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UNITED DIRECTED COMPLEX DYNAMICAL NETWORKS WITH MULTI-LINKS: NOVEL SYNCHRONIZATION STABILITY CRITERIA

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Abstract. *Networks with multi-links can be found everywhere in our daily life such as communication networks, transport networks and social networks. In view of a method of network split according to the different nature of time-delay, the model of the united directed complex dynamical networks with multi-links is considered in the paper. Then the problem of synchronization stability analysis for the united directed complex dynamical network is further discussed. Taking advantage of linear matrix inequalities (LMI), some novel Lyapunov functions are constructed. And then some new general stability criteria are proposed of synchronization state in the united directed complex dynamical networks by use of the Lyapunov stability theory. Finally, numerical simulations are provided to show the effectiveness and feasibility of the proposed theorems.*
Keywords: Complex network, Multi-links, Synchronization stability, Time delays, Linear systems

1. **Introduction.** In recent years, complex dynamical networks have attracted much attention from physics, mathematics, engineering, biology and sociology [1-8]. One of the most attractive reasons is that complex networks have applications in almost all the fields in the real world including the World Wide Web, the Internet, the food webs, the communication networks, the neural networks, the cellular and metabolic networks, the electrical power grids and the social networks, etc [1,2].

The complex network can be divided into single link and multi-links complex networks based on the properties of network edge. In fact, there are a lot of complex dynamical networks with multi-links in the real world [9,10], such as human connection networks, transportation networks, communications networks, and complex biology networks. The network is called the united complex dynamical networks with multi-links. There is more than one edge between two nodes and each of the edge has its own property in the united complex dynamical networks.

As a matter of fact, time delays commonly exist in practical systems. Some of them are trivial so that they can be ignored. However, some of them cannot be ignored because time delays may decrease the quality of the system, and even cause oscillation, instability and divergence [11-13]. Therefore, synchronization stability analysis of complex networks with time delays is not only a theoretical problem but also a practical one, and it has become an important topic in [14-18]. Gao et al. [23] and Peng et al. [9] have proposed the idea about network split according to the different nature of the network links. Time-delay was introduced into networks to affect the split and the united complex dynamical networks with multi-links were split into some sub-networks. Most of the researches of complex dynamical networks are aimed at synchronization stability analysis of complex dynamical networks with single link [1-8,19-22]. However, a few pay attention to synchronization stability analysis of the united directed complex networks with multi-links [24,28]. And, there are a lot of directed networks in real life [7,21,22]. So the synthronization stability of the united complex directed networks with multi-links needs further investigation.

In the paper, the model of the united directed complex dynamical networks with multi-links is considered. And the problem of synchronization stability is further discussed. Taking advantage of the Lyapunov stability theory, some new general stability criteria are proposed. The rest of the brief is organized as follows. In Section 2, the model of united directed complex dynamical networks with multi-links is introduced using the idea of network split and some preliminaries and assumptions are given. The novel results that guarantee the synchronised states to be asymptotically stable are presented in Section 3. Then numerical examples of united directed complex networks with multi-links are given to demonstrate the effectiveness of the proposed stability criteria in Section 4, and conclusion is presented in Section 5.

Notation: The notation used throughout the paper is fairly standard. Let R^n denote the n-dimensional Euclidean space over the reals with the norm $\|\cdot\|$. For any $u = (u_i)_{1\le i\le n}$, $v = (v_i)_{1\le i\le n}$ and $u, v \in R^n$, we define the scalar product of the vectors u and v as: $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$. Let $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, $R_+^* = (0, +\infty)$, $R_+^{*n} = \{v = (v_i)_{1\le i\le n} \in R^n, v_i \in R_+, \forall i = 1, 2, \ldots, n\}$. Let $\lambda(M)$ denote the set of eigenvalues of the matrix M, M' its transpose and M^{-1} its inverse. We define $|M| = (|m_{ij}|)_{1\le i,j\le n}$ if $M = (m_{ij})_{1\le i,j\le n}$. Let $C_n = C([-\tau, 0], R^n)$ be the Banach space of continuous functions mapping the interval with the topology of uniform convergence. For a given $\phi \in C_n$, we define $\|\phi\| = sup_{-\tau\le\theta\le 0}\|\phi(\theta)\|$, $\phi(\theta) \in R^n$. We define the function $sgn(\cdot)$ and $M^* = \left(m_{ij}^*\right)_{1\le i\le n}$ as

$$sgn(\vartheta) = \begin{cases} 1 & \vartheta \in R_+^* \\ -1 & -\vartheta \in R_+^* \\ 0 & \vartheta = 0 \end{cases}, \quad m_{ij}^* = \begin{cases} m_{ij} & \text{if } i = j \\ |m_{ij}| & \text{if } i \ne j \end{cases}$$

2. **United Directed Complex Network Model and Preliminaries.** There are lots of models that can well describe the complexity of complex dynamical networks with single link such as random networks, small-word networks, and scale-free networks [1-3,8]. The complex network model consisting of N identical nodes with linear coupling has been put forward in [1,3], which is described by

$$\dot{x_i} = f(x_i) + c\sum_{j=1}^{N} C_{ij}Gx_j(t), \quad i = 1, 2, \ldots, N \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ is a state vector representing the state variables of node i, $f(\cdot) \in R^n$ is a continuously differentiable vector function, $G = (G_{ij})_{n\times n} \in R^{n\times n}$ is a constant inner-coupling matrix between node i and node j $(i \ne j)$ for all $1 \le i, j \le N$, the constant $c > 0$ is the coupling strength, $C = (C_{ij})_{N\times N}$ is the coupling configuration matrix representing topological structure of the network, in which C_{ij} is defined as follows: if there is a connection from node i to node j $(i \ne j)$, then $C_{ij} = C_{ji} = 1$, otherwise $C_{ij} = C_{ji} = 0$, and the diagonal elements of matrix C are defined by

$$C_{ii} = -\sum_{j=1, j\ne i}^{N} C_{ij}, \quad i = 1, 2, \ldots, N \tag{2}$$

Nevertheless, these models cannot depict the characteristic of the united complex dynamical networks with multi-links. Recently, Gao et al. [23] and Peng et al. [9] have proposed the idea about network split according to the different transmission speed of links. By introducing time-delay, the united complex dynamical networks with multi-links were constructed. Then the united directed and weighted complex dynamical network consisting of N nodes with m kinds of the line properties is put forward and the model can be described as follows:

$$\begin{aligned}\dot{x}_i &= f(x_i) + c_0 \sum_{j=1}^{N} a_{(0)ij} H_0 x_j(t) + c_1 \sum_{j=1}^{N} a_{(1)ij} H_1 x_j(t-\tau_1) + c_2 \sum_{j=1}^{N} a_{(2)ij} H_2 x_j(t-\tau_2) \\ &\quad + \cdots + c_{m-1} \sum_{j=1}^{N} a_{(m-1)ij} H_{m-1} x_j(t-\tau_{m-1}), \quad i = 1, 2, \ldots, N\end{aligned} \tag{3}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ is a state vector representing the state variables of node i, $f(\cdot) \in R^n$ is a continuously differentiable vector function, $H_k = (H_{kij})_{n\times n} \in R^{n\times n}$ $(0 \le k \le m-1)$ is a constant inner-coupling matrix of the kth sub-network between node i and node j $(i \ne j)$ for all $1 \le i, j \le N$, the constant $c_k > 0$ $(0 \le k \le m-1)$ is the coupling strength, and τ_k is time delay of the kth sub-network compared with the basic network $(\tau_0 = 0)$. $A_k = (a_{kij})_{N\times N}$ is the coupling configuration matrix representing topological structure of the kth sub-network, in which a_{kij} of the kth sub-network is defined as follows: if there is a connection from node i to node j $(i \ne j)$, then $a_{kij} \ne 0$, otherwise $a_{kij} = 0$, and the diagonal elements of matrix A_l are defined by

$$a_{kii}(t) = -\sum_{j=1, j\ne i}^{N} a_{kij}(t), \quad i = 1, 2, \ldots, N \tag{4}$$

There have been various definitions of synchronization in the literature [25,26]. Hereafter, the united directed complex dynamical network (3) is said to achieve (asymptotical) synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t), \quad \text{as } t \to \infty \tag{5}$$

where $s(t) \in R^n$ is a solution of an isolate node, namely, $\dot{s}(t) = f(s(t))$.

To obtain the main results, by use of these previous studies [4,16,19,26], we can obtain the following lemma.

Lemma 2.1. *Consider the united directed complex dynamical network (3). Let*

$$0 = \lambda_{k1} \le \lambda_{k2} \le \cdots \le \lambda_{kN} \tag{6}$$

be the eigenvalues of the kth sub-network outer-coupling matrix A_k. If the following $N-1$ of n-dimensional delayed differential equations are asymptotically stable about their zero solutions:

$$\begin{aligned}\dot{w} &= (J + c_0\lambda_{0i}H_0)w(t) + c_1\lambda_{1i}H_1 w(t-\tau_1) + c_2\lambda_{2i}H_2 w(t-\tau_2) \\ &\quad + \cdots + c_{m-1}\lambda_{(m-1)i}H_{m-1}w(t-\tau_{m-1}), \quad i = 2, \ldots, N\end{aligned} \tag{7}$$

where J is the Jacobian of $f(x_i(t))$ at $s(t)$, then the synchronized states (5) are asymptotically stable for the united directed complex dynamical network (3).

3. **Synchronization Stability Criteria for United Directed Complex Dynamical Networks.** In this section, we will investigate the stability problem of united directed complex dynamical networks. And several criteria will be derived.

3.1. **Synchronization stability analysis for $m = 2$.** In this section, we discuss $m = 2$. Then the model (3) and systems (7) can be rewritten as follows:

$$\dot{x}_i = f(x_i) + c_0 \sum_{j=1}^{N} a_{(0)ij} H_0 x_j(t) + c_1 \sum_{j=1}^{N} a_{(1)ij} H_1 x_j(t-\tau_1), \quad i = 1, 2, \ldots, N \tag{8}$$

and

$$\dot{w} = (J + c_0\lambda_{0i}H_0)w(t) + c_1\lambda_{1i}H_1 w(t-\tau_1), \quad i = 2, \ldots, N \tag{9}$$

Based on the above-mentioned assumptions and definitions, we can obtain the following theorem.

Theorem 3.1. *For the dynamical system (7), if there exists* $Y = (y_{ij})_{1\le i,j\le n} = -A - \tau_1 B$ *such that*

(1) $y_{ii} > 0$, $i = 1, 2, \ldots, n$ *and* $y_{ij} \le 0$, *for* $i \ne j$, $i, j = 1, 2, \ldots, n$

(2) Successive principal minors of Y *are positive, that is,*

$$det \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1i} \\ \cdots & \cdots & \cdots & \cdots \\ y_{i1} & y_{i2} & \cdots & y_{ii} \end{pmatrix} > 0, \quad i = 1, 2, \ldots, n$$

where $A = (H + c_1\lambda_{1i}\Gamma)^*$, $B = |c_1\lambda_{1i}||\Gamma H| + (c_1\lambda_{1i})^2|\Gamma^2|$, $H = J + c_0\lambda_{0i}H_0$ *and* $\Gamma = H_1$, *then zero solution of system (9) is asymptotically stable.*

That is, the synchronized states (5) are asymptotically stable for the dynamical network (8).

Proof: Suppose that $w(t)$ is continuously differentiable when $t \ge 0$, by using the Newton-Leibniz formula, we can get

$$w(t-\tau_1) = w(t) - \int_{t-\tau_1}^{t} \dot{w}(s)ds \tag{10}$$

Substituting Equation (9) into Equation (10), we achieve

$$w(t-\tau_1) = w(t) - H\int_{t-\tau_1}^{t} w(s)ds - c_1\lambda_{1i}\Gamma\int_{t-\tau_1}^{t} w(s-\tau_1)ds \tag{11}$$

where $H = J + c_0\lambda_{0i}H_0$ and $\Gamma = H_1$. Then systems (9) can be rewritten

$$\dot{w}(t) = (H + c_1\lambda_{1i}\Gamma)w(t) - c_1\lambda_{1i}\Gamma H\int_{t-\tau_1}^{t} w(s)ds - (c_1\lambda_{1i})^2\Gamma^2\int_{t-\tau_1}^{t} w(s-\tau_1)ds \tag{12}$$

Let $v \in R^n$ with components $v_i > 0$ $(i = 1, 2, \ldots, n)$ and let us consider the radially unbound Lyapunov functional given by

$$U(t) = U_1(t) + U_2(t) + U_3(t) + U_4(t) \tag{13}$$

where

$$U_1(t) = \langle |w(t)|, v\rangle \tag{14}$$

$$U_2(t) = |c_1\lambda_{1i}|\left\langle |\Gamma H|\int_{-\tau_1}^{0}\int_{t+s}^{t} |w(\theta)|d\theta ds, v\right\rangle \tag{15}$$

$$U_3(t) = (c_1\lambda_{1i})^2\left\langle \left|\Gamma^2\right|\int_{-\tau_1}^{0}\int_{t+s}^{t} |w(\theta-\tau_1)|d\theta ds, v\right\rangle \tag{16}$$

and

$$U_4(t) = \tau_1(c_1\lambda_{1i})^2\left\langle \left|\Gamma^2\right|\int_{t-\tau_1}^{t} |w(\theta)|d\theta, v\right\rangle \tag{17}$$

Then it is obvious that

$$U(t) < \infty,\ t > 0 \tag{18}$$

The right Dini derivative of U along the solution of Equation (12) gives

$$D^+U(t)|_{(12)} = D^+U_1(t)|_{(12)} + D^+U_2(t)|_{(12)} + D^+U_3(t)|_{(12)} + D^+U_4(t)|_{(12)} \tag{19}$$

We have

$$D^+U_1(t)|_{(12)} = \left\langle \frac{d^+|w(t)|}{dt^+}, v \right\rangle \left\langle D_w(t)\frac{d^+w(t)}{dt^+}, v \right\rangle \tag{20}$$

where $D_w(t) = \mathrm{diag}\{sgn(w_1), sgn(w_2), \dots, sgn(w_n)\}$. Then we can obtain

$$\begin{aligned} D^+U_1(t)|_{(12)} &= \langle D_w(t)(H + c_1\lambda_{1i}\Gamma)w(t), v\rangle \\ &\quad - \left\langle D_w(t)\left(c_1\lambda_{1i}\Gamma H \int_{t-\tau_1}^{t} w(s)ds\right), v \right\rangle \\ &\quad - \left\langle D_w(t)\left(c_1\lambda_{1i}\right)^2 \Gamma^2 \int_{t-\tau_1}^{t} w(s-\tau_1)ds, v \right\rangle \end{aligned} \tag{21}$$

Next, by overvaluing $D^+U_1(t)|_{(10)}$, we can get

$$\begin{aligned} D^+U_1(t)|_{(12)} \le \langle (H + c_1\lambda_{1i}\Gamma)^*|w(t)|, v\rangle + \left\langle |c_1\lambda_{1i}||\Gamma H| \int_{t-\tau_1}^{t} |w(s)|ds, v \right\rangle \\ + \left\langle (c_1\lambda_{1i})^2 \left|\Gamma^2\right| \int_{t-\tau_1}^{t} |w(s-\tau_1)|ds, v \right\rangle \end{aligned} \tag{22}$$

Similarly, we have

$$D^+U_2(t)|_{(12)} = |c_1\lambda_{1i}| \left\langle |\Gamma H|(\tau_1|w(t)|), v\right\rangle - |c_1\lambda_{1i}| \left\langle |\Gamma H| \left(\int_{t-\tau_1}^{t} |w(\theta)|d\theta\right), v \right\rangle \tag{23}$$

$$\begin{aligned} D^+U_3(t)|_{(12)} &= (c_1\lambda_{1i})^2 \left\langle \left|\Gamma^2\right| (\tau_1|w(t-\tau_1)|), v\right\rangle \\ &\quad - (c_1\lambda_{1i})^2 \left\langle \left|\Gamma^2\right| \left(\int_{t-\tau_1}^{t} |w(\theta-\tau_1)|d\theta\right), v \right\rangle \end{aligned} \tag{24}$$

and

$$D^+U_4(t)|_{(12)} = (c_1\lambda_{1i})^2 \left\langle \tau_1\left|\Gamma^2\right|(|w(t)| - |w(t-\tau_1)|), v\right\rangle \tag{25}$$

From Equations (21)-(24) and Equation (13), we obtain

$$D^+U(t)|_{(12)} \le \langle -Y|w(t)|, v\rangle \tag{26}$$

where $A = (H + c_1\lambda_{1i}\Gamma)^*$, $B = |c_1\lambda_{1i}||\Gamma H| + (c_1\lambda_{1i})^2\left|\Gamma^2\right|$.

If Y satisfies the condition (1) and the condition (2), we can find a vector $\rho \in R_+^*$ [27], i.e., with components $\rho_k \in R_+^*$ satisfying the relation $Y_1 v = \rho$, $\forall v \in R_+^*$, here $\langle -Y|w(t)|, v\rangle = \langle -Y_1 v, |w(t)|\rangle$. So, we have

$$\langle -Y|w(t)|, v\rangle = \langle -\rho, |w(t)|\rangle \tag{27}$$

In the end, we can get

$$D^+U(t)|_{(12)} < -\sum_{k=1}^{n} \rho_k|w_k(t)| < 0 \tag{28}$$

Then, it follows that zero solution of system (9) is asymptotically stable. Form Lemma 2.1, we know that the synchronized states (5) are asymptotically stable for the dynamical network (8). The proof is completed.

3.2. **Synchronization stability analysis for $m = 3$.** In this section, we discuss $m = 3$. Then the model (3) and systems (7) can be rewritten as follows:

$$\begin{aligned}\dot{x_i} = f(x_i) + c_0 \sum_{j=1}^{N} a_{(0)ij} H_0 x_j(t) + c_1 \sum_{j=1}^{N} a_{(1)ij} H_1 x_j(t-\tau_1) \\ + c_2 \sum_{j=1}^{N} a_{(2)ij} H_2 x_j(t-\tau_2), \quad i = 1, 2, \ldots, N\end{aligned} \tag{29}$$

and

$$\dot{w} = (J + c_0\lambda_{0i}H_0)w(t) + c_1\lambda_{1i}H_1 w(t-\tau_1) + c_2\lambda_{2i}H_2 w(t-\tau_2), \quad i = 2, \ldots, N \tag{30}$$

Then a sufficient condition for asymptotic stability of system (30) is given as follows.

Theorem 3.2. *For the dynamical system (30), if there exists $Y = (y_{ij})_{1\le i,j\le n} = -A_1 - B_1$ such that*

(1) $y_{ii} > 0$, $i = 1, 2, \ldots, n$ and $y_{ij} \le 0$, for $i \ne j$, $i, j = 1, 2, \ldots, n$

(2) Successive principal minors of Y are positive, that is,

$$det \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1i} \\ \cdots & \cdots & \cdots & \cdots \\ y_{i1} & y_{i2} & \cdots & y_{ii} \end{pmatrix} > 0, \quad i = 1, 2, \ldots, n$$

where $A_1 = (A + B + C)^*$, $B_1 = \tau_1(|BA| + |B^2| + |BC|) + \tau_2(|CA| + |C^2| + |CB|)$, $A = J + c_0\lambda_{0i}H_0$, $B = c_1\lambda_{1i}H_1$ and $C = c_2\lambda_{2i}H_2$, then zero solution of system (30) is asymptotically stable.

That is, the synchronized states (5) are asymptotically stable for the dynamical network (29).

Proof: Suppose that $w(t)$ is continuously differentiable when $t \ge 0$, by using the Newton-Leibniz formula, we can get

$$w(t-\tau_1) = w(t) - \int_{t-\tau_1}^{t} \dot{w}(s)ds \tag{31}$$

and

$$w(t-\tau_2) = w(t) - \int_{t-\tau_2}^{t} \dot{w}(s)ds \tag{32}$$

Substituting Equation (29) into Equation (31), we achieve

$$w(t-\tau_1) = w(t) - A\int_{t-\tau_1}^{t} w(s)ds - B\int_{t-\tau_1}^{t} w(s-\tau_1)ds - C\int_{t-\tau_1}^{t} w(s-\tau_2)ds \tag{33}$$

where $A = J + c_0\lambda_{0i}H_0$, $B = c_1\lambda_{1i}H_1$ and $C = c_2\lambda_{2i}H_2$.

Similarly, we can get

$$w(t-\tau_2) = w(t) - A\int_{t-\tau_2}^{t} w(s)ds - B\int_{t-\tau_2}^{t} w(s-\tau_1)ds - C\int_{t-\tau_2}^{t} w(s-\tau_2)ds \tag{34}$$

Substituting Equation (33) and Equation (34) into Equation (30), we achieve

$$\begin{aligned}\dot{w}(t) = {} & (A + B + C)w(t) \\ & + B\left[A\int_{t-\tau_1}^{t} w(s)ds + B\int_{t-\tau_1}^{t} w(s-\tau_1)ds + C\int_{t-\tau_1}^{t} w(s-\tau_2)ds\right] \\ & + C\left[A\int_{t-\tau_2}^{t} w(s)ds + B\int_{t-\tau_2}^{t} w(s-\tau_1)ds + C\int_{t-\tau_2}^{t} w(s-\tau_2)ds\right]\end{aligned} \tag{35}$$

Let $v \in R^n$ with components $v_i > 0$ $(i = 1, 2, \ldots, n)$ and let us consider the radially unbound Lyapunov functional given by

$$U(t) = U_1(t) + U_2(t) + U_3(t) + U_4(t) + U_5(t) + U_6(t) + U_7(t) + U_8(t) \quad (36)$$

where

$$U_1(t) = \langle |w(t)|, v\rangle \quad (37)$$

$$U_2(t) = \left\langle |BA| \int_{-\tau_1}^{0} \int_{t+\theta}^{t} |w(s)| ds d\theta, v \right\rangle \quad (38)$$

$$U_3(t) = \left\langle \left|B^2\right| \int_{-\tau_1}^{0} \int_{t+\theta}^{t} |w(s-\tau_1)| ds d\theta, v \right\rangle \quad (39)$$

$$U_4(t) = \left\langle |CA| \int_{-\tau_2}^{0} \int_{t+\theta}^{t} |w(s)| ds d\theta, v \right\rangle \quad (40)$$

$$U_5(t) = \left\langle \left|C^2\right| \int_{-\tau_2}^{0} \int_{t+\theta}^{t} |w(s-\tau_2)| ds d\theta, v \right\rangle \quad (41)$$

$$U_6(t) = \left\langle |BC| \int_{-\tau_1}^{0} \int_{t+\theta}^{t} |w(s-\tau_2)| ds d\theta, v \right\rangle \quad (42)$$

$$U_7(t) = \left\langle |CB| \int_{-\tau_2}^{0} \int_{t+\theta}^{t} |w(s-\tau_1)| ds d\theta, v \right\rangle \quad (43)$$

and

$$\begin{aligned} U_8(t) = \Big\langle &\left|B^2\right|\tau_1 \int_{t-\tau_1}^{t} |w(s)| ds + \left|C^2\right|\tau_2 \int_{t-\tau_2}^{t} |w(s)| ds \\ &+ |BC|\tau_1 \int_{t-\tau_2}^{t} |w(s)| ds + |CB|\tau_2 \int_{t-\tau_1}^{t} |w(s)| ds, v \Big\rangle \end{aligned} \quad (44)$$

Then it is obvious that

$$U(t) < \infty,\ t > 0 \quad (45)$$

The right Dini derivative of U along the solution of Equation (35) gives

$$\begin{aligned} D^+U(t)|_{(35)} = &D^+U_1(t)|_{(35)} + D^+U_2(t)|_{(35)} + D^+U_3(t)|_{(35)} + D^+U_4(t)|_{(35)} \\ &+ D^+U_5(t)|_{(35)} + D^+U_6(t)|_{(35)} + D^+U_7(t)|_{(35)} + D^+U_8(t)|_{(35)} \end{aligned} \quad (46)$$

We have

$$D^+U_1(t)|_{(35)} = \left\langle \frac{d^+|w(t)|}{dt^+}, v \right\rangle = \left\langle D_w(t) \frac{d^+w(t)}{dt^+}, v \right\rangle \quad (47)$$

where $D_w(t) = \text{diag}\{sgn(w_1), sgn(w_2), \ldots, sgn(w_n)\}$.

Next, by overvaluing $D^+U_1(t)|_{(35)}$, we can get

$$\begin{aligned} D^+U_1(t)|_{(35)} \le &\left\langle (A+B+C)^*|w(t)| + |BA| \int_{t-\tau_1}^{t} |w(\theta)| d\theta + |CA| \int_{t-\tau_2}^{t} |w(\theta)| d\theta, v \right\rangle \\ &+ \left\langle |B^2| \int_{t-\tau_1}^{t} |w(\theta-\tau_1)| d\theta + \left|C^2\right| \int_{t-\tau_2}^{t} |w(\theta-\tau_2)| d\theta, v \right\rangle \\ &+ \left\langle |BC| \int_{t-\tau_1}^{t} |w(\theta-\tau_2)| d\theta + |CB|\tau_2 \int_{t-\tau_2}^{t} |w(\theta-\tau_2)| d\theta, v \right\rangle \end{aligned} \quad (48)$$

At the same time, we have

$$D^+U_2(t)|_{(35)} = \left\langle |BA| \left(\tau_1|w(t)| - \int_{t-\tau_1}^{t} |w(s)|ds \right), v \right\rangle \tag{49}$$

$$D^+U_3(t)|_{(35)} = \left\langle \left|B^2\right| \left(\tau_1|w(t-\tau_1)| - \int_{t-\tau_1}^{t} |w(s-\tau_1)|ds \right), v \right\rangle \tag{50}$$

$$D^+U_4(t)|_{(35)} = \left\langle |CA| \left(\tau_2|w(t)| - \int_{t-\tau_2}^{t} |w(s)|ds \right), v \right\rangle \tag{51}$$

$$D^+U_5(t)|_{(35)} = \left\langle \left|C^2\right| \left(\tau_2|w(t-\tau_2)| - \int_{t-\tau_2}^{t} |w(s-\tau_2)|ds \right), v \right\rangle \tag{52}$$

$$D^+U_6(t)|_{(35)} = \left\langle |BC| \left(\tau_1|w(t-\tau_2)| - \int_{t-\tau_1}^{t} |w(s-\tau_2)|ds \right), v \right\rangle \tag{53}$$

$$D^+U_7(t)|_{(35)} = \left\langle \left|C^2\right| \left(\tau_2|w(t-\tau_1)| - \int_{t-\tau_2}^{t} |w(s-\tau_1)|ds \right), v \right\rangle \tag{54}$$

and

$$\begin{aligned} D^+U_8(t)|_{(35)} = &\big\langle \big(\left|C^2\right|\tau_2 + \left|B^2\right|\tau_1 + |CB|\tau_2 + |BC|\tau_1\big)|w(t)| - \left|B^2\right|\tau_1|w(t-\tau_1)| \\ &- \left|C^2\right|\tau_2|w(t-\tau_2)| - |BC|\tau_1|w(t-\tau_2)| - |CB|\tau_2|w(t-\tau_1)|, v\big\rangle \end{aligned} \tag{55}$$

From Equations (48)-(55) and Equation (36), we obtain

$$D^+U(t)|_{(35)} \leq \langle -Y|w(t)|, v\rangle \tag{56}$$

where $A_1 = (A + B + C)^*$, $B_1 = \tau_1(|BA| + |B^2| + |BC|) + \tau_2(|CA| + \left|C^2\right| + |CB|)$.

If Y satisfies the condition (1) and the condition (2), we can find a vector $\rho \in R_+^*$ [27], i.e., with components $\rho_k \in R_+^*$ satisfying the relation $Y_1 v = \rho$, $\forall v \in R_+^*$, here $\langle -Y|w(t)|, v\rangle = \langle -Y_1 v, |w(t)|\rangle$. So, we have

$$\langle -Y|w(t)|, v\rangle = \langle -\rho, |w(t)|\rangle \tag{57}$$

In the end, we can get

$$D^+U(t)|_{(35)} < -\sum_{k=1}^{n} \rho_k |w_k(t)| < 0 \tag{58}$$

Then, it follows that zero solution of system (30) is asymptotically stable. Form Lemma 2.1, we know that the synchronized states (5) are asymptotically stable for the united complex dynamical network (29). The proof is completed.

4. **Numerical Simulation.** In this section, we use two examples to illustrate the results derived in this work. The above asymptotically stable conditions can be applied to networks with different topologies and different sizes. In order to illustrate the main results, we consider a lower-dimensional network model.

4.1. **Example 1.** We consider a lower-dimensional network model with five nodes, in which each node is a simple three-dimensional stable linear system described in [16].

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -2x_2 \\ \dot{x}_3 = -3x_3 \end{cases}$$

which is asymptotically stable at $s(t) = 0$, and its Jacobian is $J(t) = \text{diag}\{-1, -2, -3\}$.

Assume that the outer-coupling matrices A_0, A_1 are given as follows:

$$A_0 = \begin{pmatrix} -1 & 0.25 & 0.25 & 0 & 0.5 \\ 0 & -1 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} -1 & 0.25 & 0.25 & 0 & 0.5 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & -1 \end{pmatrix}$$

The inner-coupling matrices is often assumed to be the unit matrix [23,28]. And here the inner-coupling matrices $H_0 = H_1$ are given as follows:

$$H_0 = \begin{pmatrix} 1.5267 & 1.2382 & 0 \\ -1.8824 & 0 & 1.0337 \\ 0 & 1.4472 & 0 \end{pmatrix}$$

The eigenvalues of A_0 are $\lambda_{0i} = 0, -0.6910, -1, -1.5, -1.809$ and the eigenvalues of A_1 are $\lambda_{1i} = 0, -0.6910, -1, -1.5, -1.809$. For clearer visions, we take the coupling strength $c_0 = 0.01$, $c_1 = 0.02$ and $\tau_1 = 0.05$.

In terms of Theorem 3.1, if the condition (1) and the condition (2) are satisfied, then it is inferred that the synchronization of the complex network (1) can be achieved. When $\lambda_{0i} = -0.6910, -1, -1.5, -1.809$ and $\lambda_{1i} = -0.6910, -1, -1.5, -1.809$, we can get

$$Y = \begin{pmatrix} 1.0306 & -0.0274 & 0 \\ -0.0404 & 2 & -0.0236 \\ 0 & -0.032 & 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 1.0443 & -0.0387 & 0 \\ -0.0584 & 2 & -0.0341 \\ 0 & -0.0463 & 3 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1.0664 & -0.0596 & 0 \\ -0.0877 & 1.9999 & -0.0512 \\ 0 & -0.0695 & 2.9999 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1.0801 & -0.0719 & 0 \\ -0.1058 & 1.9999 & -0.0617 \\ 0 & -0.0838 & 2.9999 \end{pmatrix}$$

respectively. Obviously, the condition (1) of Theorem 3.1 is satisfied. At the same time, we observe that eigenvalues of the above four matrices are $d_1 = (1.0295, 2.0004, 3.0070)$, $d_2 = (1.0419, 2.0008, 3.0015)$, $d_3 = (1.0608, 2.0020, 3.0035)$ and $d_4 = (1.0719, 2.0030, 3.0050)$, respectively. Therefore, the condition (2) of Theorem 3.1 is satisfied and the synchronized states (5) of network (8) are asymptotically stable. In Figure 1, we plot the curves of the synchronization errors between the states of node i and node $i+1$ (that is, $e_{ij}(t) = x_{ij}(t) - x_{i+1,j}(t)$), for $i = 1, 2, 3, 4$, $j = 1, 2, 3$, with the coupling strength $c_0 = 0.01$, $c_1 = 0.02$ and time delay $\tau = 0.05$. When coupling strength $c_0 = 1.5$, $c_1 = 0.6$ and time delay $\tau_1 = 0.2$, it is obvious that the condition (1) and the condition (2) of Theorem 3.1 are not satisfied. The curves of the synchronization errors are shown as in Figure 2. When coupling strength $c_0 = 0.2$, $c_1 = 0.25$ and time delay $\tau_1 = 0.18$, we can obtain that the condition (1) and the condition (2) of Theorem 3.1 are satisfied. The curves of the synchronization errors are shown as in Figure 3.

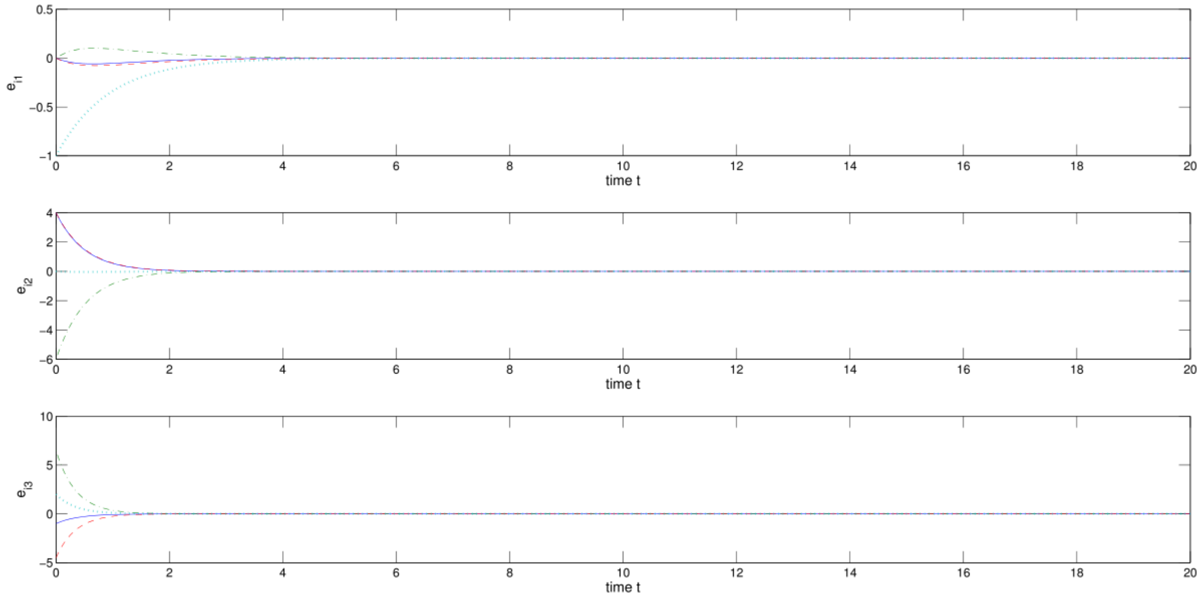

FIGURE 1. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 0.01$, $c_1 = 0.02$ and time delay $\tau_1 = 0.05$

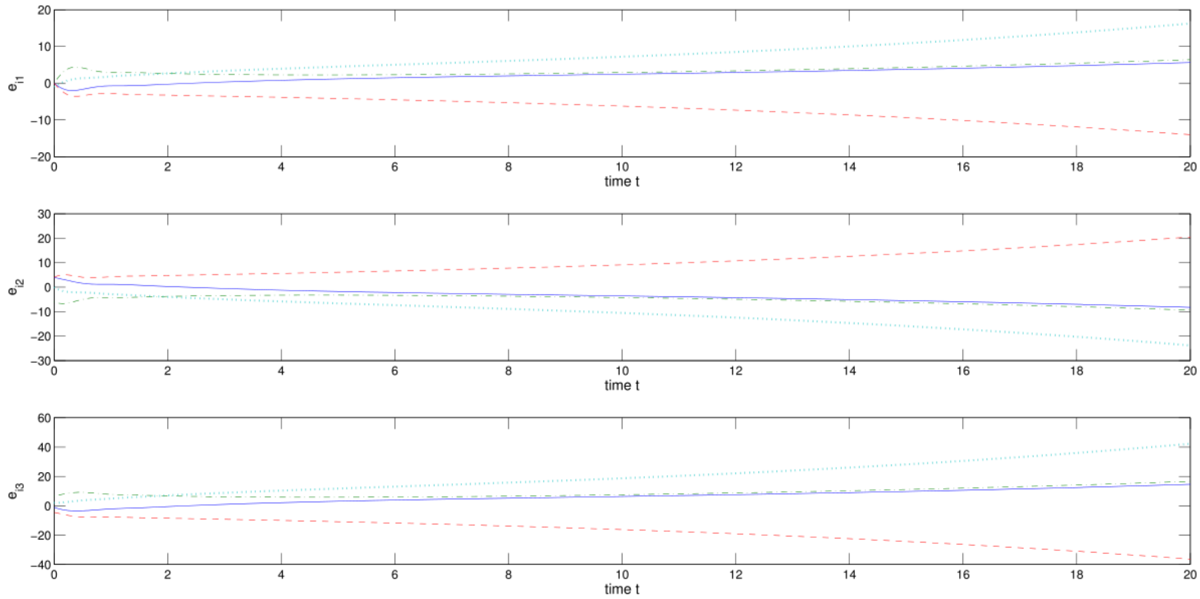

FIGURE 2. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 1.5$, $c_1 = 0.6$ and time delay $\tau_1 = 0.2$

4.2. **Example 2.** In the example, we start to consider the asymptotically stability for the dynamical network (29). At the same time, we consider a lower-dimensional network model with five nodes, in which each node is the three-dimensional stable linear system as above. Here the inner-coupling matrices $H_0 = H_1 = H_2$ are given as follows:

$$H_0 = \begin{pmatrix} 1.5267 & 1.2382 & 0 \\ -1.8824 & 0 & 1.0337 \\ 0 & 1.4472 & 0 \end{pmatrix}$$

and the outer-coupling matrices A_0 and A_1 are given as above in Example 1, and $A_2 = A_0$.

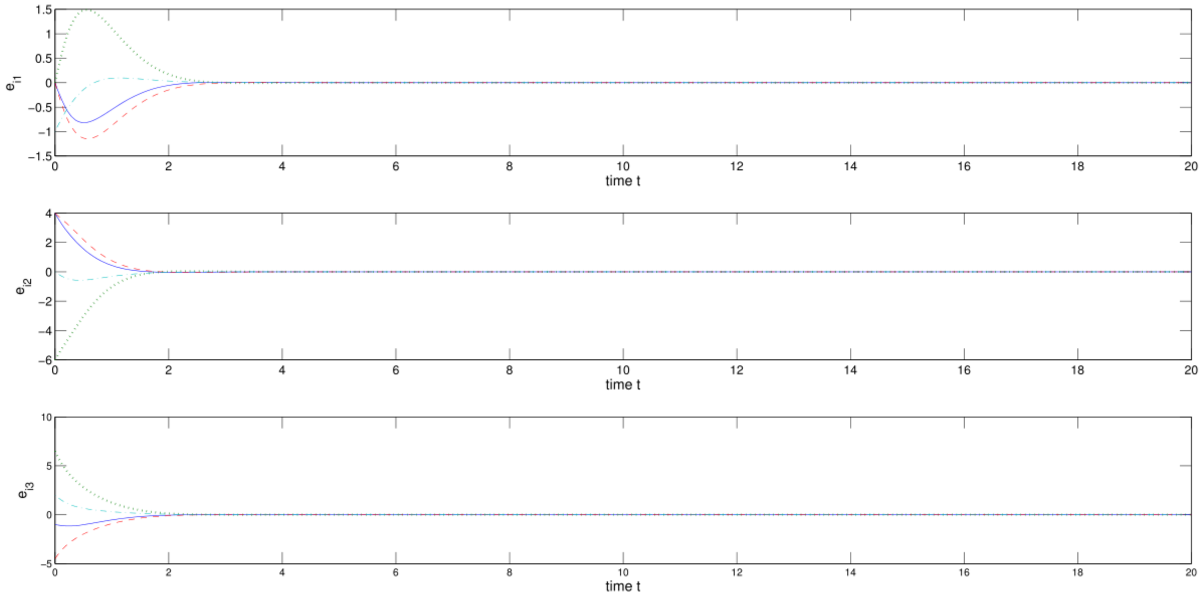

FIGURE 3. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 0.2$, $c_1 = 0.25$ and time delay $\tau_1 = 0.18$

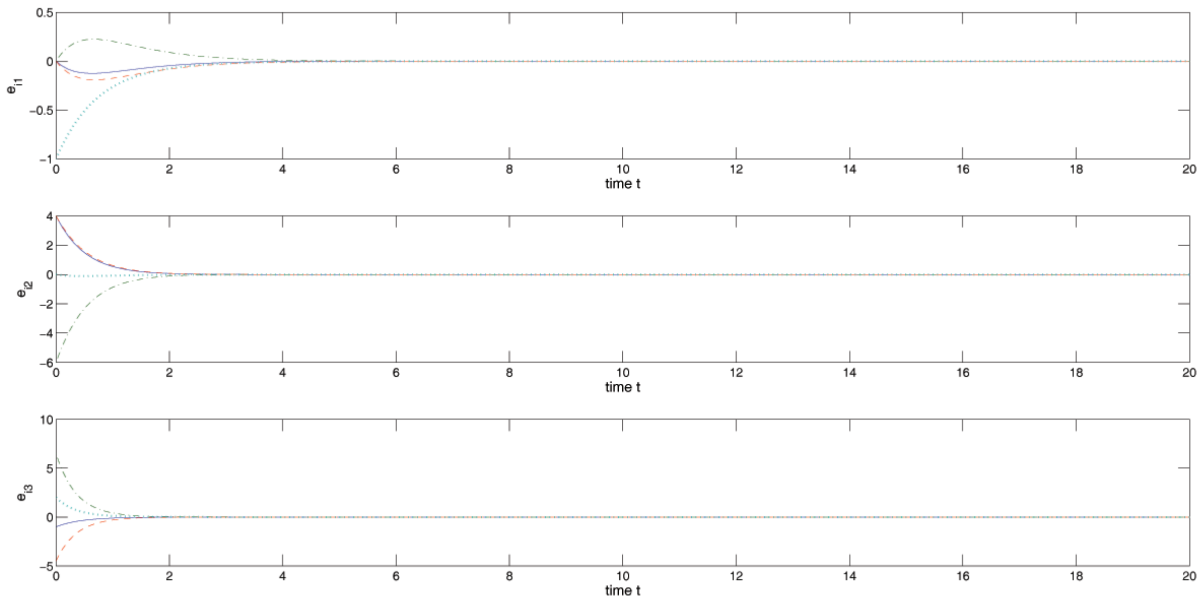

FIGURE 4. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 0.04$, $c_1 = 0.01$, $c_2 = 0.03$ and time delay $\tau_1 = 0.01$, $\tau_2 = 0.02$

When coupling strength $c_0 = 0.04$, $c_1 = 0.01$, $c_2 = 0.03$ and time delay $\tau_1 = 0.01$, $\tau_2 = 0.02$, we can obtain that the condition (1) and the condition (2) of Theorem 3.2 are satisfied. In Figure 4, we plot the curves of the synchronization errors between the states of node i and node $i + 1$. However, the dynamical network (29) is not asymptotically stable when coupling strength $c_0 = 0.4, c_1 = 0.3, c_2 = 0.3$ and time delay $\tau_1 = 0.1$, $\tau_2 = 1$ as shown in Figure 5. Similarly, the dynamical network (29) is asymptotically stable as shown in Figure 6 when coupling strength $c_0 = 0.04$, $c_1 = 0.1$, $c_2 = 0.03$ and time delay $\tau_1 = 0.01$, $\tau_2 = 0.1$. From Figure 4 to Figure 6, we can see that the dynamical

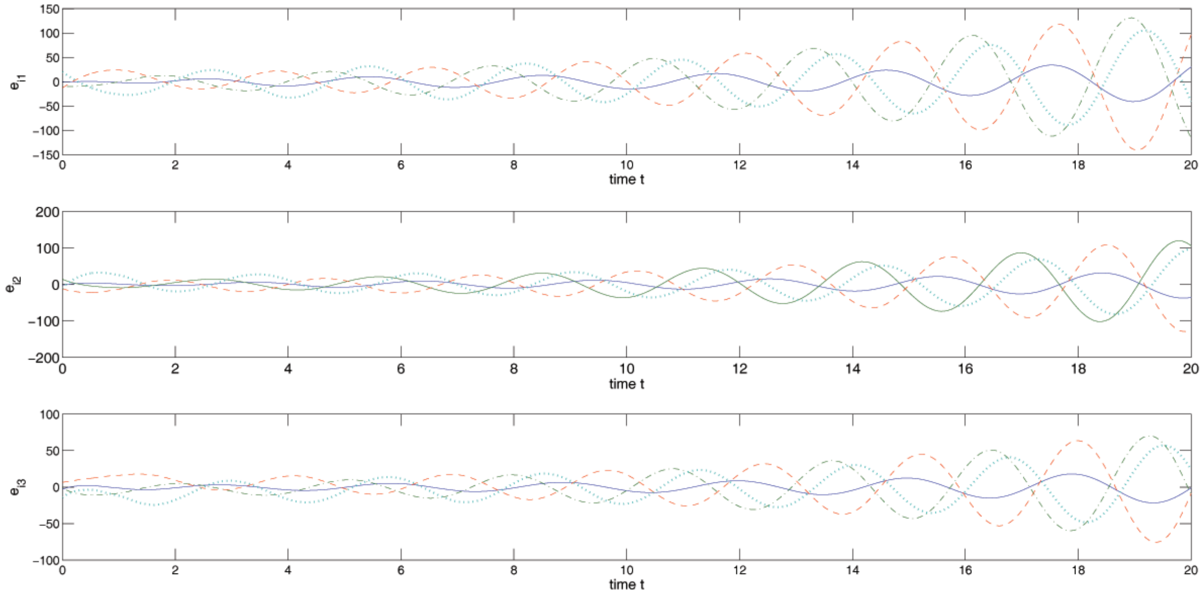

FIGURE 5. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 0.4$, $c_1 = 0.3$, $c_2 = 0.3$ and time delay $\tau_1 = 0.1$, $\tau_2 = 1$

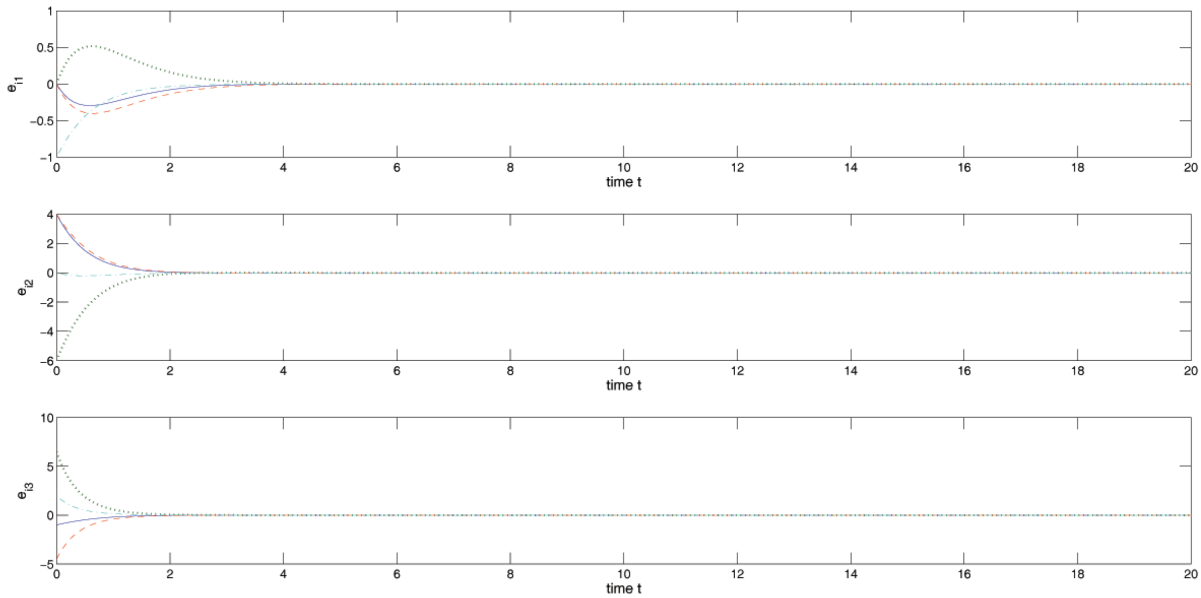

FIGURE 6. (color online) Synchronization errors for the united directed complex dynamical networks when coupling strength $c_0 = 0.04$, $c_1 = 0.1$, $c_2 = 0.03$ and time delay $\tau_1 = 0.01$, $\tau_2 = 0.1$

network (29) is asymptotically stable when conditions of Theorem 3.2 are satisfied. Here the stability criteria of dynamical networks are discussed. And a dynamic network can be synchronized only if the network parameters meet certain conditions. There are few studies on this subject. However, there are many dynamical networks and the stability conditions of these networks are not satisfied. So some controllers are added to ensure the synchronization of complex dynamical networks. For example, Hu et al. [28] have discussed the synchronization of united complex dynamical networks via pinning control.

By use of adaptive periodically intermittent control, exponential synchronisation of united complex dynamical networks has been studied [24].

5. **Conclusion.** In this paper, the problem of synchronization stability for the united directed complex networks with multi-links has been investigated in detail. The idea about network split is presented according to the different nature of time-delay. Time-delay is introduced into the integration process. Based on the Lyapunov function combined with linear matrix inequalities, some general criteria for ensuring united directed complex networks synchronization have been derived. Finally, numerical simulations are provided to show the effectiveness and feasibility of the proposed theorems. In fact, there are many united complex dynamical networks with multi-links such as the World Wide Web, relationship networks and transportation networks. And the designed methods may be very useful to study dynamical behaviors real-life networks for the theoretical works and applications.

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