal f of an LA-ring of R is fuzzy completely prime if and only if* $\max\{f(x), f(y)\} = f(xy)$ *and* $\max\{f(x), f(y)\} \geq f(x - y)$, *for all $x, y \in R$.*

Proof: It is straightforward by Definition 3.1. □

Theorem 3.1. *Let R be an LA-ring. Then f is fuzzy LA-subring of R if and only if $1 - f$ is fuzzy completely prime.*

Proof: ($\Rightarrow$) Assume that f is a fuzzy LA-subring of R. Then $f(xy) \geq \min\{f(x), f(y)\}$ and $f(x - y) \geq \min\{f(x), f(y)\}$ so that $1 - f(xy) \leq 1 - \min\{f(x), f(y)\}$ and $1 - f(x - y) \leq 1 - \min\{f(x), f(y)\}$, for all $x, y \in R$. If $f(x) \leq f(y)$, then $1 - f(x) \geq 1 - f(y)$. Then

$$\begin{aligned} \max\{1 - f(x), 1 - f(y)\} &= 1 - f(x) \\ &= 1 - \min\{f(x), f(y)\} \\ &\geq 1 - f(xy) \end{aligned}$$

and

$$\begin{aligned}\max\{1-f(x),1-f(y)\} &= 1-f(x)\\ &= 1-\min\{f(x),f(y)\}\\ &\geq 1-f(x-y)\end{aligned}$$

so that $1-f$ is fuzzy completely prime. If $f(x)>f(y)$, we have the same result. Thus,

$$\begin{aligned}\max\{1-f(x),1-f(y)\} &= 1-f(y)\\ &= 1-\min\{f(x),f(y)\}\\ &\geq 1-f(xy)\end{aligned}$$

and

$$\begin{aligned}\max\{1-f(x),1-f(y)\} &= 1-f(y)\\ &= 1-\min\{f(x),f(y)\}\\ &\geq 1-f(x-y)\end{aligned}$$

so that $1-f$ is fuzzy completely prime.

($\Leftarrow$) Suppose that $1-f$ is fuzzy completely prime of R. Then

$$\max\{1-f(x),1-f(y)\}\geq 1-f(xy)$$

and $\max\{1-f(x),1-f(y)\}\geq 1-f(x-y)$, so that $1-\max\{1-f(x),1-f(y)\}\leq 1-(1-f(xy))=f(xy)$ and $1-\max\{1-f(x),1-f(y)\}\leq 1-(1-f(x-y))=f(x-y)$, for all $x,y\in R$. If $1-f(x)\leq 1-f(y)$, then $f(xy)\geq 1-(1-f(y))=f(y)$ and $f(x-y)\geq 1-(1-f(y))=f(y)$. If $1-f(x)>1-f(y)$, then $f(xy)\geq 1-(1-f(x))=f(x)$ and $f(xy)\geq 1-(1-f(x))=f(x)$. Thus, $f(xy)\geq\min\{f(x),f(y)\}$ and $f(x-y)\geq\min\{f(x),f(y)\}$. Hence, f is a fuzzy LA-subring of R. □

Definition 3.2. *A fuzzy subset f of an LA-ring of R is called weakly fuzzy completely prime if*

$$\max\{f(x),f(y)\}\geq f(xy)$$

and $\max\{f(x),f(y)\}\geq f(x-y)$, *for all $x,y\in R$ such that $xy\neq 0$.*

Remark 3.1. *It is easy to see that every fuzzy completely prime subset is weakly fuzzy completely prime.*

Example 3.2. *Let $R=\{0,1,2,3,4,5\}$ be a set under the binary operations defined as follows,*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	1	2	3	4
2	4	5	0	1	2	3
3	3	4	5	0	1	2
4	2	3	4	5	0	1
5	1	2	3	4	5	0

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Then R is an LA-ring. We define a fuzzy subset $f:R\to[0;1]$ by

$$f(x)=\begin{cases}1; & x=0\\ 0; & \textit{otherwise.}\end{cases}$$

It is easy to see that f is a weakly fuzzy completely prime subset of R. However, f is not a fuzzy completely prime subset of R, since $\max\{f(4), f(3)\} = \max\{0, 0\} = 0$, *while* $f(4 \cdot 3) = f(0) = 1$.

Theorem 3.2. *Let R be an LA-ring. If P_i, $i \in \beta$ are fuzzy completely prime (weakly fuzzy completely prime) subsets of R, then $\bigcup_{i\in\beta} P_i$ is fuzzy completely prime (weakly fuzzy completely prime) subset of R.*

Proof: Suppose that P_i, $i \in \beta$ are fuzzy completely prime subset of R. Then $P_i(xy) \leq \max\{P_i(x), P_i(y)\}$ and $P_i(x-y) \leq \max\{P_i(x), P_i(y)\}$, for all $x, y \in R$, and for $i \in \beta$. Since

$$\max\left\{\bigcup_{i\in\beta} P_i(x), \bigcup_{i\in\beta} P_i(y)\right\} \geq P_i(xy)$$

and $\max\left\{\bigcup_{i\in\beta} P_i(x), \bigcup_{i\in\beta} P_i(y)\right\} \geq P_i(x-y)$, for all $i \in \beta$, we get

$$\max\left\{\bigcup_{i\in\beta} P_i(x), \bigcup_{i\in\beta} P_i(y)\right\} \geq \bigcup_{i\in\beta} P_i(xy)$$

and

$$\max\left\{\bigcup_{i\in\beta} P_i(x), \bigcup_{i\in\beta} P_i(y)\right\} \geq \bigcup_{i\in\beta} P_i(x-y).$$

Hence, $\bigcup_{i\in\beta} P_i$ is a fuzzy completely prime subset of R. □

Theorem 3.3. *Let P be a left ideal of LA-ring R. Then P is a completely quasi-prime (weakly completely quasi-prime) ideal of R if and only if the fuzzy subset f_P is a fuzzy completely prime (weakly fuzzy completely prime) left ideal of R.*

Proof: $(\Rightarrow)$ Suppose that P is a completely prime ideal of R. Obviously, f_P is a fuzzy left ideal of R. Let $x, y \in R$. If $xy \notin P$, then $f_P(xy) = 0 \leq \max\{f_P(x), f_P(y)\}$. If $xy \in P$, then $x \in P$ or $y \in P$. Thus, $f_P(x) = 1$ or $f_P(y) = 1$, so $\max\{f_P(x), f_P(y)\} = 1$. Therefore, the fuzzy left ideal P is a fuzzy completely prime ideal of R.

$(\Leftarrow)$ Suppose that f_P is a fuzzy completely prime ideal of R. Let $x, y \in R$ such that $xy \in P$. Then $f_P(xy) = 1$. Since f_P is a fuzzy completely prime ideal of R, we have $f_P(xy) \leq \max\{f_P(x), f_P(y)\}$. Thus, $f_P(x) = 1$ or $f_P(y) = 1$ and so $x \in P$ or $y \in P$. Therefore, P is a completely prime ideal of R. □

Definition 3.3. *Let R be an LA-ring. A fuzzy left ideal f of R is said to be a fuzzy quasi-prime if $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $th_B \subseteq f$, for the left ideals A and B in R and for all $t \in (0, 1]$.*

Definition 3.4. *Let R be an LA-ring. A fuzzy left ideal f of R is said to be a weakly fuzzy quasi-prime if $0_t \neq tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $th_B \subseteq f$, for the left ideals A and B in R and for all $t \in (0, 1]$.*

Remark 3.2. *It is easy to see that every fuzzy quasi-prime is weakly fuzzy quasi-prime.*

Lemma 3.2. *Let A, B be any non-empty subset of an LA-ring R. Then for any $t \in (0, 1]$ the following statements are true.*

1) $tf_A \circ tf_B = tf_{AB}$.

2) $tf_A \cap tf_B = tf_{A\cap B}$.
3) $tf_A \cup tf_B = tf_{A\cup B}$.
4) $tf_A = \bigcup_{a\in A} a_t$.
5) $R \circ tf_A = tf_{RA}$, $tf_A \circ R = tf_{AR}$ *and* $R \circ (tf_A \circ R) = tf_{R(AR)}$.
6) If A *is a left ideal (right, ideal) of* R*, then* tf_A *is a fuzzy left ideal (fuzzy left, fuzzy ideal) of* R*.*

Proof: We leave the straightforward proof to the reader. □

Theorem 3.4. *Let* P *be a fuzzy left ideal of an LA-ring with left identity* R*. Then the following statements are equivalent:*
1) P *is a fuzzy quasi-prime (weakly fuzzy quasi-prime) of* R*.*
2) For any $x, y \in R$ *and* $t \in (0,1]$*, if* $x_t \circ (R \circ y_t) \subseteq P$ $(0_t \neq x_t \circ (R \circ y_t) \subseteq P)$*, then* $x_t \in P$ *or* $y_t \in P$*.*
3) For any $x, y \in R$ *and* $t \in (0,1]$*, if* $tf_x \circ tf_y \subseteq P$ $(0_t \neq tf_x \circ tf_y \subseteq P)$*, then* $x_t \in P$ *or* $y_t \in P$*.*
4) If A *and* B *are left ideals of* R *such that* $tf_A \circ tf_B \subseteq P$ $(0_t \neq tf_A \circ tf_B \subseteq P)$*, then* $tf_A \subseteq P$ *or* $tf_B \subseteq P$*.*

Proof: $(1 \Rightarrow 2)$ Let P be a fuzzy quasi-prime of R. For any $x, y \in R$ and $t \in (0,1]$, if $x_t \circ (R \circ y_t) \subseteq P$, then

$$\begin{aligned} tf_{(xe)R} \circ tf_{(ye)R} &= (tf_{(xe)} \circ R) \circ (tf_{ye} \circ R) \\ &= (tf_{(xe)} \circ tf_{ye}) \circ (R \circ R) \\ &= ((tf_x \circ tf_e) \circ (tf_y \circ tf_e)) \circ (R \circ R) \\ &= ((tf_x \circ tf_y) \circ (tf_e \circ tf_e)) \circ (R \circ R) \\ &= ((tf_e \circ tf_e) \circ (tf_y \circ tf_x)) \circ (R \circ R) \\ &= (tf_{ee} \circ (tf_y \circ tf_x)) \circ (R \circ R) \\ &= (tf_y \circ (tf_e \circ tf_x)) \circ (R \circ R) \\ &= (tf_y \circ tf_{ex}) \circ (R \circ R) \\ &= (R \circ R) \circ (tf_x \circ tf_y) \\ &= R \circ (tf_x \circ tf_y) \\ &= tf_x \circ (R \circ tf_y) \\ &= x_t \circ (R \circ y_t) \\ &\subseteq P. \end{aligned}$$

Since P is a fuzzy quasi-prime, we get $x_t = tf_x = tf_{(ee)x} = tf_{(xe)e} \subseteq tf_{(xe)R} \subseteq P$ or $y_t = tf_y = tf_{(ee)y} = tf_{(ye)e} \subseteq tf_{(ye)R} \subseteq P$. Hence, $x_t \in tf_x \subseteq P$ or $y_t \in tf_y \subseteq P$.

$(2 \Rightarrow 3)$ Let $x, y \in R$, $t \in (0,1]$ and $tf_x \circ tf_y \subseteq P$. Then

$$\begin{aligned} x_t \circ (R \circ y_t) &\subseteq tf_x \circ (R \circ tf_y) \\ &= R \circ (tf_x \circ tf_y) \\ &\subseteq R \circ P \\ &\subseteq P. \end{aligned}$$

Thus, by hypothesis $x_t \in P$ or $y_t \in P$.

$(3 \Rightarrow 4)$ Let A and B be two ideals of R. Then, by Lemma 3.2, we get tf_A and tf_B are fuzzy left ideals of R. Suppose that $tf_A \circ tf_B \subseteq P$, and $tf_A \not\subseteq P$, and then there exists $x \in A$ such that $x_t \notin P$. For any $y \in B$, by Lemma 3.2 and hypothesis,

$$\begin{aligned} tf_x \circ tf_y &= tf_{xy} \\ &\subseteq tf_{AB} \\ &= tf_A \circ tf_B \\ &\subseteq P. \end{aligned}$$

Since $x_t \notin P$, which implies $tf_y \subset P$, and so $y_t \in P$. By Lemma 3.2, it follows that $tf_B = \bigcup_{y \in B} y_t$.

(4 ⇒ 1) We leave the straightforward proof to the reader. □

Corollary 3.1. *Let P be a fuzzy left ideal of an LA-ring with left identity R. Then the following statements are equivalent:*

1) P is a fuzzy quasi-prime (weakly fuzzy quasi-prime) ideal of R.

2) For any $x, y \in R$ and $t \in (0,1]$, if $x_t \circ y_t \in P$ ($0_t \neq x_t \circ y_t \in P$), then $x_t \in P$ or $y_t \in P$.

Proof: It is straightforward by Theorem 3.4. □

Example 3.3. *Let $R = \{0,1,2,3,4,5,6,7\}$ be a set under the binary operations defined as follows,*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Then R is an LA-ring. We define a fuzzy subset $f : R \to [0;1]$ by

$$f(x) = \begin{cases} 1; & x \in \{0\} \\ 0.5; & x \in \{4\} \\ 0; & \textit{otherwise.} \end{cases}$$

It is easy to see that f is a weakly fuzzy quasi-prime ideal of R. However, f is not a fuzzy quasi-prime ideal of R, since $5_{0.5} \circ 3_1 \subseteq f$ while $5_{0.5} \notin f$ and $3_1 \notin f$.

Theorem 3.5. *Let R be an LA-ring with left identity. If f is a fuzzy quasi-prime of R, then $\inf \{f(a^2(Rb^2))\} = \max \{f(a^2), f(b^2)\}$, for all $a, b \in R$.*

Proof: Suppose that $\inf \{f(a^2(Rb^2))\} \neq \max \{f(a^2), f(b^2)\}$. Since f is fuzzy left ideal of R, we get $f(a^2(rb^2)) \geq f(rb^2) \geq f(b^2)$ and

$$f\left(a^2\left(rb^2\right)\right) \geq f\left(\left(b^2 r\right) a^2\right) \geq f(a^2),$$

for all $r \in R$. Then $\max \{f(a^2), f(b^2)\} < \inf \{f(a(Rb))\}$. Let $\inf \{f(a(Rb))\} = m$. Define two fuzzy subsets g and h of R as follows:

$$g(x) = \begin{cases} m; & x \in a^2R; \\ 0; & \text{otherwise;} \end{cases}$$

and

$$h(x) = \begin{cases} m; & x \in b^2R; \\ 0; & \text{otherwise.} \end{cases}$$

Then g and h are two fuzzy left ideals of R by Theorem 2.2. If

$$g \circ h(x) = \bigcup_{x=yz} \min\{g(y), h(z)\} = m,$$

then there exist $u \in a^2R$, $v \in b^2R$ such that $uv = x$. Put $u = a^2t$ and $v = b^2k$, for some $t, k \in R$. Then

$$\begin{aligned} f(x) &= f(uv) \\ &= f((a^2t)(b^2k)) \\ &= f((a^2b^2)(tk)) \\ &= f((kt)(b^2a^2)) \\ &\geq f(b^2a^2) \\ &= f(a^2b^2) \\ &= f(a^2(eb^2)) \\ &\geq inf\{f(a^2(Rb^2))\} \\ &= m \end{aligned}$$

so that $g \circ h \subseteq f$. Since f is a quasi-prime ideal, we get $g \subseteq f$ or $h \subseteq f$. Thus, $g(a^2) = g((ee)a^2) = g(a^2e) = m$ or $h(b^2) = h((ee)b^2) = h(b^2e) = m$. However, from $m = \max\{f(a^2), f(b^2)\} < inf\{f(a(Rb))\} = m$, we have a contradiction. □

Theorem 3.6. *Let R be an LA-ring with left identity. A fuzzy ideal P of an LA-ring R is fuzzy quasi-prime ideal if and only if* $\max\{P(x), P(y)\} = P(xy)$, *for all $x, y \in R$.*

Proof: ($\Rightarrow$) Suppose that P is a fuzzy ideal of R. Then $P(xy) \geq f(x)$ and $P(xy) \geq f(y)$, so $P(xy) \geq \max\{f(x), f(y)\}$, for all $x, y \in R$. On the other hand, if $P(xy) > \max\{P(x), P(y)\}$, then there exists $t \in (0, 1)$ such that

$$P(xy) > t > \max\{P(x), P(y)\}.$$

Thus, $x_t \circ (R \circ y_t) = R \circ (x_t \circ y_t) \subseteq R \circ (xy)_t \in R \circ P \subseteq P$, for all $x, y \in R$. Since P is a fuzzy quasi-prime ideal of R, we get $x_t \in P$ or $y_t \in P$, but $x_t \notin P$ and $y_t \notin P$, which is impossible. Therefore, $P(xy) = \max\{P(x), P(y)\}$, for all $x, y \in R$.

($\Leftarrow$) Suppose that x_t, y_t ($t \in (0, 1]$) are the fuzzy points of R such that $x_t \circ (R \circ y_t) \subseteq P$. Since $R \circ (xy)_t \subseteq R \circ (x_t \circ y_t) = x_t \circ (R \circ y_t) \subseteq P$ and $P(xy) = \max\{P(x), P(y)\}$, we have $P(xy) \geq t$, which implies that $P(x) \geq t$ or $P(y) \geq t$. Then $x_t \in P$ or $y_t \in P$ and hence P is a fuzzy quasi-prime ideal of R. □

Theorem 3.7. *Let R be an LA-ring with left identity. If P is a fuzzy completely prime, then P is a fuzzy quasi-prime ideal of R.*

Proof: We leave the straightforward proof to the reader. □

Let R_1 and R_2 be two LA-rings. Then $R_1 \times R_2 := \{(x, y) \in R_1 \times R_2 | x \in R_1, y \in R_2\}$ and for any $(a, b), (c, d) \in R_1 \times R_2$ we define

$$(a, b) + (c, d) := (a + c, b + d) \text{ and } (a, b)(c, d) := (ac, bd)$$

and then $R_1 \times R_2$ is an LA-ring as well. Let $f : R_1 \to [0, 1]$ and $g : R_2 \to [0, 1]$ be two fuzzy subsets of LA-rings R_1 and R_2 respectively. Then the Cartesian product of fuzzy subsets is denoted by $f \times g$ and defined as $f \times g : R_1 \times R_2 \to [0, 1]$, where $(f \times g)(x, y) = \min\{f(x), g(y)\}$.

Lemma 3.3. *If f and g be two fuzzy LA-subrings of R_1 and R_2 respectively, then $f \times g$ is a fuzzy LA-subring of $R_1 \times R_2$.*

Proof: We leave the straightforward proof to the reader. □

Lemma 3.4. *If f and g be two fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of R_1 and R_2 respectively, then $f \times g$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $R_1 \times R_2$.*

Proof: We leave the straightforward proof to the reader. □

Corollary 3.2. *Let $f_1, f_2, \ldots, f_n$ are fuzzy subsets of LA-rings $R_1, R_2, \ldots, R_n$ respectively.*

1) If $f_1, f_2, f_3, \ldots, f_n$ are fuzzy LA-subrings of $R_1, R_2, R_3, \ldots, R_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy LA-subring of $\prod_{i=1}^{n} R_i$.

2) If $f_1, f_2, f_3, \ldots, f_n$ are fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of $R_1, R_2, R_3, \ldots, R_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $\prod_{i=1}^{n} R_i$.

Proof: One can easily show by induction method. □

Lemma 3.5. *Let f, g be two fuzzy subsets of LA-rings R_1, R_2 respectively such that $f \times g$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $R_1 \times R_2$. Then f or g is fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of R_1 or R_2 respectively.*

Proof: We leave the straightforward proof to the reader. □

Corollary 3.3. *Let $f_1, f_2, \ldots, f_n$ are fuzzy subsets of LA-rings $R_1, R_2, \ldots, R_n$ respectively. If $\prod_{i=1}^{n} f_i$ is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $\prod_{i=1}^{n} R_i$, then f_1 or f_2 or f_3 or $\ldots$ or f_n is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of $R_1, R_2, R_3, \ldots, R_n$ respectively.*

Proof: We leave the straightforward proof to the reader. □

Lemma 3.6. *Let f, g be two fuzzy subsets of LA-rings R_1, R_2 respectively and $t \in (0,1]$. Then $(f \times g)_t = f_t \times f_t$.*

Proof: We leave the straightforward proof to the reader. □

Corollary 3.4. *Let $f_1, f_2, \ldots, f_n$ be fuzzy subsets of LA-rings $R_1, R_2, \ldots, R_n$ respectively and $t \in (0,1]$. Then $\left(\prod_{i=1}^{n} f_i\right)_t = \prod_{i=1}^{n} (f_i)_t$.*

Proof: One can easily show by induction method. □

Theorem 3.8. *Let R_1 and R_2 be two LA-ring with left identity. A fuzzy left ideal f is a fuzzy quasi-prime ideal of an LA-ring R_1 if and only if $f \times R_2$ is a fuzzy quasi-prime ideal of $R_1 \times R_2$.*

Proof: Suppose that f is a fuzzy quasi-prime ideal of R_1. Let $(a,b), (c,d) \in R_1 \times R_2$ such that $(ac, bd)_t = (a,b)_t \circ (c,d)_t \in f \times R_2$. Then $f(ac) = \min\{f(ac), 1\} = \min\{f(ac), R_2(bd)\} = f \times R_2(ac, bd) \geq t$, so $f(ac) \geq t$. Obviously, $a_t \circ c_t = (ac)_t \in f$. By Corollary 3.1, we get $a_t \in f$ or $c_t \in f$. Thus, $f(a) \geq t$ or $f(c) \geq t$. It is easy to see that

$$\begin{aligned} f \times R_2(a,b) &= \min\{f(a), R_2(b)\} \\ &= \min\{t, 1\} \\ &= t \end{aligned}$$

or

$$\begin{aligned} f \times R_2(b,d) &= \min\{f(b), R_2(d)\} \\ &= \min\{t, 1\} \\ &= t. \end{aligned}$$

Therefore, $(a,b)_t \in f \times R_2$ or $(c,d)_t \in f \times R_2$. By Corollary 3.1, we have $f \times R_2$ is a fuzzy quasi-prime ideal of $R_1 \times R_2$. Conversely, assume that $f \times S_2$ is a fuzzy quasi-prime ideal of $R_1 \times R_2$. Let $a, c \in R_1$ such that $(ac)_t = a_t \circ c_t \in f$. Then $t \leq f(ac) = \min\{f(ac), 1\} = \min\{f(ac), R_2(bd)\} = f \times R_2(ac, bd)$, so $f \times R_2(ac, bd) \geq t$. Obviously, $(a,b)_t \circ (c,d)_t = (ac, bd)_t \in f \times R_2$. By Corollary 3.1, we get $(a,b)_t \in f \times R_2$ or $(c,d)_t \in f \times R_2$. Thus, $f \times R_2(a,b) \geq t$ or $f \times R_2(c,d) \geq t$. It is easy to see that

$$\begin{aligned} f(a) &= \min\{f(a), 1\} \\ &= \min\{f(a), R_2(b)\} \\ &= f \times R_2(a,b) \\ &\geq t \end{aligned}$$

or

$$\begin{aligned} f(b) &= \min\{f(b), 1\} \\ &= \min\{f(b), R_2(d)\} \\ &= f \times R_2(b,d) \\ &\geq t. \end{aligned}$$

Therefore, $a_t \subseteq f$ or $c_t \subseteq f$. By Corollary 3.1, we have f is a fuzzy quasi-prime ideal of R_1. □

Corollary 3.5. *Let R_1 and R_2 be two LA-rings with left identity. A fuzzy left ideal f is a fuzzy quasi-prime ideal of an LA-ring R_2 if and only if $R_1 \times f$ is a fuzzy quasi-prime ideal of $R_1 \times R_2$.*

Proof: It is straightforward by Theorem 3.8. □

Corollary 3.6. *Let R_i be an LA-ring with left identity. A fuzzy left ideal f_i is a fuzzy quasi-prime ideal of an LA-ring R_i if and only if $R_1 \times \ldots \times R_i \times R_{i+1} \times \ldots \times R_n$ is a fuzzy quasi-prime ideal of $\prod_{i=1}^{n} R_i$.*

Proof: One can easily show by induction method. □

Theorem 3.9. *Let f, g be two fuzzy ideals of LA-rings with left identity R_1, R_2 respectively such that $f \times g$ is a fuzzy completely prime (weakly fuzzy completely prime) ideal of $R_1 \times R_2$. Then f or g is fuzzy completely prime (weakly fuzzy completely prime) of R_1 or R_2 respectively.*

Proof: We know that $\min\{f(0_1), g(0_2)\} = (f \times g)(0_1, 0_2) \geq (f \times g)(x,y) = \min\{f(x), g(y)\}$, for all $(x,y) \in R_1 \times R_2$. Then $f(x) \leq f(0_1)$ or $g(y) \leq g(0_2)$. If $f(x) \leq f(0_1)$, then $f(x) \leq g(0_2)$ or $g(y) \leq g(0_2)$. Let $f(x) \leq g(0_2)$. Then $(f \times g)(x, 0_2) = f(x)$. So

$$\begin{aligned} f(x_1x_2) &= (f \times g)(x_1x_2, 0_2) \\ &= (f \times g)((x_1, 0_2)(x_2, 0_2)) \\ &\leq \max\{(f \times g)(x_1, 0_2), (f \times g)(x_2, 0_2)\} \\ &= \max\{f(x_1), f(x_2)\} \end{aligned}$$

for all $x_1, x_2 \in R_1$. Therefore, f is a fuzzy completely prime of R_1. Now suppose that $f(x) \leq g(0_2)$ is not true for all $x \in R_1$. If $f(x) > g(0_2)$ for some $x \in R_1$, then $g(y) \leq g(0_2)$, for all $y \in R_2$. Therefore, $(f \times g)(0_1, y) = g(y)$, for all $y \in R_2$. Similarly

$$\begin{aligned} g(y_1y_2) &= (f \times g)(0_1, y_1y_2) \\ &= (f \times g)((0_1, y_1)(0_1, y_1)) \\ &\leq \max\{(f \times g)(0_1, y_1), (f \times g)(0_1, y_2)\} \\ &= \max\{g(y_1), g(y_2)\} \end{aligned}$$

for all $y_1, y_2 \in R_2$. Hence, g is fuzzy completely prime of R_2. □

Theorem 3.10. *Let f, g be two fuzzy subsets of LA-rings R_1, R_2 respectively. Then $f \times g$ is a fuzzy completely prime (weakly fuzzy completely prime) ideal of $R_1 \times R_2$ if and only if $(f \times g)_t$, $t \in Im(f \times g)$ of $f \times g$ is a completely prime (weakly completely prime) ideal of $R_1 \times R_2$, for every $t \in (0, 1]$.*

Proof: $(\Rightarrow)$ Suppose that $f \times g$ is a fuzzy completely prime ideal of $R_1 \times R_2$. Let $(x, y), (m, n) \in R_1 \times R_2$ such that $(x, y)(m, n) \in (f \times g)_t$. Then $(f \times g)((x, y)(m, n)) \geq t$ so that $(f \times g)(xm, yn) \geq t$. Since $f \times g$ is a fuzzy completely prime ideal of $R_1 \times R_2$, we have

$$(f \times g)((x, y)(m, n)) \leq \max \{(f \times g)(x, y), (f \times g)(m, n)\}.$$

If $(f \times g)(x, y) \leq (f \times g)(m, n)$, then $t \leq \max \{(f \times g)(x, y), (f \times g)(m, n)\} = (f \times g)(m, n)$, and $(f \times g)(m, n) \geq t$, so $(m, n) \in (f \times g)_t$. If $(f \times g)(x, y) > (f \times g)(m, n)$, then

$$t \leq \max \{(f \times g)(x, y), (f \times g)(m, n)\} = (f \times g)(x, y),$$

and $(f \times g)(x, y) \geq t$, so $(x, y) \in (f \times g)_t$.

$(\Leftarrow)$ Suppose that $(f \times g)_t$ is a weakly completely prime ideal of $R_1 \times R_2$, for every $t \in (0, 1]$. Let $(x, y), (m, n) \in R_1 \times R_2$. Then $(f \times g)((x, y)(m, n)) \geq 0$. Since $(x, y)(m, n) \in (f \times g)_{(f\times g)((x,y)(m,n))}$, by hypothesis, we have $(x, y) \in (f \times g)_{(f\times g)((x,y)(m,n))}$ or $(m, n) \in (f \times g)_{(f\times g)((x,y)(m,n))}$. Thus, $(f \times g)(x, y) \geq (f \times g)((x, y)(m, n))$ or $(f \times g)(m, n) \geq (f \times g)((x, y)(m, n))$ and hence $\max \{(f \times g)(x, y), (f \times g)(m, n)\} \geq (f \times g)((x, y)(m, n))$. □

Corollary 3.7. *Let $f_1, f_2, \ldots, f_n$ be fuzzy subsets of LA-rings $R_1, R_2, \ldots, R_n$ respectively and $t \in (0, 1]$. Then $\prod_{i=1}^{n} f_i$ is a fuzzy completely prime (weakly fuzzy completely prime) ideal of $\prod_{i=1}^{n} R_i$ if and only if $\left(\prod_{i=1}^{n} f_i\right)_t$, $t \in Im\left(\prod_{i=1}^{n} R_i\right)$ is a completely prime (weakly completely prime) ideal of $\prod_{i=1}^{n} R_i$.*

Proof: One can easily show by induction method. □

4. **Conclusions.** Many new classes of LA-rings have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. In this paper, we have introduced and studied the concepts of fuzzy quasi-prime and weakly fuzzy quasi-prime ideals in LA-rings and their interrelations. Further study can be made for obtaining some characterizations of fuzzy quasi-prime, weakly fuzzy quasi-prime, fuzzy completely prime and weakly fuzzy completely prime ideals in LA-rings by using the properties of these fuzzy left ideals.

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