International Journal of Innovative Computing, Information and Control
Volume **13**, Number **5**, October **2017**
ICIC International ©2017 ISSN 1349-4198
pp. 1669–1679

NUMERICAL SCHEME FOR GLOBAL OPTIMIZATION OF FRACTIONAL OPTIMAL CONTROL PROBLEM WITH BOUNDARY CONDITIONS

Najeeb Alam Khan, Oyoon Abdul Razzaq, Tooba Hameed and Muhammad Ayaz

Department of Mathematics
University of Karachi
Karachi 75270, Pakistan
{ njbalam; oyoon.abdulrazzaq }@yahoo.com; { toobahameed; ayaz_maths }@hotmail.com

Received February 2017; revised June 2017

Abstract. *This paper executes an optimization schema for a class of fractional optimal control problems (FOCPs), subjected to boundary conditions. This novel technique, named as the Taylor optimization method (TOM), is based on the simulated annealing method (SA) together with the generalized Taylor's series. It constitutes the approximation of state and control functions using generalized Taylor series expansions. While, simulated annealing seeks out the values of unknown terms of the series along with the global minimum value of performance index of FOCPs. Elaborately, error and convergence analysis are also discussed for the proposed scheme. Furthermore, graphical and tabulated results of some examples of FOCPs are exemplified using TOM. A comparative study is also carried out with exact and the existing solutions in the literature, asserting the effectiveness and accuracy of this simulating algorithm.*

Keywords: Generalized Taylor's series, Simulated annealing, Fractional optimal control problems

1. **Introduction.** Fractional calculus has become a considerable area of study in different theoretical and applied fields of mathematics, nowadays. In view of the fact that it deals with the study of arbitrary orders of derivatives and integrals, this theory has been potentially applied in modeling various real-world physical problems [1-3]. Among many significant fractional order dynamical models, in this exertion, we might take following class of fractional control problems into consideration.

Let $\alpha \in (0,1)$ be the fractional order, Ψ, $L : [a,T] \times \mathfrak{R}^2 \to \mathfrak{R}$ be two differentiable functions and $x(t)$, $u(t)$ be the state and control functions, respectively, and then the performance index function can be outlined as,

$$\min J(x,u,T) = \int_a^T L(t, x(t), u(t))dt \tag{1}$$

with the dynamical constraints,

$$M_1\dot{x}(t) + M_2 D_t^\alpha x(t) = \Psi(t, x(t), u(t)) \tag{2}$$

and the boundary conditions,

$$x(a) = x_a, \;\; x(T) = x_T \tag{3}$$

where M_1, M_2 are nonzero constants and T, x_a and x_T are fixed real numbers. These paradigms describe a set of integral and differential equations that define the paths for the control and state variables while minimizing performance index function. The FOCPs are intended to endure the capability of inducing the optimization of the dynamical systems. For this reason, these models have been widely contemplated in designing different biological, physical and management sciences problems. In literature, many research works are found that contain discussions and several applications of FOCPs [4-7].

Most commonly, FOCPs are assessed by means of the Hamiltonian function [8-12] or Rayleigh-Ritz method [8,13], for the necessary conditions of optimality. As an alternative, here, we consider the Taylor optimization method in which simulated annealing algorithm [14,15] is undertaken in place of these optimality conditions for the FOCPs. SA is a probabilistic technique, which searches the feasible solutions of the functions with the global optimum of the objective function of an optimization problem, instead of local optimum. It is a derivative free and therefore has the capability to expeditiously execute the results. The computational time required by SA is principally the time entailed to evaluate the objective function. Specifically, SA is highly effective for the problems where locating an approximate global optimum is more imperative than detecting a precise local optimum in a fixed amount of time. The SA algorithm offers the following preferences in comparison with the other optimality conditions.

a) The optimality search proceeds within the entire domain, without using the Lagrange multiplier or the derivatives of the functions.
b) Without neglecting any, all the calculated results at each point of the domain are considered and assigned a probability.
c) The result with the greatest probability, i.e., 1, suggests the feasible solution.
d) In a short period of time, rapidly provide the solutions together with the global optimum of the optimization problems, without being trapped in local minima.

The major contribution of this description is to bring in a scheme for the literature, which approximates and globally optimizes the problems, which can be in the form of FOCP with the objective function or the differential equations with an error function.

Methodically, TOM approximates the arbitrary functions through Taylor's series expansion [16-22], and then simulates the unknown terms of the series, while optimizing the performance index function, by simulated annealing algorithm. Sequentially, after some basic notations and preliminaries of fractional calculus, this paper comprises the elaboration of the methodology with the error bound assessment. Moreover, convergence analysis of the method is also carried out, along with the illustration of some examples of FOCP. The effective concluding remarks on the validity and precision of TOM verify its capability for being applicable to solve many other differential equations of different categories.

2. **Preliminaries.**

In this section, we represent some necessary definitions and properties of fractional calculus with some notations of generalized Taylor's series expansion.

2.1. **Riemann-Liouville fractional integral.**

The Riemann-Liouville fractional integral of order $\alpha > 0$, for any arbitrary function $x : [a, b] \to \mathfrak{R}$ is given by [1,2],

$$I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau$$

as the left Riemann-Liouville fractional integral and

$$I_b^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau) d\tau$$

as the right Riemann-Liouville fractional integral. Moreover, the operator I^α also satisfies the following properties. Let $\mu > 0$, then

$$I^\alpha I^\mu x(t) = I^{\alpha+\mu} x(t),$$
$$I^\alpha I^\mu x(t) = I^\alpha I^\mu x(t),$$
$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} t^{\beta+\alpha}$$

2.2. **Caputo fractional derivative.** The Caputo fractional derivative of order $n-1 < \alpha \leq n$, where n is any positive integer, of any arbitrary function $x : [a,b] \to \mathfrak{R}$ is defined as [1,2],

$$D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau$$

as the left Caputo fractional derivative and

$$D_b^\alpha x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} x^{(n)}(\tau) d\tau$$

as the right Caputo fractional derivative. Furthermore, the operator D^ρ also satisfies the following properties

$$D_t^\alpha K = 0, \ (K \text{ is constant})$$
$$I^\alpha D_t^\alpha x(t) = x(t) - \sum_{i=0}^{m-1} \frac{x^{(i)}(0)}{i!} t^i,$$
$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha},$$
$$D_t^\alpha (\lambda x(t) + \eta y(t)) = \lambda D_t^\alpha x(t) + \eta D_t^\alpha y(t)$$

2.3. **The generalized Taylor's series.** Consider the continuous function $x(t) : \mathfrak{R} \to \mathfrak{R}$ that has a continuous fractional derivative of order $i\alpha$, for any integer $i > 0$ and $0 < \alpha \leq 1$, then the subsequent equality holds,

$$x(t) = \sum_{i=0}^{\infty} \frac{D_{t_0}^{i\alpha}}{\Gamma(i\alpha+1)} (t-t_0)^{i\alpha}, \ t_0 \in [a,b] \tag{4}$$

Comprehensively, this series can be written as,

$$x(t) = \sum_{i=0}^{N} \frac{D_{t_0}^{i\alpha}}{\Gamma(i\alpha+1)} (t-t_0)^{i\alpha} + R_N^\alpha(t,t_0) \tag{5}$$

where

$$R_N^\alpha(t,\xi) = \frac{(t-t_0)^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} D_\xi^{(N+1)\alpha} x(\xi), \ \xi \in [a,t], \ \forall t \in (a,T] \tag{6}$$

is the remainder function of generalized Taylor's series [20].

3. The Taylor Optimization Method (TOM).

3.1. **The Taylor's series approximation.** Assume a square integrable function $y(t)$, described over an interval $[a,T)$. Then, the function $y(t)$ can be approximated about a point a with the truncated generalized Taylor's series as,

$$y(t,s) = s_0 + s_1\frac{(t-1)^{\alpha}}{\Gamma(\alpha+1)} + \cdots + s_N\frac{(t-1)^{N\alpha}}{\Gamma(N\alpha+1)}, \quad \alpha \in \Re \tag{7}$$

where $s_i = D_a^{i\alpha}y(a)$, for $0 \le i \le N$, are said to be the coefficients of the series [20]. Analogous to Equation (7), the state function $x(t)$ can also be approximated as,

$$x(t,s) = s_0 + s_1\frac{(t-1)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{N} s_i\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} \tag{8}$$

where $s_0 = x(a)$, $s_1 = D_a^{\alpha}x(a)$ and so on. Let $t = T$, and on taking the boundary conditions specified in Equation (3) and simplifying we get

$$s_1 = \frac{1}{w_1(T)}\left(x_T - x_a - \sum_{i=2}^{N} s_i w_i(T)\right)$$

where $w_i(T) = \frac{(T-a)^{i\alpha}}{\Gamma(i\alpha+1)}$. On substituting the above value of s_1 in Equation (8), the trial solution for the state function is acquired, i.e.,

$$x_{trial}(t,s) = x_a + \frac{w_1(t)}{w_1(T)}\left(x_T - x_a - \sum_{i=2}^{N} s_i w_i(T)\right)\frac{(t-1)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{N} s_i w_i(t) \tag{9}$$

where s_i are the unknown coefficients of the series that are to be determined. The trial solution for control function $u(t)$ is attained by using Equation (9) in Equation (2), i.e.,

$$u_{trial}(t,s) = M_1\dot{x}_{trial}(t,s) + M_2 D_t^{\alpha}x_{trial}(t,s) - \Psi(t, x_{trial}(t,s)) \tag{10}$$

Finally, substituting Equations (9) and (10) in Equation (1), the approximation of performance index J is constructed as,

$$\min J\left(x_{trial}, u_{trial}, T\right) = \int_a^T L\left(t, x_{trial}(t,s), u_{trial}(t,s)\right) \tag{11}$$

3.2. **The simulated annealing (optimization).** After erecting the trial solutions, we utilize SA algorithm to compute the unknown terms of the series and optimize the performance index. SA is the most popular metaheuristic method which employs random search technique and the Boltzmann probability distribution [23] in simulation process. This process encompasses the following features:

a) A space of random points in a specified domain, which is used as the sampling points for the iteration,
b) The unknown parameters whose values are to be determined,
c) A conditional equation or an objective function. Its values play crucial part in measuring the probability,
d) Boltzmann probability distribution, which is an exponential function that assigns probability to each value of the conditional equation.

Hence, for the governing optimizing problem, let

$$t_i = \frac{(T-a)i}{M}, \quad i = 0, 1, \ldots, M \tag{12}$$

define the space of random points, s_i are the unknown parameters and the objective function is expressed as,

$$\min J\left(x_{trial}, u_{trial}, T\right) = \sum_{i=1}^{M} L\left(t_i, x_{trial}(t_i,s), u_{trial}(t_i,s)\right) \le 10^{-\varepsilon} \tag{13}$$

where ε is any positive integer. A point is selected randomly from Equation (12) and the trial solutions of state and control functions are generated from Equations (9) and (10), respectively. These trial solutions are then substituted in Equation (13) and a probability is calculated using its value in the probability distribution. This probability will suggest to which extent the approximated values of the unknowns are accurate. In the similar manner, after exercising all the sampling points of the given space, the value of the objective function with the greatest probability, i.e., 1, is accepted to be the global minimum value. In addition, the values of the unknown parameters at that point will be considered to be the most accurately convergent approximations towards the exact solutions among the other calculated values within the space.

Thus, the algorithmic process of TOM for immediate implementation on FOCPs, is as follows.

Step 1.

i) Set $N \geq 2$, for $i = 1, 2, \ldots, N$.

ii) Construct the trial solution of state function

$$x_{trial}(t,s) = x_a + \frac{w_1(t)}{w_1(T)}\left(x_T - x_a - \sum_{i=2}^{N} s_i w_i(T)\right)\frac{(t-1)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=2}^{N} s_i w_i(t)$$

Step 2.

i) Set $0 < \alpha \leq 1$.

ii) Compute all the components of given dynamical system, i.e.,

$$D_t^\alpha x_{trial}(t,s) = \frac{1}{w_1(T)}\left(x_T - x_a - \sum_{i=2}^{N} s_i w_i(T)\right) D_t^\alpha w_1(t) + \sum_{i=2}^{N} s_i \left(D_t^\alpha w_i(t)\right)$$

Step 3.

Substitute all components in Equation (2) and construct trial solution of the control function,

$$u_{trial}(t,s) = M_1 \dot{x}_{trial}(t,s) + M_2 D_t^\alpha x_{trial}(t,s) - \Psi(t, x_{trial}(t,s))$$

Step 4.

i) Set sampling points,

$$t_i = \frac{(T-a)i}{M},\ i = 0, 1, \ldots, M \text{ for } M \geq N$$

ii) Substitute the trial solutions in objective function, defined in Equation (13).

$$\min J\left(x_{trial}, u_{trial}, T\right) = \sum_{i=1}^{M} L\left(t_i, x_{trial}(t_i,s), u_{trial}(t_i,s)\right)$$

iii) Set the probability distribution $P_d = e^{\frac{-J}{k\theta}}$, where $k = 1.38 \times 10^{-23}$ is Boltzmann constant and $\theta = 1$ is the initial state.

Step 5.

Organize simulated annealing process.

Input

i) Randomly select.

ii) Substitute in objective function.

$$\min J(s_2, s_3, \ldots, s_N) = \sum_{i=1}^{M} L(s_2, s_3, \ldots, s_N)$$

and use *Mathematica* software for calculations.

Output(i):

Minimum value of J and the values of all unknown terms s_i.
iii) Put the value of J in P_d and calculate the probability.
iv) Repeat the process until all the defined sampling points of the space are utilized.
Output(ii):
All the minimum values of J with assigned probabilities P_d.
Step 6.
Input
i) Select that minimum J, which is producing the greatest probability, i.e., $P_d = 1$ or nearest neighboring of 1.
ii) Take the values of all unknown terms s_i at selected J.
iii) Substitute the values of s_i in trial solutions $x_{trial}(t, s)$ and $u_{trial}(t, s)$.
Output:
Global minimum value and approximate solutions of $x_{trial}(t)$ and $u_{trial}(t)$.

4. **Error Bounds and Convergence.** In this sequel, an assessment for the accuracy and boundedness of the solutions of $x(t)$ and $u(t)$ is carried out. Since the Taylor's series expansion (9) is contemplated as the approximate solution of state function $x(t)$, it must satisfy the following error function, i.e., for $t = t_i \in [a, b]$, $i = 0, 1, 2, \ldots$,

$$E_N(t_i) =\mid x^*(t) - x_{trial}^N(t_i) \mid \cong 0$$

where $x^*(t)$ is the exact solution of the governing problem (1)-(3). Correspondingly, if $E_N(t_i) \leq 10^{-\gamma}$ with γ being any positive integer, then the truncation limit N is enlarged until $E_N(t_i)$, at each point becomes diminutive compared with the specified $10^{-\gamma}$.

Convergence of Taylor series

Assume a square integrable function $x(t)$, continuous in interval $[a, T]$. Then the series expansion (4) for $x(t)$, is said to be convergent, if R_N^α in Equation (5) approaches zero as $N \to \infty$, i.e.,

$$x(t) \leq \frac{1}{\Gamma(i\alpha + 1)} WH(T - t_0) \tag{14}$$

where $t_0 \in [a, T]$ and W, H are positive integers.

Proof: Assume $(t - t_0)^{i\alpha}$ and $D_{t_0}^{i\alpha}$ are bounded and continuous in $[a, T]$. Therefore, let $|(t - t_0)^{i\alpha}| \leq H$ and $|{}^C D_{t_0}^{i\alpha}| \leq H$. Thus, using the association of summation and integrals, the absolute value of Equation (4) becomes,

$$|x(t)| = \left| \int_{t_0}^{T} \frac{(t - t_0)^{i\alpha}}{\Gamma(i\alpha + 1)} D_{t_0}^{i\alpha} x(t_0) dt + R_N^\alpha \right| \tag{15}$$

Since

$$R_N^\alpha = \frac{(t - t_0)^{(N+1)\alpha}}{\Gamma((N + 1)\alpha + 1)} D_\xi^{(N+1)\alpha} x(\xi) \leq \frac{1}{\Gamma((N + 1)\alpha + 1)} (WH)^{(N+1)\alpha} \tag{16}$$

as $|1| < |\Gamma((N + 1)\alpha + 1)|$, when $N \to \infty$, we get

$$|x(t)|_{N=\infty} \leq \frac{1}{\Gamma(i\alpha + 1)} WH(T - t_0) \tag{17}$$

5. **Illustrative Experiments.** This section encompasses some numerical experiments of FOCPs for the efficiency and reliability manifestation of TOM. Tabulated and graphical results of each problem are carried out by using *Mathematica 10.*

Experiment 1

Consider the subsequent FOCP

$$MinJ(x, u) = \int_0^1 (tu(t) - (\alpha + 2)x(t))^2 dt \tag{18}$$

subjected to the dynamical system,

$$\dot{x}(t) + D_t^{\alpha} x(t) = u(t) + t^2 \tag{19}$$

with the boundary conditions $x(0) = 0$, $x(1) = \frac{2}{\Gamma(3+\alpha)}$. The exact solution of control and state functions are given as

$$x^*(t) = \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}$$

The FOCP mentioned in Equations (19) and (20) have also been studied in [8], where Chebyshev spectral method along with the Hamiltonian function and Rayleigh-Ritz method is utilized to attain the approximate solutions and minimum value of the objective function. Here, we use TOM to attain the optimum solutions at different α. Following the schematic algorithm, the measured numerical solutions of $x(t)$ and $u(t)$, at different values of α are shown in Figures 1 and 2, respectively. These figures reveal the significant precision of approximate solutions with the exact values of the state and control functions.

Furthermore, Table 1 ascertains the performance index J at different values of N and α. In view of the manifestation given in Section 3.2, for each fractional order and number of generalized Taylor's series expansion, $J \leq 10^{-\epsilon}$ for positive integers $\epsilon = 12, \ldots, 33$,

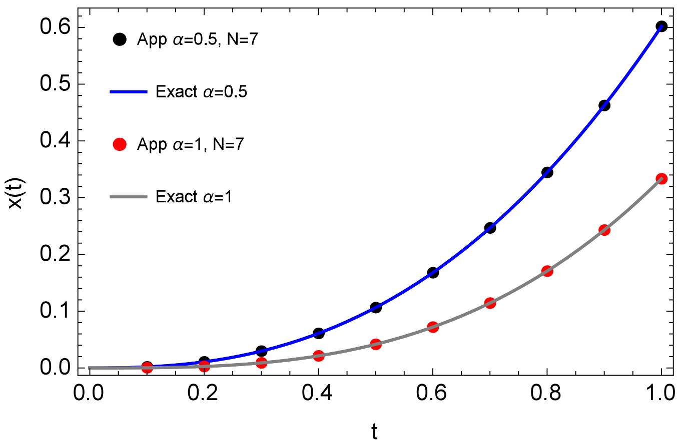

FIGURE 1. Approximate versus exact solutions of state function $x(t)$ for Experiment 1

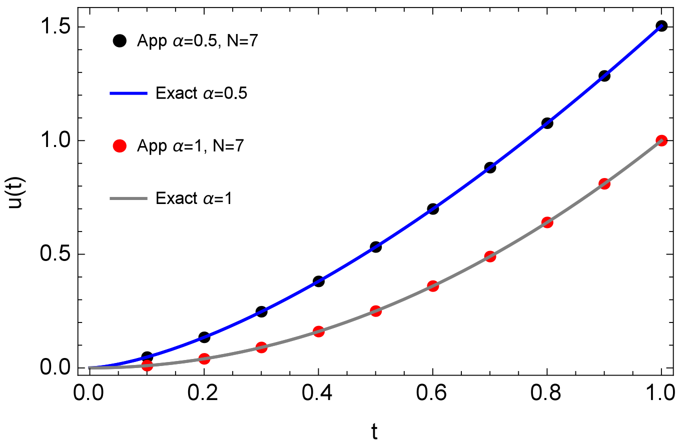

FIGURE 2. Approximate versus exact solutions of control function $u(t)$ for Experiment 1

TABLE 1. Performance index J of Experiment 1 for different values of $t \in [0, 1]$

α	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
0.1	8.92678×10^{-1}	7.87512×10^{-2}	3.24812×10^{-3}	7.10997×10^{-5}	9.82089×10^{-7}
0.3	5.17084×10^{-1}	2.49407×10^{-2}	4.4126×10^{-4}	2.6647×10^{-6}	4.27772×10^{-9}
0.5	2.77182×10^{-1}	6.28842×10^{-3}	2.37611×10^{-5}	1.68549×10^{-26}	6.21342×10^{-32}
0.7	1.34642×10^{-1}	1.01127×10^{-3}	1.58065×10^{-7}	1.08878×10^{-9}	1.84107×10^{-11}
0.9	5.93413×10^{-2}	4.42928×10^{-5}	3.3696×10^{-7}	9.66229×10^{-9}	4.87333×10^{-10}
1.0	3.80518×10^{-2}	6.47112×10^{-32}	4.11672×10^{-29}	4.13026×10^{-32}	1.12099×10^{-33}

TABLE 2. Comparison between TOM and Chebyshev spectral method [8] for Experiment 1 at $\alpha = 1$ and $t \in [0, 1]$

Max Error	TOM		Alg. A of Method [8]		Alg. B of Method [8]	
	$N = 3$	$N = 5$	$N = 3$	$N = 5$	$N = 3$	$N = 5$
$x(t)$	7.87512×10^{-9}	7.10997×10^{-12}	3.4641×10^{-3}	2.6415×10^{-4}	3.4641×10^{-3}	2.6416×10^{-4}
$u(t)$	2.49407×10^{-8}	2.6647×10^{-11}	4.1878×10^{-2}	7.7493×10^{-3}	4.8393×10^{-2}	8.0532×10^{-3}

which signify the worth mentioning accuracy of the proposed approach. Whereas, Table 2 interprets the error analysis comparison between TOM and the method proposed in [8] for control function $u(t)$ and state function $x(t)$. Maximum errors obtained from TOM, at $\alpha = 1$ and different values of N, are less than that of [8], which further elucidates the competency and effectiveness of TOM to solve FOCP.

Experiment 2

Consider another example of FOCP, which is a linear-quadratic problem, outlined as

$$MinJ(x, u) = \int_0^1 (u(t) - x(t))^2 dt \tag{20}$$

subjected to the dynamical system,

$$\dot{x}(t) + D_t^\alpha x(t) = u(t) - x(t) + \frac{6t^{\alpha+2}}{\Gamma(3+\alpha)} + t^3 \tag{21}$$

with the boundary conditions $x(0) = 0$, $x(1) = \frac{6}{\Gamma(4+\alpha)}$. The exact solution of control $x(t)$ and $u(t)$ state functions is given as,

$$x^*(t) = \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \quad u^*(t) = \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}$$

The deliberated experiment is solved using TOM for different values of $N = 3, 4, 5, \dots, 10$ and $\alpha = 0.1, 0.3, \dots, 1$. Graphical solutions of $x(t)$ and $u(t)$, at different values of α, are shown in Figures 3 and 4, respectively. These figures illustrate the correspondence of the calculated solutions with the exact solutions.

Additionally, Table 3 exhibits the performance index J at different values of N and α. Considering that in Section 3.2, for each fractional order and number of generalized Taylor's series expansion, $J \le 10^{-\epsilon}$ for positive integers $\epsilon = 1, 2, \dots, 33$, which signify the fastidiousness of the proposed tactic. On the other hand, Table 4 elucidates the error analysis comparison between TOM and the method proposed in [8] for control function

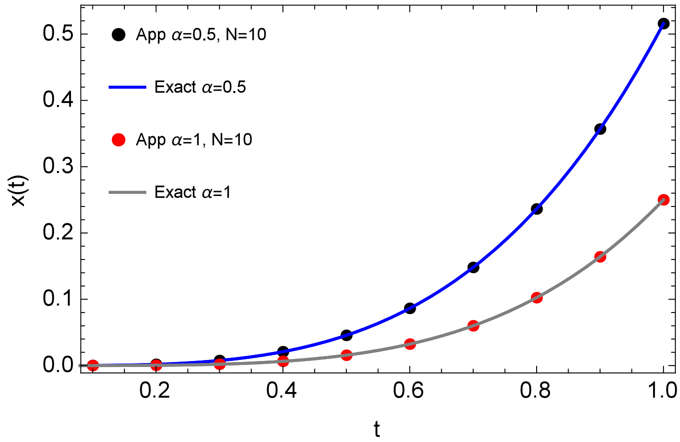

FIGURE 3. Approximate versus exact solutions of state function $x(t)$ for Experiment 2

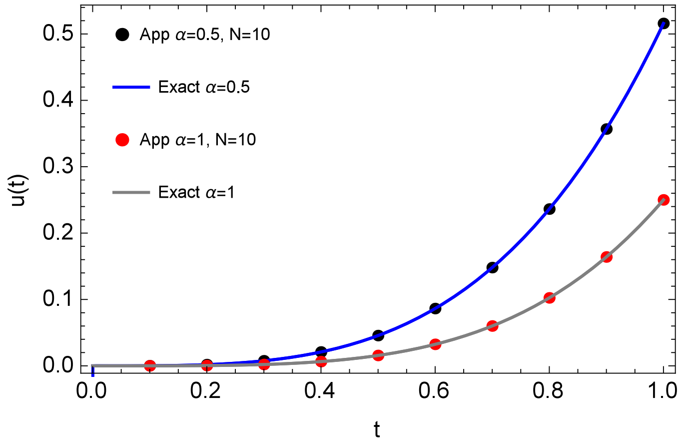

FIGURE 4. Approximate versus exact solutions of control function $u(t)$ for Experiment 2

TABLE 3. Performance index J of Experiment 2 for different values of $t \in [0, 1]$

α	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$
0.1	8.73326×10^{-1}	8.59181×10^{-2}	4.78182×10^{-3}	1.63294×10^{-4}	2.96342×10^{-5}
0.3	4.09059×10^{-1}	2.40337×10^{-2}	6.68911×10^{-4}	8.77275×10^{-6}	5.32087×10^{-8}
0.5	1.73417×10^{-1}	5.02095×10^{-3}	4.76968×10^{-5}	7.3741×10^{-8}	1.21731×10^{-31}
0.7	5.64116×10^{-2}	5.8359×10^{-4}	2.68538×10^{-7}	4.27279×10^{-10}	3.09083×10^{-12}
0.9	1.22628×10^{-2}	1.43764×10^{-5}	8.04005×10^{-8}	1.12374×10^{-9}	3.86691×10^{-11}
1.0	4.93827×10^{-3}	1.2326×10^{-31}	4.80788×10^{-32}	5.0421×10^{-33}	5.63078×10^{-33}

$u(t)$ and state function $x(t)$. Maximum errors obtained from TOM, at $\alpha = 1$ and different values of N, are less than that of [8], which further clarifies the proficiency and practicality of TOM to solve FOCP.

6. **Conclusions.** In this endeavor, we expounded Taylor optimization method for the assessment of fractional optimal dynamical models considered with boundary conditions.

Table 4. Comparison between TOM and Chebyshev spectral method [8] for Experiment 2 at $\alpha = 1$ and $t \in [0, 1]$

Max Error	TOM		Alg. A of Method [8]		Alg. B of Method [8]	
	$N = 3$	$N = 5$	$N = 3$	$N = 5$	$N = 3$	$N = 5$
$x(t)$	7.10997×10^{-4}	5.10997×10^{-12}	7.6404×10^{-3}	7.8604×10^{-5}	1.1943×10^{-2}	1.0304×10^{-4}
$u(t)$	2.6647×10^{-3}	2.6647×10^{-13}	7.6404×10^{-3}	7.8604×10^{-5}	1.6339×10^{-1}	1.0600×10^{-3}

Innovatively, generalized Taylor's series expansion was incorporated with simulated annealing method to approximate and optimize the solutions. Also, a comprehensive discussion on convergence and error analysis had been part of this study. Thus, examining this method on some experiments of FOCPs, we summarize the whole paper with the following constructive outcomes.

- The potential ability of generalized Taylor's series in approximating functions, enables to erect continuous and differentiable analytical approximations of the state and control functions of the dynamic systems.
- Simulated annealing algorithm, depending on the probability, does not only accept the changes which decreases the minimizing objective function, but also accepts the changes that increase the objective function with a probability. Hence, it considers all the possible values of the function.
- The random search ability of SA and using large number of sampling points, increases the convergence of the approximate solutions, without any computational complexity.
- Efficient and easy coding of a simulated annealing algorithm makes the systematic scheme effortless and straightforward that extends its advantageous applicability to solve a wide class of dynamical problems.
- Unlike other Taylor based method, TOM calculates large number of unknown terms more rapidly and conveniently, without being trapped in local minima.
- TOM is deemed to be a very effective and robust method to approximate functions and globally optimize any linear or nonlinear differential and integral equations of fractional as well as integer order.

Additionally, for the reason of being a competent approximating method, in the future, we will further corroborate the applicability of TOM on different ordinary and partial differential problems appearing in several fields of applied sciences.

REFERENCES

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[2] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.
[3] N. A. Khan, O. A. Razzaq, A. Ara and F. Riaz, Numerical solution of system of fractional differential equations in imprecise environment, *Numerical Simulation-From Brain Imaging to Turbulent Flow*, InTech, 2016.
[4] R. T. Hernandez, V. R. Ramirez and R. R. Martinez, A fractional calculus approach to the dynamic optimization of biological reactive systems, Part II: Numerical solution of fractional optimal control problems, *Chem. Eng. Sci.*, vol.117, no.27, pp.239-247, 2014.
[5] E. Tohidi and H. Saberi, A Bessel collocation method for solving fractional optimal control problems, *Appl. Math. Model.*, vol.39, no.2, pp.455-465, 2015.
[6] N. H. Sweilam and T. M. Al-ajami, Legendre spectral-collocation method for solving some types of fractional optimal control problems, *J. Adv. Res.*, vol.6, no.3, pp.393-403, 2015.

[7] T. X. Jun and C. Kai, A Chebyshev-Gauss pseudospectral method for solving optimal control problems, *Acta Auto. Sin.*, vol.41, no.10, pp.1778-1787, 2015.
[8] N. H. Sweilam, T. M. Al-Ajami and R. H. W. Hoppe, Numerical solution of some types of fractional optimal control problems, *Sci. World J.*, Article ID 306237, 2013.
[9] J. Sabouri, K. S. Effati and M. Pakdaman, A neural network approach for solving a class of fractional optimal control problems, *Neur. Proc. Lett.*, vol.45, no.1, pp.59-74, 2016.
[10] L. Chen, F. Hu and W. Zhu, Stochastic dynamics and fractional optimal control of quasi integrable Hamiltonian systems with fractional derivative damping, *Int. J. Theory App.*, vol.16, no.1, pp.189-225, 2013.
[11] S. Ganjefar and S. Rezaei, Modified homotopy perturbation method for optimal control problems using the Padé approximant, *Appl. Math. Model.*, pp.1-20, 2016.
[12] S. A. Rakhshan, A. V. Kamyad and S. Effati, An efficient method to solve a fractional differential equation by using linear programming and its application to an optimal control problem, *J. Vib. Cont.*, vol.22, no.8, pp.2120-2134, 2015.
[13] M. H. Heydari, M. R. Hooshmandasl, F. M. M. Ghaini and C. Cattani, Wavelets method for solving fractional optimal control problems, *Appl. Math. Comput.*, vol.286, pp.139-154, 2016.
[14] S. P. Brooks and B. J. T. Morgan, Optimization using simulated annealing, *J. Royal Stats. Soc.*, vol.44, no.2, pp.241-257, 1995.
[15] X. S. Yang, Chapter 4 – Simulated annealing, in *Nature-Inspired Optimization Algorithms*, Elsevier, Oxford, 2014.
[16] M. Gülsu and M. Sezer, On the solution of the Riccati equation by the Taylor matrix method, *Appl. Math. Comput.*, vol.176, no.2, pp.414-421, 2006.
[17] Y. Öztürk, A. Anapah, M. Gülsu and M. Sezer, A collocation method for solving fractional Riccati differential equation, *J. Appl. Math.*, 2013.
[18] B. Bülbül and M. Sezer, Numerical solution of duffing equation by using an improved Taylor matrix method, *J. Appl. Math.*, 2013.
[19] B. B. Aslan, B. Gurbuz and M. Sezer, A Taylor matrix-collocation method based on residual error for solving Lane-Emden type differential equations, *New Trend. Math. Sci.*, vol.3, no.2, pp.219-224, 2015.
[20] S. Ghosh, A. Deb and G. Sarkar, Taylor series approach for function approximation using estimated higher derivatives, *Appl. Math. Comput.*, vol.284, pp.89-101, 2016.
[21] B. Bülbül and M. Sezer, A Taylor matrix method for the solution of a two-dimensional linear hyperbolic equation, *Appl. Math. Lett.*, vol.24, no.10, pp.1716-1720, 2011.
[22] A. E. Ajou, O. A. Arqub and M. A. Smadi, A general form of the generalized Taylor's formula with some applications, *Appl. Math. Comput.*, vol.256, pp.851-859, 2015.
[23] V. A. Shim, K. C. Tan, C. Y. Cheong and J. Y. Chia, Enhancing the scalability of multi-objective optimization via restricted Boltzmann machine-based estimation of distribution algorithm, *Info. Sci.*, vol.248, no.1, pp.191-213, 2013.