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SUITABLE INVENTORY ASSET MANAGEMENT USING ROUTE-DEPENDENT OPTIONS IN MATHEMATICAL FINANCE

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Abstract. *Small- and medium-sized manufacturers have no financial resources. Therefore, making a profit at the fiscal year-end is a high priority. In order to manage a manufacturing process, we make discussion from a mathematical finance point of view, not from a conventional management engineering point of view. That is, an idea of a production level corresponding to a look-back option level being discussed in finance engineering is introduced. One profit-controlling factor in the manufacturing business is inventory-asset management. This study reports a method that manages inventory assets at the end of the fiscal year. The method is based on the route-dependent options of mathematical finance and is validated in a theoretical verification based on inventory assets over five years (from 2007 to 2011). Suitable inventory asset management is essential to keep a profit. There is no case applying mathematical finance thinking to inventory management.*
Keywords: Route dependent options, Inventory asset management, Fokker-Plank equation, Stochastic partial differential equation of log-normal type, Production process

1. **Introduction.** Many studies of economic process management have assumed an exponential distribution of the failure rate of production processes [1]. The optimum production cycle time and economic production quantity that minimizes the total expected costs of setup and inventory maintenance are determined under the given demand and production rates [2, 3]. In general, inventory management is considered as part of a logistics system. Logistics integrates the storage, transportation, distribution, and processing subsystems into a total system. Therefore, optimization cannot be based on the inventory-management system alone. Although inventory-management expenses are necessary for corporate management, measuring the number of inventories in actual companies is a difficult task. The difficult points of inventory management are to predict a demand forecast. Then, many companies are carrying out stochastic prediction. Also, it is difficult to order at what point so as not to cause loss of opportunity due to excess inventory and out of stock.

On the other hand, there are several reports on evaluation and risk management of production processes utilizing mathematical finance. With respect to financial analysis, a rate of return and volatility at the time of long term investment was researched to compare a rate of return and volatility of short term investment [4, 5]. In this research, Monte Carlo method was utilized in order to simulate a rate of return. Further, there is a report saying that, as a result of investigation of long-term return on investment and its dispersion characteristics, geometric Brownian motion models describing a price of risk assets differ substantially from actual phenomenon [6].

With respect to a stochastic model, a stochastic Euler-Lagrange equation in order to characterize both of dynamics of a total investment amount and impact of various political measures on capital accumulation based on data of a Japanese gas company in the period from 1981 to 1995 has been introduced. In addition, it has proposed a dynamic factor demand model in order to analyze a dynamic cost structure [7].

With respect to stochastic analysis in our previous research, we have reported that production elements in manufacturing processes are treated as stochastic production operation. In particular, in order to analyze a manufacturing process as a stochastic process, we have introduced an idea of a production level corresponding to an energy level being discussed in physics [9]. To achieve the production system goals, we propose the use of a mathematical model that focuses on the selection process and adaptation mechanism of the production lead time [8]. We model the throughput time of the production demand/production system in the production stage by using a stochastic differential equation of the log-normal type, which is derived from its dynamic behavior. By applying Black Scholes equation in mathematical finance using this model and risk-neutral integral, we have defined and computed the evaluation equation for the compatibility condition of the production lead time [21, 22]. Furthermore, we apply the synchronization process and show that the throughput of the production process is reduced [8, 9].

This study introduces the idea of finance into inventory-asset management. The conventional inventory control method is an important research subject in the field of operations research (OR), and there are most stochastic and statistical methods. We present stochastic differential equations and there is no dynamic method utilizing mathematical finance yet. For example, the relation among the stochastic processes of inventory assets is assumed to derive from the fluctuation of demand amount. We model the stochastic process of stock management as a stochastic diffusion process. In other words, stock management is described by a partial differential equation of lognormal type. We evaluate the asset process based on the option evaluation value after the stock asset value has passed its minimum under the demand fluctuation. This approach is equivalent to the route-dependent look-back option in option evaluation theory. According to the route-dependent look-back option, we evaluate the corresponding asset inventory on actual average and variance data collected from 2007 to 2011. Although it is difficult to construct a stochastic model, it is possible to apply option theory from treating inventory as assets. In particular, in small and medium-sized enterprises, we are concentrating on the fact that inventory management is stochastic, that is, not being excessive inventory as much as possible due to capital relationships and avoiding opportunity losses. Considering these items, we report the paper as one approach.

2. **Production Business of a Small-to-Midsize Firm.**

2.1. **Production systems in the production equipment industry.**

The production methods used in equipment are briefly covered in this paper (refer to Figure 1). We refer to the production system in manufacturing equipment industry studied in this paper. This is not a special system but "Make-to-order system with version control". Make-to-order system is a system which allows necessary manufacturing after taking orders from clients, resulting in "volatility" according to its delivery date and lead time. In addition, "volatility" occurs in lead time depending on the contents of make-to-order products (production equipment).

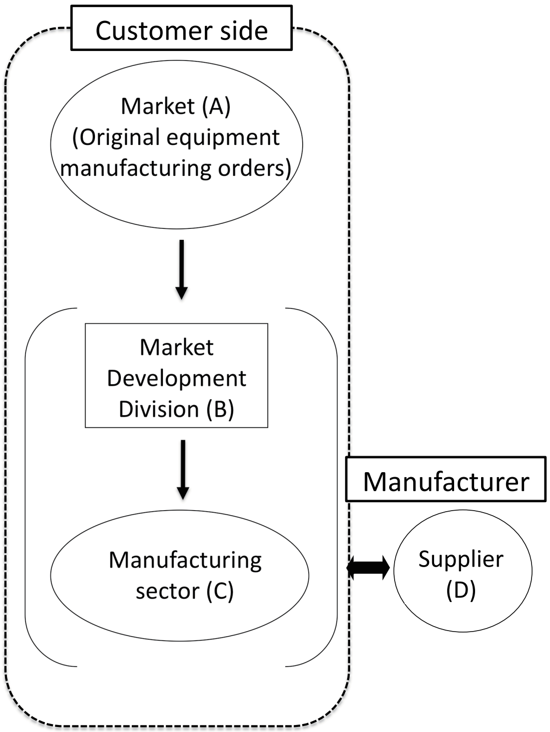

Figure 1. Business structure of company of research target

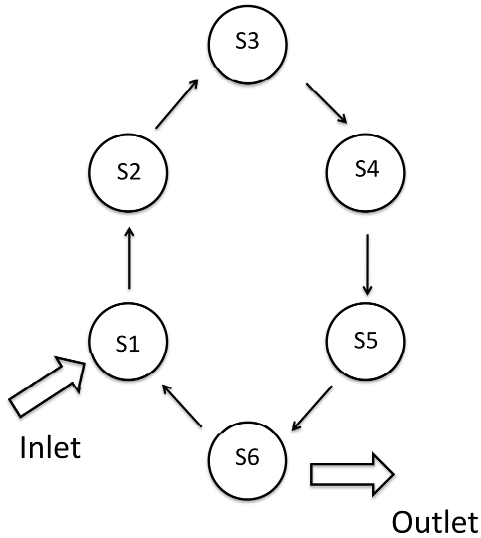

Figure 2. Production flow process

However, effective utilization of the production forecast information on the orders may suppress certain amount of "variation", but the complete suppression of variation will be difficult. In other words, "volatility" in monthly cash flow occurs and of course influences a rate of return in these companies. Production management systems, suitable for the separate make-to-order system which is managed by numbers assigned to each product upon order, is called as "product number management system" and is widely used.

All productions are controlled with numbered products and instructions are given for each numbered product.

Thus, ordering design, logistics and suppliers are conducted for each manufacturer's serial numbers in most cases except for semifinished products (unit incorporated into the final product) and strategic stocks.

Therefore, careful management of the lead time or production date may not suppress "volatility" in manufacturing (production).

2.2. **Production flow process.** A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the production process. In Figure 2, the processes consist of six stages. In each step S1-S6 of the manufacturing process, materials are being produced.

Figure 2 represents a manufacturing process called a flow production system, which is a manufacturing method employed in the production of control equipment. The flow production system, which in this case has six stages, is commercialized by the production of material in steps S1-S6 of the manufacturing process.

The direction of the arrow represents the direction of the production flow. In this system, production materials are supplied from the inlet and the end product will be shipped from the outlet.

2.3. **Nonlinear characteristic of net sales.** Figures 3-5 display graphs in which no significant difference is apparent between cumulative revenues related to production costs and revenues related to production throughput.

Figures 3-5 plot the rate of return on net sales of specific control equipment produced by some domestic enterprises from 1996 through 1998. The rate of return on sales gives rise to the nonlinear characteristics.

The dashed line in the figures is the fitted curve representing the relationship between the rate of return on sales and sales volume fee. In the data, the return rate plummeted from 0.3 at a sales fee of 480 to 0.15 at a sales fee of 440 (see Figure 3). This sharp drop represents the relationship in Equation (7).

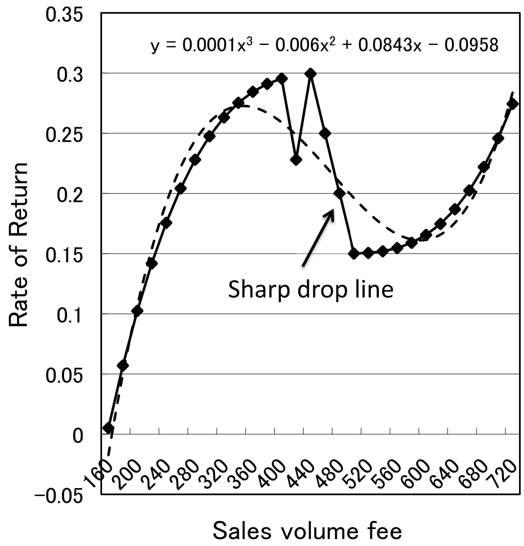

FIGURE 3. Rate of return on sales volume 1

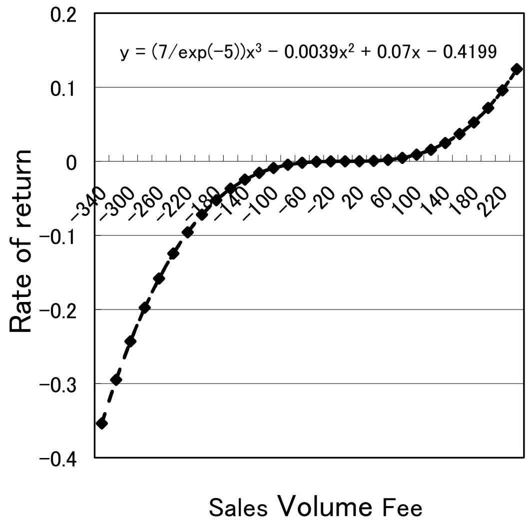

FIGURE 4. Rate of return on sales volume 2

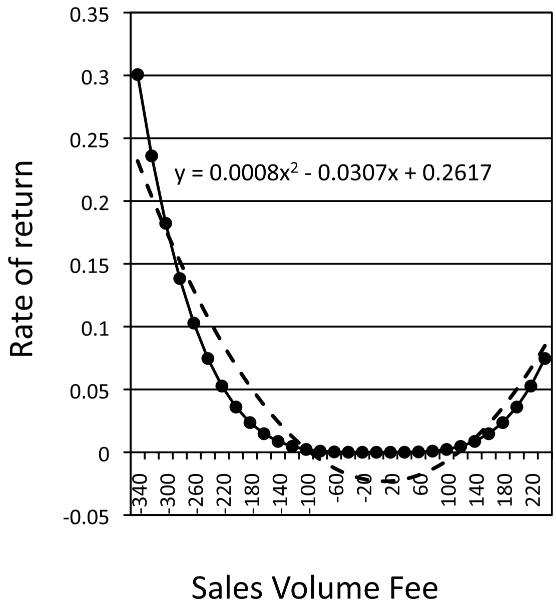

FIGURE 5. Rate of return on sales volume 3

The resulting straight line appears in the vicinity of the phase transition and is equivalent to the oscillation point of the reference line in elements displaying nonlinear characteristics (such as the Esaki diode) [14].

$$h_s(S) = F(S) + \xi(h_{s_0}) \tag{1}$$

where $F(S)$ represents the basic characteristics of the return rate, and $\xi(h_{s_0})$ is a neighborhood of local nonlinearity around h_{s_0}. The following mathematical model is derived from the data plotted in Figures 3-5 [14].

$$a\frac{dh_s}{dt} + bh_s + S = S_E \tag{2}$$

$$h_s = h_{s_1} + h_{s_2} \tag{3}$$

$$h_{s_2} = \tilde{F}(S') \tag{4}$$

$$S' = c\int h_{s_1} dt \tag{5}$$

$$\tilde{F}(S') = F(S) + \xi(h_{S_0}) \tag{6}$$

$$S_E - bh_{s_0} = S' \tag{7}$$

where a, b, and c are cost coefficients, h_s is the rate of return and h_{s_1} is the rate of return contributing to the sales volume. h_{s_2} is a nonlinear characteristic of the rate of return (introduced by costs that cannot contribute directly to sales and that lead to production delays), and (h_{s_0}, S_0) is the median of the nonlinear characteristic.

Physically, Equation (2) represents the temporal variation of the rate of return h_s; that is, the relationship between the deviation of the rate of return and the sales or rate of return. Although sales are essentially proportional to production costs, not all of the production cost can be invested in sales.

Equation (3) is the sum of the rate of return and the nonlinear element. In other words, it embodies the cost of production and nonproduction costs that make no contribution to sales [15].

Assumption 2.1. *The production structure is nonlinear.*

Assumption 2.2. *The production structure is a closed structure; that is, the production is driven by a cyclic system (production flow system).*

Assumption 2.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the throughput generation structure in a stochastic manufacturing process (hereafter called the manufacturing field). Because such a structure is at least dependent on the demand, it is considered to have a nonlinear structure.

Because the value of such a product depends on the throughput, its production structure is nonlinear. Therefore, Assumption 2.1 reflects the realistic production structure and is somewhat valid. Assumption 2.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption 2.2 is reasonable because new production starts from S1.

Based on the control equipment, the product can be manufactured in one cycle. The production throughput required to maintain 6 pieces of equipment/day is as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25 \text{ (min)} \tag{8}$$

where the throughput of the previous process is set as 20 (min). In Equation (8), "28" represents the throughput of the previous process plus the idle time for synchronization.

"8" is the number of processes and the total number of all processes is "8" plus the previous process. "60" is given by 20 (min) $\times$ 3 (cycles).

One process throughput (20 min) in full synchronization is

$$T_s = 3 \times 120 + 40 = 400 \text{ (min)} \tag{9}$$

Therefore, a throughput reduction of about 10% can be achieved. However, the time between processes involves some asynchronous idle time.

As a result, the above test run is as follows. Tables 3-7 are shown in Appendix B.

- (test run1): Each throughput in every process (S1-S6) is asynchronous, and its process throughput is asynchronous. Table 3 represents the manufacturing time (min) in each process. Table 4 represents the variance in each process performed by workers. Table 3 represents the target time, and the theoretical throughput is given by $3 \times 199 + 2 \times 15 = 627$ (min).

  In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. Figure 21 is a graph illustrating the measurement data in Table 3, and it represents the total working time for each worker (K1-K9). The graph in Figure 22 represents the variance data for each working time in Table 3.
- (test run2): Set to synchronously process the throughput.

  The target time in Table 5 is 500 (min), and the theoretical throughput (not including the synchronized idle time) is 400 (min). Table 6 represents the variance data of each working process (S1-S6) for each worker (K1-K9).
- (test run3): The process throughput is performed synchronously with the reclassification of the process. The theoretical throughput (not including the synchronized idle time) is 400 (min) in Table 7.

  Table 8 represents the variance data of Table 7. "WS" in the measurement tables represents the standard working time. This is an empirical value obtained from long-term experiments.

3. **Distribution System and Diffusion Equation of the Production Process.** Figure 6 schematizes the network interprocess division. The network capacity (i.e., the statically acceptable amount of production) in an interprocess network (a production field) is denoted as R. Once the current process is complete, the interprocess network indicates the sequential flow to the next process. The production density function $S_i(x,t)$ for the i-th equipment is given by

$$[J(x,t)dt - J(x+dx,t)dt]R = [S_i(x,t+dt) - S_i(x,t)]Rdx \tag{10}$$

where J is the production flow, t is the time variable and x is the spatial variable [10, 11].

$$\frac{\partial S_i(x,t)}{\partial t} = D\frac{\partial^2 S_i(x,t)}{\partial x^2}. \tag{11}$$

where D is the diffusion coefficient.

This equation is equivalent to the diffusion equation derived from the minimization condition of free energy in a production field. Therefore, the connections between processes can be treated as a diffusive propagation of products (see Figure 6) [10, 11].

The production process, which is connected in one dimension, is modeled as follows. During the production process, the production units are moved from one process (node) to another. This production flow is equivalent to the transmission rate in communications engineering, defined as the rate of data flow between connected nodes.

Accordingly, the production model is treated similarly to heat propagation in physics. Mathematically, the production process is modeled by a continuous-diffusion type of partial differential equation comprising temporal and spatial variables [10].

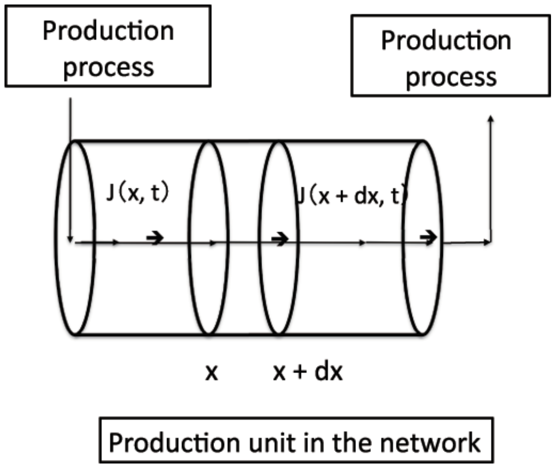

FIGURE 6. Network inter-process division of worker

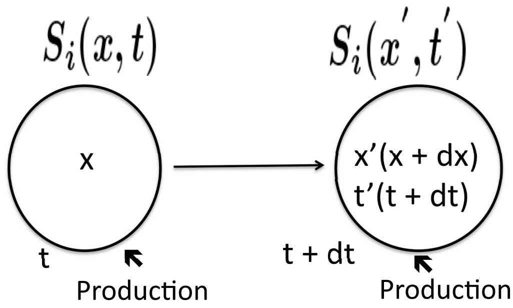

FIGURE 7. Unit of production by changing the excitation force

Setting the network capacity (available static production volume) to R in an interprocess network (production field, equivalent to a stochastic field), we obtain the following:

$$[J(x)dt - J(X + dx)dt]R = [S(t + dt) - S(t)]Rdx \tag{12}$$

where J is the production flow and S is the production density.

In the present model, the production flow indicates the displacement of production processes in the direction related to the production density. In other words, the production cost per production is as follows.

Definition 3.1. *Production cost per unit production*

$$J = -D\frac{\partial S}{\partial x} \tag{13}$$

where D is a diffusion coefficient.

From Equation (12), we obtain

$$-\frac{\partial J}{\partial x} = \frac{\partial S}{\partial t} \tag{14}$$

From Equations (13) and (14), we obtain

$$\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} \tag{15}$$

where $t \in [0, T]$, $x \in [0, L] \equiv \Omega$, $S(0, x) = S_0(x)$, $B_x S(t, x)|_{x=\partial\Omega}$.

This equation is equivalent to the diffusion equation derived from the minimization condition of free energy in a production field [10]. The connections between processes can be treated as a diffusive propagation of products (refer to Figure 6).

As shown in Figure 7, X represents the production elements that constitute a unit production and varies $X \to X'$ at $[t + dt]$. In other words, the unit production varies by exciting the external force and is the basis for revenue generation (an increase of potential energy). Therefore, in the transition $S_i(t, x) \to S_i(t, x')$, the production cost, which is the cumulated external force, increases. The connections between production processes are referred to as "joints".

In the general idea of production flow, we define the joint propagation model at multiple stages in the production process and the potential energy in the production field.

Thereafter, we can construct a control system, which increases the process throughput, by calculating the gradient function in the autonomous distributed system. The gradient function is described in the next opportunity.

$$\frac{\partial S}{\partial t} + \Delta(v \cdot S) = \frac{1}{2}\Delta(D^2 S) + \lambda \tag{16}$$

where λ denotes a forced external force function and v denotes a production propagation speed. Here, λ is omitted here.

We assume that S defines as follows: S represents a production density with a fluctuation, and v also causes a fluctuation in throughput. As a result, a production is proportional to the slope of production density.

Definition 3.2. *Mathematical model of each stage*

$$dx(t) = \left\{a(t,x)dt + c(t,x)d\tilde{B}(t)\right\} + D(t,x)dB(t) \tag{17}$$

where $\tilde{B}$ and B denote an independent Brownian motion. c denotes a fluctuation term, which follows a stochastic differential equation.

The first term on the right-hand side of Equation (17) denotes the flow of the medium, and the second term represents the fluctuation of diffusion. Moreover, $a(t,\cdot)$ denotes an average lead-time and $c(t,x)d\tilde{B}(t)$ denotes a fluctuation around processes [12, 13].

We report a stochastic approach for a production process based on the production density equation [10], i.e., a fluctuation is induced by a stochastic characteristic of a lead-time function. In this case, we apply stochastic analysis to evaluating the manner in which the production density is constrained.

Generally, Equation (16) with constraint such as Equation (17) can be derived as follows:

$$\begin{aligned} \partial S(t,x) = &\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x) + c^2(t,x)\right\}S(t,x) - \frac{\partial}{\partial x}\Big(a(t,x)S(t,x)\Big)\right]dt \\ &+ \frac{\partial}{\partial x}\Big\{c(t,x)S(t,x)\Big\}\partial\tilde{B}_t \end{aligned} \tag{18}$$

where $S(t,x)$ denotes a production density and is derived as follows [10]:

$$S(t, I_h^x) = \int_0^t P(\tau, x_0; t, I_h^x) S(\tau, x_0) d\tau \tag{19}$$

where $I_h^x \equiv [x, x+h]$.

From Equation (19), a production density distribution varies according to increasing a production density.

$S(t,x)$ satisfies a Fokker-Plank equation as follows [17, 18, 19, 20].

$$\frac{\partial S(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x)S(t,x)\right\} - \frac{\partial}{\partial x}\{a(t,x)S(t,x)\} \tag{20}$$

where $x(t)$ satisfies Equation (17).

According to Okazaki's analysis, we obtain as follows [16]:

$$\begin{aligned} \partial S(t,x) = &\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x) + c^2(t,x)\right\}S(t,x) - \frac{\partial}{\partial x}\big(a(t,x)S(t,x)\big)\right]dt \\ &+ \frac{\partial}{\partial x}\Big\{c(t,x)S(t,x)\Big\}\partial\tilde{B}_t \end{aligned} \tag{21}$$

where $D^2(t,x) + c^2(t,x)$ denotes a trend, $a(t,x)S(t,x)$ denotes a fluctuation of stages and $c(t,x)S(t,x)$ denotes also a fluctuation of lead-time.

Definition 3.3. *Trend of a production density distribution*

$$m(t,x) = E[S(t,x)] \tag{22}$$

According to Equation (16), $m(t,x)$ is derived as follows:

$$\frac{\partial}{\partial t}m(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[\left\{D^2(t,x)+c^2(t,x)\right\}m(t,x)\Big] - \{a(t,x)m(t,x)\} \tag{23}$$

where the dispersion covariance of a production density $\chi(t,x,x')$ is defined as follows.

Definition 3.4. *Dispersion covariance of a production density* $\chi(t,x,x')$

$$\chi[t,x,x'] = E\left[S(t,x)\cdot S(t,x')\right],\ t\in R,\ x'\in R \tag{24}$$

where R denotes Euclidean space.

From Equation (22), we obtain as follows:

$$Cov.\left[S(t,x)\cdot S(t,x')\right] = \chi(t,x,x') - m(t,x)\cdot m(t,x') \tag{25}$$

According to a stochastic process theory, the following equation holds.

$$\begin{aligned} d\left\{S(t,x)\cdot S(t,x')\right\} &= S(t,x)\cdot dS(t,x') + S(t,x')\cdot dS(t,x) \\ &\quad + \frac{1}{2}\cdot 2\cdot d < S(\bullet,x), S(\bullet,x') >_t \end{aligned} \tag{26}$$

$$\begin{aligned} \chi[t,x,x'] &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left\{D^2(t,x)+c^2(t,x)\right\}S(t,x') - \frac{\partial}{\partial x'}a(t,x')S(t,x')\right]dt \\ &+ S(t,x')\frac{1}{2}\frac{\partial^2}{\partial x'^2}\Big[\{D^2(t,x)+c^2(t,x)\}S(t,x) - \frac{\partial}{\partial x}a(t,x)S(t,x)\Big]dt \\ &+ \frac{\partial}{\partial x}\{c(t,x)S(t,x)\}\frac{\partial}{\partial x'}\left\{c(t,x')S(t,x')\right\} + S(t,x)\frac{\partial}{\partial x'}\left\{c(t,x')S(t,x')\right\}d\tilde{B}_t \\ &+ S(t,x')\frac{\partial}{\partial x}\{c(t,x)S(t,x)\}d\tilde{B}_t \end{aligned} \tag{27}$$

Then, we obtain the dispersion covariance of a production density between stages as follows by taking the average value.

$$\begin{aligned} \frac{\partial}{\partial t}\chi[t,x,x'] &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{D^2(t,x)+c^2(t,x)\right\}\chi(t,x,x') \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x'^2}\left\{D^2(t,x')+c^2(t,x')\right\}\chi(t,x,x') \\ &- \frac{\partial}{\partial x}a(t,x)\chi(t,x,x') - \frac{\partial}{\partial x'}\left\{a(t,x')\chi(t,x,x')\right\} \end{aligned} \tag{28}$$

where $a(t,x) > 0$, $b(t,x) > 0$ and $c(t,x) > 0$.

Definition 3.5. *Correlation function of lead-time function between stages*

$$dx^{i+1}(t) = \left\{a\left(t,x^{i+1}\right)dt + \int_R c\left(t,x^i,x^{i+1}\right)d\tilde{B}\left(dt,dx^{i+1}\right)\right\} + b\left(t,x^{i+1}(t)\right)dB_t^i \tag{29}$$

The production density distribution satisfies as follows based on Equation (29):

$$\begin{aligned} dS(t,x) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{b^2(t,x) + \int_R c^2(t,x,z)dzS(t,x)\right\} - \frac{\partial}{\partial x}\{a(t,x)S(t,x)\} \\ &+ \int_R \frac{\partial}{\partial x}\left\{c(t,x,x')S(t,x)\right\}\tilde{B}(dt,dx') \end{aligned} \tag{30}$$

4. Preparation for Numerical Calculation.

4.1. **Trend function of production density distribution.** We present an example for numerical parameters such as follows: $a > 0$, $b > 0$ and $c > 0$ are constant parameters. Let $S(0, x) = \delta(x)$, which denotes as follows:

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(- \frac{x^2}{2\sigma^2} \right), \quad x \in R \tag{31}$$

Under the parameter conditions $a > 0$, $b > 0$, and $c > 0$, a production density exists between any two stages.

Then, according to Equation (23), we obtain as follows:

$$\frac{\partial}{\partial t} m(t, x) = \frac{1}{2}\left(r\frac{\partial^2}{\partial x^2} \right) - am(t, x) \tag{32}$$

According to Equation (23), we obtain as follows:

$$\begin{aligned} \frac{\partial \chi(t, x, x^{'})}{\partial t} &= \frac{1}{2} r \left\{ \frac{\partial^2 \chi(t, x, x^{'})}{\partial x^2} + \frac{\partial^2 \chi(t, x, x^{'})}{\partial x'^2} \right\} \\ &\quad - a \left\{ \frac{\partial \chi(t, x, x^{'})}{\partial x} + \frac{\partial \chi(t, x, x^{'})}{\partial x^{'}} \right\} \end{aligned} \tag{33}$$

From Equation (32), we obtain as follows:

$$m(t, x) = \frac{1}{\sqrt{2\pi} rt} \exp\left(-\frac{(x - at)^2}{2r^2t^2} \right) \tag{34}$$

Similarly, according to Equation (33), we obtain as follows:

$$\begin{aligned} \chi(t, x, x^{'}) = \frac{1}{2\pi(r^2 - c^4)t} \exp\Bigg(&- \frac{1}{2(r^2 - c^4)t} \\ &\times \left\{ r(x - at)^2 - 2c^2(x - at)(x^{'} - at) + r(x^{'} - at)^2 \right\} \Bigg) \end{aligned} \tag{35}$$

where $r = D^2 + c^2$.

From Equation (35), the numerical data of correlation function can be calculated for x and $x^{'}$ of production density.

$$\begin{aligned} dS(t, x) &= \frac{1}{2}\left[\left\{ D^2 + \int_R c^2(t, x, x^{'}) \right\} \frac{\partial^2 S(t, x)}{\partial x^2} - a\frac{\partial S(t, x)}{\partial x} \right] dt \\ &\quad + \frac{\partial}{\partial x}\left[\int_R c(t, x, x^{'}) S(t, x) \tilde{B}(dt, dx^{'}) \right] \end{aligned} \tag{36}$$

where $\tilde{B}(dt, dx^{'})$ denotes any of the k interval $F_1 = I_1 \times J_1$, $F_2 = I_2 \times J_2, \cdots$, $F_k = I_k \times J_k \subset R^2$. $(B(F_1), B(F_2), \cdots, B(F_k))^{'}$ in $\tilde{B}(dt, dx^{'})$ denotes a k-dimensional normal distribution with average zero. However, from Equation (18) in case of a single Brownian motion, we obtain as follows:

$$\partial S(t, x) = \frac{1}{2}\left[\left(D^2 + c^2 \right) \frac{\partial^2 S(t, x)}{\partial x^2} \right] \partial t - a\frac{\partial}{\partial x} S(t, x)\partial t + c\frac{\partial}{\partial x} S(t, x)\tilde{B}(t) \tag{37}$$

The aforementioned calculation clarifies that the production density distribution fluctuation follows a normal distribution (see Equation (34)). In the case of single Brownian motion, this trend denotes a stochastic diffusion partial differential equation (Equation (37)). In other words, the motion of the trend is affected by the coefficient c, which is generated by lead-time fluctuations.

With respect to the lead-time distribution, we derive the following expression from Equation (17):

$$dx(t) = \left\{adt + cd\tilde{B}(t)\right\} + DdB(t) \tag{38}$$

In the derivation of Equation (39), we derive the stochastic model of a production density distribution is as Equation (37).

Letting $\tilde{B}(t) \approx B(t)$, Equation (39) simplifies to

$$dx(t) = adt + (c + D)dB(t) \tag{39}$$

4.2. **Rate of return.**

Log-normal distribution characteristics of rate of return. For a small-to-midsize firm, it is of the upmost importance not to cause default in a cash flow, and it is necessary for business continuity. We also analyzed a return acquisition rate defined by Equation (40). The result is shown in Figure 8.

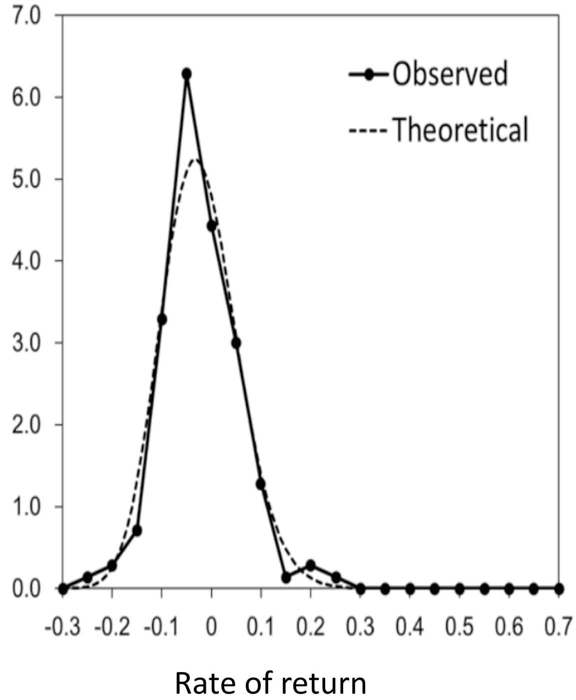

FIGURE 8. Probability density function of rate-of-return deviation: actual data (solid line) and data based on theoretical equation (dotted line)

From the data of monthly rate of return observed, its probability density function was calculated (Figure 8). As a result, it was found that the probability density function conforms to log-normal distribution (Figure 8, Theoretical).

Theoretical curve was calculated using EasyFit software (http://www.mathwave.com/), and as a result of Kolmogorov and Smirnov test, the observed values conformed to a log-normal type probability density function. Because, in the goodness-of-fit test of Kolmogorov-Smirnov, a null hypothesis that it is "log-normal" was not rejected with rejection rate 0.2, and this data conforms to "log-normal" distribution. P-value was 0.588. The parameters of a theoretical curve were: $\mu_p = -0.134$ (average), $\sigma_p = 0.0873$ (standard deviation), $\gamma_p = -0.900$. The theoretical curve is given by the following equation.

$$f(x) = \frac{1}{\sqrt{2\pi}(x - \gamma_p)\sigma_p} \exp\left\{-\frac{1}{2}\left(\frac{(\ln x - \gamma_p) - \mu}{\sigma_p}\right)^2\right\} \tag{40}$$

Definition 4.1. *Rate of return deviation* $Q(t)$

$$\frac{dQ(t)}{Q(t)} = \mu(t)dt + \sigma(t)dZ(t) \tag{41}$$

The acquisition rate model is described as follows [22]:

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dZ(t) \tag{42}$$

where $S(t)$ is assumed that $(Q(s), \sigma(s); s \le t)$ is measurable variables.

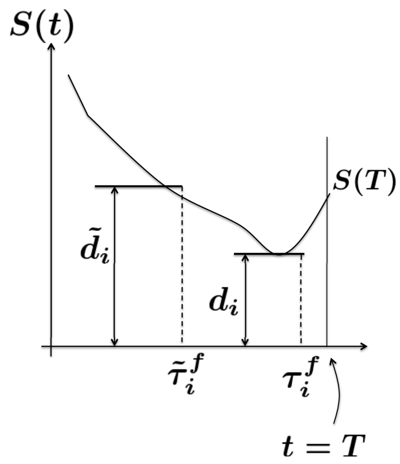

Figure 9. Retention of production processes

4.3. **Process retention recovery analysis.** Figure 9 shows the process retention situation. According to this figure, the acquisition rate process, which starts from the initial value $S(0) = S_0$ of the acquisition rate $S(t)$, reaches the setting value d_i at time τ_i^f. We obtain as follows:

$$d_i = \min_{0 \le \tau \le t} S(\tau) \tag{43}$$

Here we have

$$\begin{aligned} \tau_i^s &= \min_{\tau_i \in [0,T]} [S(\tau_i) - d_i \mid S(0) = S_0] \\ &= \max_{\tau_i \in [0,T]} \left[\tilde{d}_i - S(\tau_i) \mid S(0) = S_0\right] \end{aligned} \tag{44}$$

where $\tilde{d}_i$ and d_i are called the first and second lower limits, respectively.

In this algorithm, the process is transited to the congested state under the influence of the above-described input burst from the no-retention state [25] and arrives first at $\tilde{d}$.

Thus, the retention state is assumed to depend on the previous path of $S(t)$.

We define the arrival times of the first and second lower limits as $\tilde{\tau}_i^s$ and τ_i^s, respectively.

Definition 4.2. *Second lower limit arrival time* $\tilde{\tau}_i^s$

$$\begin{aligned} 0 \le \tilde{\tau}_i^f \le t, \\ t < \tau_i^s \le T \end{aligned} \tag{45}$$

Definition 4.3. $\tilde{\tau}_i^s$ *and* τ_i^s

$$\tilde{\tau}_i^s = \max_{0 \le \tilde{\tau}_i \le t} \left[\tilde{d}_i - S(\tilde{\tau}_i) \mid f(0) = S_0\right] \tag{46}$$

$$\tau_i^s = \max_{t<\tau_i\leq T} [d_i - S(\tau_i) \mid S(0) = S_0] \tag{47}$$

At this time, $\tilde{d}_i$ and d_i are evaluated as follows:

$$\tilde{d}_i = \min_{0\leq\tilde{\tau}_i^s\leq t} S\left(\tilde{\tau}_i^s\right) \tag{48}$$

$$d_i = \min_{t<\tau_i^s\leq T} S\left(\tau_i^s\right) \tag{49}$$

The expected gain under the look back option in Finance is as follows [25]:

$$S(T) - \min\left\{\tilde{d}_i, d_i\right\} \tag{50}$$

The look-back option in the production processes corresponds to the evaluation of the stock volume at the final time, when the option was exercised at the time of minimum stock.

Therefore, to derive the call option price $\Theta\left[S(t), \tilde{d}_i\right]$ at time t, we apply option pricing theory and hence evaluate the stock volume by the risk-neutral method [25].

Definition 4.4. *Call option price* $\Theta\left(f, \tilde{d}_i\right)$

$$\begin{aligned} &\Theta\left(f, \tilde{d}_i\right) \\ &= e^{-r\tau} E\left[\tilde{f}(T) - \min\left\{\tilde{d}_i, d_i\right\} \mid f(t) = S_0\right] \\ &= e^{-r\tau} E\left[\tilde{f}(T) - \tilde{d}_i \mid d_i > \tilde{d}_i\right] P\left\{d_i > \tilde{d}_i\right\} + E\left[\tilde{f}(T) - d_i \mid d_i \leq \tilde{d}_i\right] P\left\{d_i \leq \tilde{d}_i\right\} \\ &= e^{-r\tau}\left(E\left[\tilde{f}(T) \mid f(t) = S_0\right] - \tilde{d}_i \cdot P\left\{d_i > \tilde{d}_i\right\} - E\left[d_i \mid d_i \leq \tilde{d}_i\right] P\left\{d_i \leq \tilde{d}_i\right\}\right) \end{aligned} \tag{51}$$

where $E[\cdot]$ *denotes the expected value when the average is* $\left(r - \frac{\sigma_s^2}{2}\right)\tau$ *and the volatility is* $\sigma_s^2\tau$. r *denotes the rate of risk neutral.*

Definition 4.5. *Distribution function* $F\left(\tilde{d}_i\right)$ *of* d_i *in log-normal process*

$$\begin{aligned} F\left(\tilde{d}_i\right) = P\left(\tilde{d}_i \leq d_i\right) &= \Phi\left[\frac{\ln\left(\tilde{d}_i/S_0\right) - (r - \sigma_s^2/2)\,T}{\sigma_s\sqrt{T}}\right] \\ &+ \left(\frac{S_0}{\tilde{d}_i}\right)^{(-2r/\sigma_s^2)} \times \Phi\left[\frac{\ln\left(\tilde{d}_i/S_0\right) + (r - \sigma_s^2/2)\,T}{\sigma_s\sqrt{T}}\right] \end{aligned} \tag{52}$$

where $\Phi(\cdot)$ *denotes the normal distribution with the average zero and volatility* σ_s [25].

Thus, we obtain the following equation by omitting the calculation.

$$\begin{aligned} \Theta\left[S, \tilde{d}_i\right] &= e^{-r\tau} E\left[S(\tau) - \min\left\{d_i, \tilde{d}_i\right\} \mid S_i(t) = S_0\right] \\ &= S_0 - \tilde{d}_i e^{-r\tau}\{\Phi(d_1) - V\Phi(d_2)\} - S_0\left(1 + \frac{\sigma_s^2}{2r}\right)\Phi(d_3) \end{aligned} \tag{53}$$

Please refer to the Appendix A for detailed equation derivation process [25].

$$d_1 = \frac{\ln\left(\tilde{d}_i/S_0\right) + \left(r - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}} \tag{54}$$

$$d_2 = \frac{-\ln\left(\tilde{d}_i/S_0\right) + \left(r - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}} \tag{55}$$

$$d_3 = \frac{-\ln\left(\tilde{d}_i/S_0\right) - \left(r + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}} \tag{56}$$

$$V = \frac{\sigma_s^2}{2r}\left(\frac{S_0}{\tilde{d}_i}\right)^{1-\frac{2r}{\sigma_s^2}} \tag{57}$$

This evaluation value is expressed as the difference between the minimum value $\left\{d_i, \tilde{d}_i\right\}$ during the period and the value $S(\tau)$ at the end of the period.

5. **Numerical Results.** Table 1 shows the gross profit margins from 2007 to 2011 derived from the actual average μ_s and variance σ_s data. μ_s and σ_s represent the normalized average value of product inventory and volatility for one year in each fiscal year respectively.

The gross profit margin was the highest in 2011. Table 2 gives the gross profit margin against the average value. Consistent with Table 1, the highest gross margin was achieved in 2011. That is, as $\Theta\left(S, \tilde{d}_i\right)$ increases, and R_s is pushed down as an excess inventory asset. Incidentally, the gross profit margin of the Lehman stock was the lowest in 2009. In other words, when the inventory assets decrease, they can be managed at the end of the period by calculating the look-back option assets. However, the correlation between the determination of an appropriate look-back option and the gross profit margin must also be calculated.

Table 1. Gain under the look-back option

	μ_s	σ_s	$\Theta\left(S, \tilde{d}_i\right)$	Gross profit margin
2007	0.509	0.244	0.5282	0.405
2008	0.667	0.166	0.6098	0.378
2009	0.715	0.119	0.6349	0.360
2010	0.454	0.141	0.513	0.419
2011	0.485	0.172	0.5261	0.42
Average	0.566	0.168	0.5665	0.396

Table 2. Gross margin on average

	Gross margin on average
2007	1.022
2008	0.954
2009	0.909
2010	1.058
2011	1.060

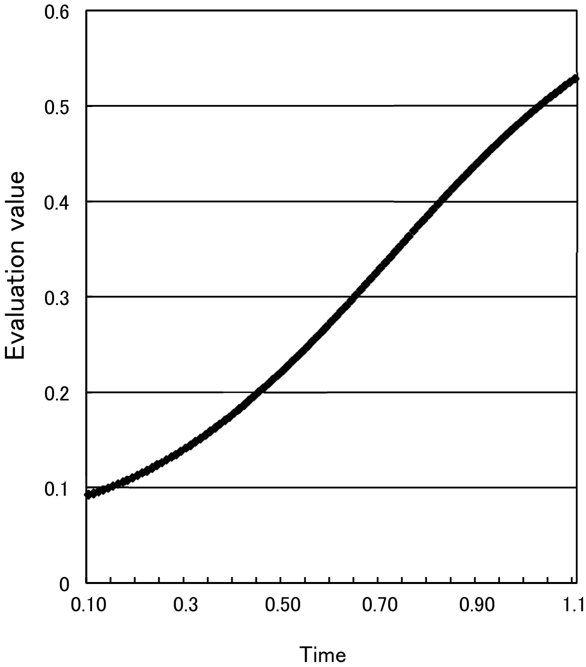

FIGURE 10. System evaluation for stochastic inventory process (2007)

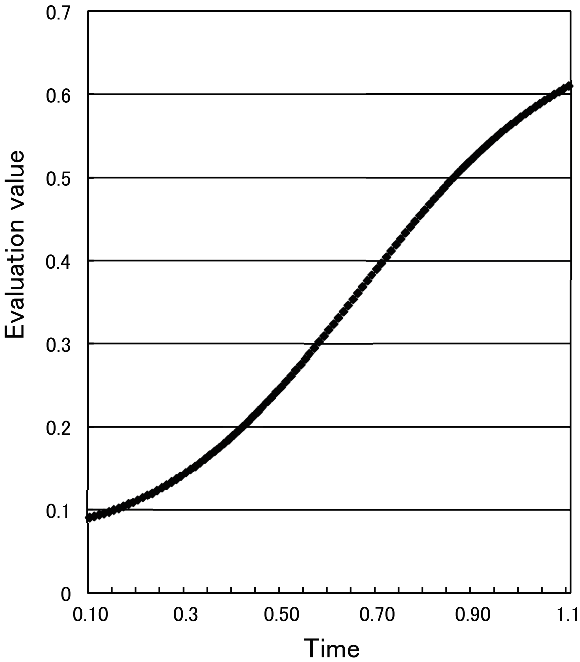

FIGURE 11. System evaluation for stochastic inventory process (2008)

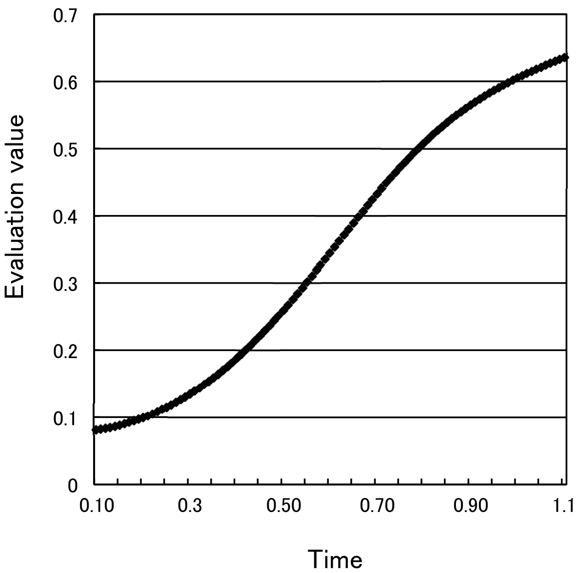

FIGURE 12. System evaluation for stochastic inventory process (2009)

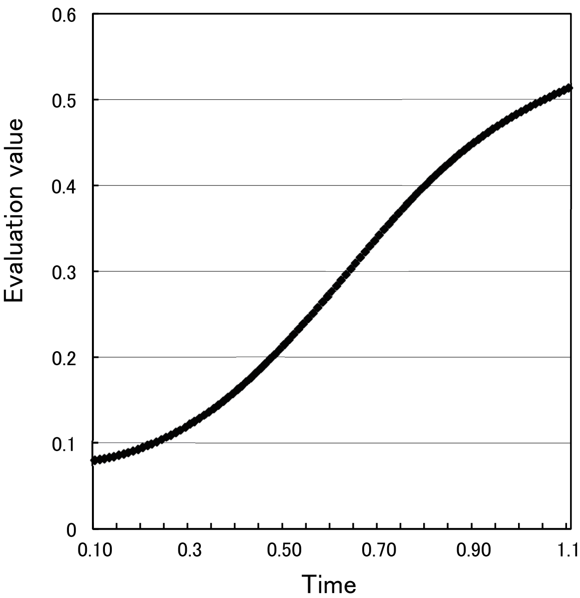

FIGURE 13. System evaluation for stochastic inventory process (2010)

Figures 10 through 14 show the system evaluation of the stochastic inventory process from 2007 to 2011, obtained by calculating Equation (53) under the parameter settings of Table 1. Figure 15 shows the 5-yearly average (2007 through 2011) of this evaluation.

Figures 16 to 20 are actual examples of some parts inventory for manufacturing control equipment, which is the actual data from 2007 to 2011.

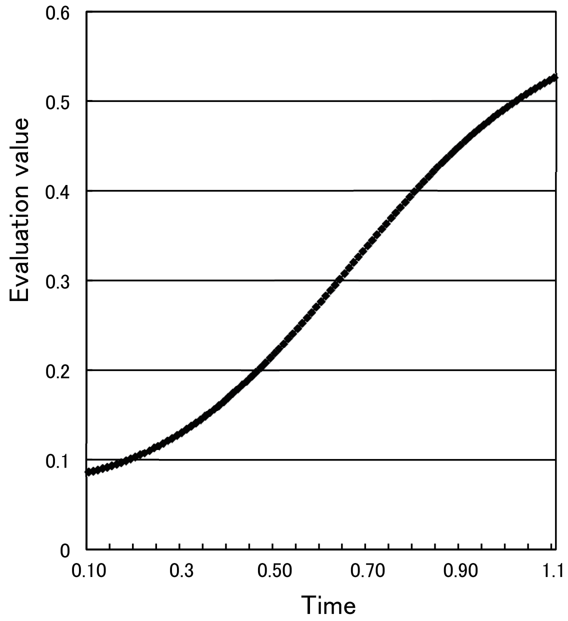

Figure 14. System evaluation for stochastic inventory process (2011)

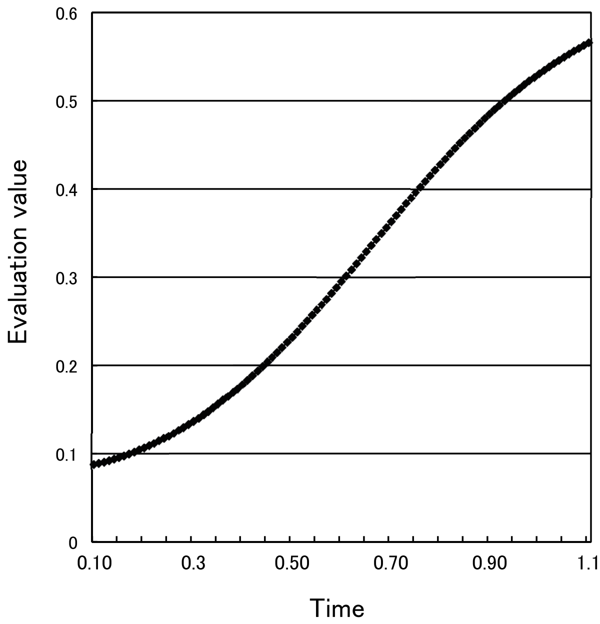

Figure 15. System evaluation for stochastic inventory process (average)

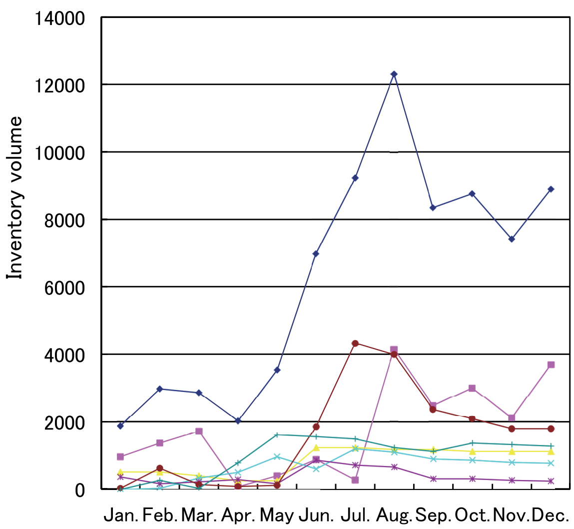

Figure 16. Inventory volume (2007)

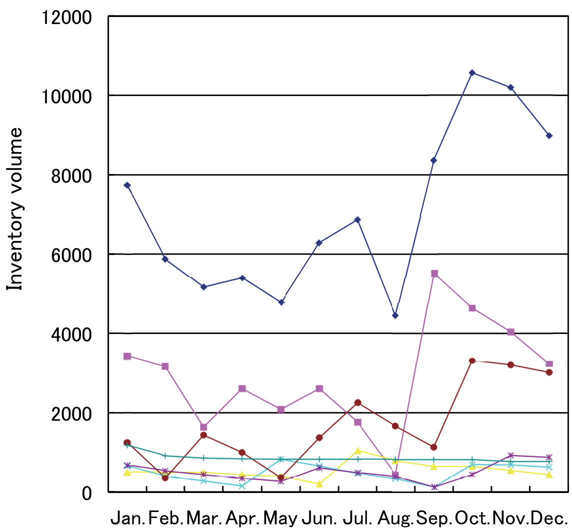

Figure 17. Inventory volume (2008)

6. **Conclusions.** Management of inventory assets at the end of the fiscal year is crucial for small- and medium-sized manufacturers. Such management can be achieved by utilizing the path-dependent option evaluation theory of mathematical finance. In a later study, we will report on the relation between the path-dependent option and phase transition.

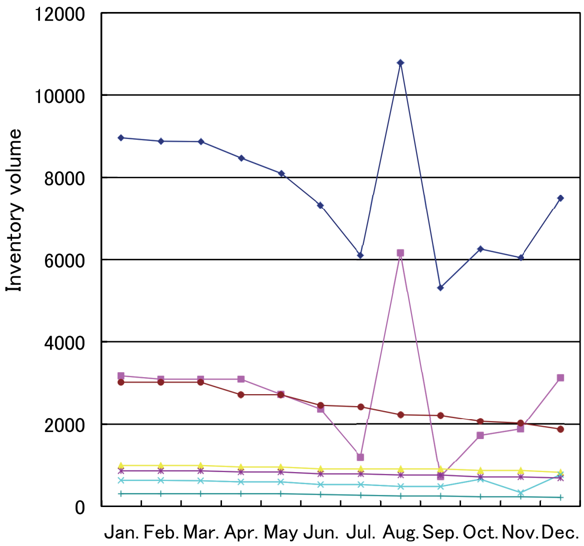

FIGURE 18. Inventory volume (2009)

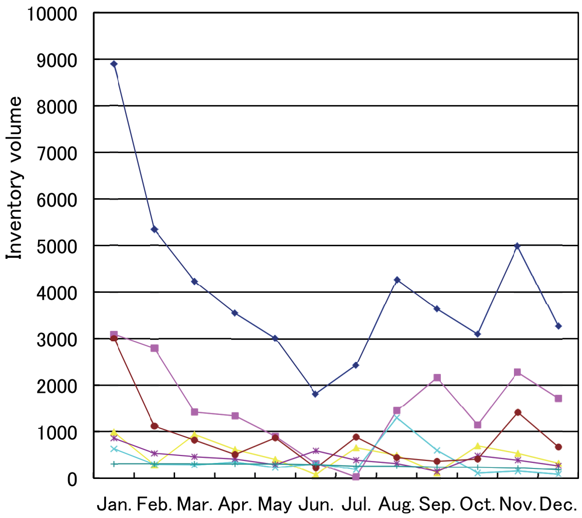

FIGURE 19. Inventory volume (2010)

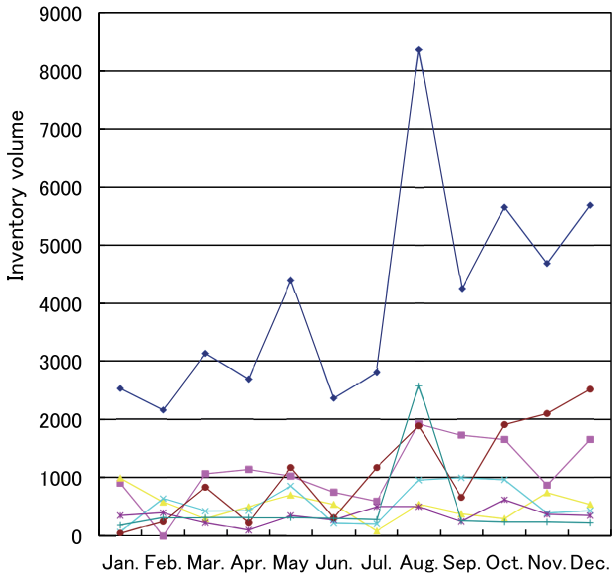

FIGURE 20. Inventory volume (2011)

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Appendix A. **Calculation According to the Refer.** We calculate Equation (51) by using the third term of right hand in Equation (52) as follows [23, 24, 25]:

$$E\left[\tilde{d}_i \leq d_i\right] \cdot P\left(\tilde{d}_i \leq d_i\right)$$

$$= \int_0^{\tilde{d}_i} lF(l)$$

$$
\begin{aligned}
&= \tilde{d}_i F\left(\tilde{d}_i\right) - \int_0^{\tilde{d}_i} \Phi\left(\frac{\ln(l/S) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) dl \\
&\quad - \int_0^{\tilde{d}_i} \left(\frac{S}{l}\right)^{(1-2\mu_s)/\sigma_s^2} \Phi\left(\frac{\ln(l/S) + \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) dl \\
&= \tilde{d}_i \left\{\Phi\left(\frac{\ln(l/S) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) + \left(\frac{S}{\tilde{d}_i}\right)^{-\left(2\mu_s - \sigma_s^2\right)/\sigma_s^2}\right. \\
&\quad \left.\times \Phi\left(\frac{-\ln(l/S) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right\} - \tilde{d}_i \Phi\left(\frac{-\ln(l/S) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) \\
&\quad + Se^{\mu_s\tau}\Phi\left(\frac{-\ln(l/S) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) - S\left(\frac{S}{\tilde{d}_i}\right)^{-2\mu_s/\sigma_s^2} \cdot \frac{\sigma_s^2}{2\mu_s} \\
&\quad \times \left\{\Phi\left(\frac{\ln(l/S) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) - e^{\mu_s\tau}\left(\frac{S}{\tilde{d}_i}\right)^{2\mu_s/\sigma_s^2} \times \Phi\left(\frac{\ln(l/S) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right\}
\end{aligned} \tag{58}
$$

According to Equation (51),

$$
\begin{aligned}
&\Theta\left(S, \tilde{d}_i\right) \\
&= S - e^{-\mu_s\tau}\left\{\tilde{d}_i\left[\Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) + \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right.\right. \\
&\quad \left. + \left(\frac{S}{\tilde{d}_i}\right)^{1-\frac{2\mu_s}{\sigma_s^2}} \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) - \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right] \\
&\quad - S\left(\frac{S}{\tilde{d}_i}\right)^{-2\mu_s/\sigma_s^2} \cdot \left(\frac{\sigma_s^2}{2\mu_s}\right) e^{\mu_s\tau} \times \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) \\
&\quad \left. + Se^{\mu_s\tau}\Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right\} \\
&= S - e^{-\mu_s\tau}\tilde{d}_i\left[\Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right. \\
&\quad + \left(\frac{S}{\tilde{d}_i}\right)^{1-\frac{2\mu_s}{\sigma_s^2}} \Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) - \left(\frac{S}{\tilde{d}_i}\right)^{-\frac{2\mu_s}{\sigma_s^2}} \left(\frac{2\mu_s}{\sigma_s^2}\right) \\
&\quad \left.\times \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right] - S\left(\frac{2\mu_s}{\sigma_s^2}\right)\Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- S\Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) \\
&= S - e^{-\mu_s\tau}\tilde{d}_i\left[\Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) + \left(\frac{S}{\tilde{d}_i}\right)^{1-\frac{2\mu_s}{\sigma_s^2}} - \left(\frac{S}{\tilde{d}_i}\right)^{-\frac{2\mu_s}{\sigma_s^2}}\left(\frac{\sigma_s^2}{2\mu_s}\right)\right] \\
&\times \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) \\
&- S\left\{1 + \left(\frac{\sigma_s^2}{2\mu_s}\right)\right\}\Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)
\end{aligned}
\tag{59}
$$

Then, according to Equation (59),

$$
\begin{aligned}
&\Theta\left(S, \tilde{d}_i\right) \\
&= S - \tilde{d}_i e^{-\mu_s\tau}\left[\Phi\left(\frac{\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right) + \left\{\left(\frac{S}{\tilde{d}_i}\right)^{1-\frac{2\mu_s}{\sigma_s^2}} - \left(\frac{S}{\tilde{d}_i}\right)^{-\frac{2\mu_s}{\sigma_s^2}}\left(\frac{\sigma_s^2}{2\mu_s}\right)\right\}\right. \\
&\left.\times \Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) + \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)\right] \\
&- S\left(1 + \frac{\sigma_s^2}{2\mu_s}\right)\Phi\left(\frac{-\ln\left(\tilde{d}_i/S\right) - \left(\mu_s - \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}}\right)
\end{aligned}
\tag{60}
$$

Appendix B. **Analysis of Actual Data in the Production Flow System.**

TABLE 3. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

TABLE 4. Volatility of Table 3

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

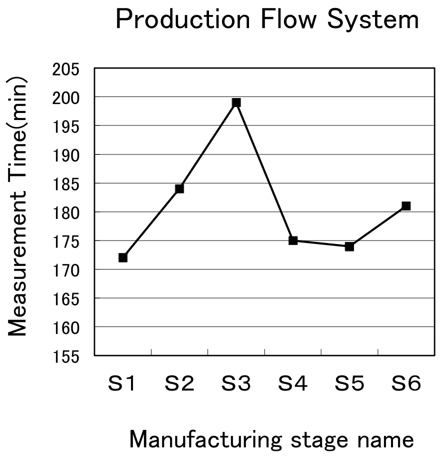

FIGURE 21. Total work time for each stage (S1-S6) in Table 3

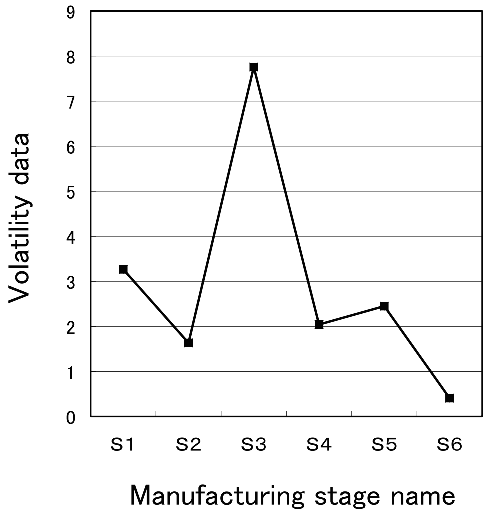

FIGURE 22. Volatility data for each stage (S1-S6) in Table 3

TABLE 5. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

TABLE 6. Volatility of Table 5

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 7. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	21	21	21	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	22	22	20	20	20	20
K9	20	25	25	25	20	20	20
Total	180	165	164	161	180	180	180

TABLE 8. Variance of Table 7

K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.33	3	3	0	0	0
K5	1.33	1.33	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.33	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0