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RISK-REWARD STRATEGIES FOR THE NON-ADDITIVE TWO-OPTION ONLINE LEASING PROBLEM

Xiaoli Chen and Weijun Xu*

School of Business Administration
South China University of Technology
No. 381, Wushan Road, Tianhe District, Guangzhou 510641, P. R. China
chenxiaoli871122@163.com; *Corresponding author: xuweijun75@163.com

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Abstract. *We consider the non-additive two-option leasing problem, which is common in the leasing market. In this problem there are two payment options to lease a piece of equipment, where each Option i (for $i = 1, 2$) has two kinds of costs: the one-time cost b_i to start using Option i and the corresponding rental price a_i of Option i. Without loss of generality, we assume that $a_1 > a_2 \geq 0$, $b_2 > b_1 \geq 0$. And if we switch from Option 1 to Option 2, we should pay a transition cost c, where $c \geq b_2 - b_1$. As the decision-maker must make decisions at once without knowing the exact length of using the equipment, this problem is online. In this paper we give the optimal deterministic strategy and its competitive ratio by the method of competitive analysis. We also obtain the risk-reward algorithms and strategies by taking the risk tolerance and probabilistic forecasts of the decision-maker into consideration. In addition, we use numerical analysis to show the influence of the parameters on the risk-reward strategies and the sensitivity of the traditional strategy to the parameters, which may help make decisions.*

Keywords: Non-additive two-option online leasing, Competitive analysis, Risk tolerance, Probabilistic forecasts, Risk-reward strategies

1. **Introduction.** The leasing industry as a sunrise industry has demonstrated its resilience since the global economic crisis and the outlook are cautiously optimistic [1]. A company or an individual without enough money to buy certain equipment can own the right to use the equipment by leasing. To decide whether leasing is a beneficial way to use the equipment or not, we should determine the length of using the equipment. However, in practice, it is hard to know the exact duration. This shows the online feature of leasing. Fortunately, researchers explored the competitive analysis [2, 3] to study the online problems and evaluate their strategies. We would find the appropriate strategy through competitive ratio, which is the ratio of the cost paid by our online strategy to the cost paid by the optimal offline strategy (which is obtained when everything is known in advance).

The classical instance for online leasing problem is the "ski rental" problem [4, 5]: a person plans to go skiing, but he has no idea of the exact ski duration. So he has to decide whether to rent or buy a pair of skis. To rent the skis, he must pay 1 per day; to buy the skis, he has to pay s ($s > 1$) and does not need to pay the rental fee any longer. Then which is the optimal strategy, to rent, to buy, or to rent at first then to buy? By means of competitive analysis, the optimal deterministic strategy can be obtained. That is to rent the skis for the first $s - 1$ days, and then to buy the skis if he continues to ski in the s-th day. The competitive ratio of this strategy is $2 - 1/s$, which means the online strategy never pays more than $2 - 1/s$ times the optimal offline cost [5]. Considering randomization can sometimes improve the performance ratio, Karlin et al. [6] gave an $e/(e-1)$-competitive online randomized algorithm. Since then, many researchers extended the "ski rental" problem and considered into the extended online rental problems practical economic factors, such as interest rate [7, 8], tax rate [9], and price fluctuation [10, 11].

In the financial market, not all the decision-makers are risk averse. They are sometimes willing to undertake the risk moderately to obtain higher reward. So the risk preference of the decision-makers cannot be ignored. Al-Binali [12] introduced the decision-makers' risk tolerance and forecast for the future into the "ski rental" problem. He defined the risk and reward of a competitive algorithm and built the famous risk-reward model. In this model, if the input σ is an instance of the problem Σ and the cost ratio of an online algorithm A and the optimal algorithm OPT is denoted by $R_A(\sigma)$, then the competitive ratio of A on the problem Σ is $R_A = \sup\limits_{\sigma \in \Sigma}(R_A(\sigma))$ and the optimal competitive ratio for the problem Σ is $R^* = \inf\limits_{A} R_A$. The risk of A is defined as R_A/R^*. If the decision-maker's risk tolerance is λ ($\lambda \geq 1$), then the set of risk tolerable strategies is $I_\lambda = \{A | R_A \leq \lambda R^*\}$. And if the decision-maker has a forecast $F \subset \Sigma$, the restricted ratio of A is $\bar{R}_A = \sup\limits_{\sigma \in F}(R_A(\sigma))$ and the reward of A is $R^*/\bar{R}_A$ when the forecast is correct. The risk-reward model is to maximize $R^*/\bar{R}_A$ subject to $A \in I_\lambda$. Then a lot of researchers studied the rental problem based on this risk-reward framework. For example, Zhang et al. [13] analyzed the risk-reward strategy for the online leasing of depreciable equipment with the interest rate. Wang et al. [14] considered the online financial leasing problem and presented its risk-reward model. Considered that the decision-maker can have more than one forecast, Dong et al. [15] put forward a more flexible risk-reward model where each forecast has a probability. This model contains Al-Binali's risk-reward model, so in this paper we call Al-Binali's model the traditional risk-reward model and call the model of Dong et al. [15] the general risk-reward model. Based on these two risk-reward frameworks, Zhang et al. [16] gave the traditional and general risk-reward model for the online leasing of depreciable equipment.

The aforementioned studies just analyzed the case that there were two options: pure rental and pure buying options. However, more options can be chosen in the leasing market. Considering the case with no pure buying option, Lotker et al. [17] studied the ski rental problem with two general options: one is pure rental option and the other is to pay a one-time cost and then to rent with a lower price. They gave a randomized algorithm and proved its optimality. Further, Chen and Xu [18] continued the analysis of the problem in [17] and presented the risk-reward strategy with compound interest rate. Moreover, Fujiwara et al. [19] considered the ski rental problem with more than two options and termed it the multislope ski rental problem. By mathematical programming they obtained the infimum and supremum of the competitive ratio for the best possible deterministic strategy. And Augustine et al. [20] regarded the "rent" and "buy" options as the energy-consumption modes of a system and studied the power-down strategies with more than one low-power state, where each state has its own power-consumption rate and one-time cost. They discussed two variants of this problem: one is the additive case, where the transition cost from one state to another is the difference of the corresponding one-time cost; the other is the non-additive case, where the transition cost is arbitrary. Further, Lotker et al. [21] took randomization into the multislope ski rental problem and studied the randomized algorithms for this problem. They put forward the best possible online randomized algorithm for the additive instance and an e-competitive randomized algorithm for any instance. However, few papers study the non-additive online leasing problem and give the analytical form of the competitive ratio even for the two-option case.

Motivated by [20, 21], Levi and Patt-Shamir [22] studied a non-additive two-option online leasing problem, and we call it NTOLP for short. They gave the optimal deterministic and randomized algorithms for this problem. In this problem the two rental options are such that each Option i (for $i = 1, 2$) is characterized by the one-time cost b_i to start using Option i and the corresponding rental cost a_i per unit of time for using Option i. And it is assumed that $b_2 > b_1 \geq 0$, $a_1 > a_2 \geq 0$. However, transition from Option 1 to Option 2 costs c, which means if we start with Option 1 at time 0 and switch to Option 2 at time $t > 0$, then the total cost at time $T \geq t$ is $b_1 + a_1 t + c + a_2(T - t)$. Besides, $c \geq b_2 - b_1$, otherwise the leasing problem is simplified to the additive version [22]. In this paper we mark this problem as $(a_1, b_1; a_2, b_2; c)$-NTOLP. There is a simplified variant in the study of [22], where the parameters satisfy $0 \leq b_1 < b_2 \leq 1$, $c \geq b_2 - b_1$ and $a_i = 1 - b_i$ for $i = 1, 2$. We mark this simplified variant as (b_1, b_2, c)-NTOLP.

Considering that Levi and Patt-Shamir [22] did not take decision-maker's risk preference and estimation of market demand into account, and the studies based on the risk-reward frameworks just analyzed the additive leasing problems, in this paper we introduce decision-maker's risk tolerance and forecasts for the duration into the NTOLP and give the risk-reward strategies based on [12, 15]. We first present the optimal deterministic competitive ratio of $(a_1, b_1; a_2, b_2; c)$-NTOLP and the optimal competitive ratio is $\min\left\{1 + \frac{c(a_1-a_2)}{a_1 b_2 - a_2 b_1}, \frac{b_2}{b_1}, \frac{a_1}{a_2}\right\}$. Then we obtain the optimal traditional and general risk-reward strategies and algorithms for $(a_1, b_1; a_2, b_2; c)$-NTOLP. And for (b_1, b_2, c)-NTOLP, we also get more simplified risk-reward strategies. Through these strategies the decision-maker can know when to switch to the other option based on his own risk tolerance and forecasts for the future.

The remainder of this paper is organized as follows. In Section 2, we provide the optimal deterministic strategy for $(a_1, b_1; a_2, b_2; c)$-NTOLP without any forecast by means of competitive analysis. In Section 3, we obtain the traditional and general risk-reward strategies for $(a_1, b_1; a_2, b_2; c)$-NTOLP and (b_1, b_2, c)-NTOLP based on the risk-reward framework of [12, 15]. In Section 4, we give numerical analysis. Finally, a summary of this paper is presented in Section 5.

2. Deterministic Online Leasing Strategy.

A company needs a piece of equipment, but there is not enough cash to buy it. Then the decision-maker decides to rent it. When facing the NTOLP, what is the optimal leasing method? In this section, we provide an optimal deterministic online leasing strategy and its competitive ratio for the NTOLP.

Assume the length of using the equipment is T, which is known to the offline adversary and unknown to the online decision-maker. Let $T^* = \frac{b_2 - b_1}{a_1 - a_2}$, and then for the offline adversary, the cost of the optimal offline algorithm, OPT, is

$$Cost_{OPT}(T) = \begin{cases} b_1 + a_1 T, & T < T^*; \\ b_2 + a_2 T, & T \geq T^*. \end{cases}$$

For the online decision-maker the cost is related to the switching time and unknown duration. We define an online strategy S_t, which starts to use Option 2 at time t ($0 \leq t \leq \infty$).

Firstly, when $t = 0$, the online strategy S_0 uses Option 2 at the beginning and till the end. Then the cost of S_0 is $Cost_S(T) = b_2 + a_2 T$. S_0 is optimal when $T \geq T^*$. When $0 \leq T < T^*$, the competitive ratio of S_0 is $R(0) = \sup\limits_{T<T^*} \frac{b_2 + a_2 T}{b_1 + a_1 T} = \frac{b_2}{b_1}$.

Next, when $t > 0$, the cost of the online strategy S_t is

$$Cost_S(t; T) = \begin{cases} b_1 + a_1 T, & T < t; \\ b_1 + a_1 t + c + a_2(T - t), & T \geq t. \end{cases} \tag{1}$$

Let $R(t;T) = \frac{Cost_S(t;T)}{Cost_{OPT}(T)}$ and $R(t) = \sup\limits_{T} R(t;T)$. Then $R(t)$ is the competitive ratio of the online strategy S_t according to the definition of competitive ratio. Now we discuss the optimal strategy in two cases.

Case 1: If $0 < t < T^*$, then

$$R(t;T) = \begin{cases} 1, & 0 < T < t; \\ \dfrac{b_1 + a_1 t + c + a_2(T-t)}{b_1 + a_1 T}, & t \le T < T^*; \\ \dfrac{b_1 + a_1 t + c + a_2(T-t)}{b_2 + a_2 T}, & T \ge T^*. \end{cases}$$

Through derivation we can find that $R(t;T)$ decreases with respect to T in both the second and third intervals. Then the maxima are obtained in the left endpoint of these two intervals. So the competitive ratio of S_t is

$$R(t) = \max\left\{\frac{b_1 + a_1 t + c}{b_1 + a_1 t}, \frac{b_1 + a_1 t + c + a_2(T^* - t)}{b_2 + a_2 T^*}\right\} = \frac{b_1 + a_1 t + c}{b_1 + a_1 t}.$$

To obtain a competitive ratio as small as possible, the online decision-maker will choose $t \to T^*$ as $R(t)$ decreases with respect to t. Then the optimal competitive ratio in this case is $\lim\limits_{t \to T^*} R(t) = 1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}$.

Case 2: If $t \ge T^*$, then

$$R(t;T) = \begin{cases} 1, & 0 < T < T^*; \\ \dfrac{b_1 + a_1 T}{b_2 + a_2 T}, & T^* \le T < t; \\ \dfrac{b_1 + a_1 t + c + a_2(T-t)}{b_2 + a_2 T}, & T \ge t. \end{cases}$$

According to the monotonicity of this piecewise function in each interval, we know that $R(t;T)$ approaches its greatest value at $T = t$ and the competitive ratio of S_t is $R(t) = \frac{b_1 + a_1 t + c}{b_2 + a_2 t}$. By differentiating $R(t)$ with respect to t we obtain

$$\frac{dR(t)}{dt} = \frac{a_1(b_2 + a_2 t) - a_2(b_1 + a_1 t + c)}{(b_2 + a_2 t)^2} = \frac{a_1 b_2 - a_2 b_1 - a_2 c}{(b_2 + a_2 t)^2}.$$

Then $R(t)$, as a function of t, is increasing if $c \le \frac{a_1 b_2 - a_2 b_1}{a_2}$ and decreasing if $c > \frac{a_1 b_2 - a_2 b_1}{a_2}$. If $c \le \frac{a_1 b_2 - a_2 b_1}{a_2}$, the competitive ratio can reach its lower bound when $t = T^*$. Then the optimal competitive ratio in this case is $R(T^*) = \frac{b_1 + a_1 T^* + c}{b_2 + a_2 T^*} = 1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}$. If $c > \frac{a_1 b_2 - a_2 b_1}{a_2}$, the competitive ratio is lower bounded by the limit as $t \to \infty$. Then the optimal competitive ratio in this case is $\lim\limits_{t \to \infty} R(t) = \frac{a_1}{a_2}$.

In conclusion, we can obtain a theorem as follows:

Theorem 2.1. *The deterministic optimal competitive ratio of the NTOLP is*

$$R^* = \min\left\{1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}, \frac{b_2}{b_1}, \frac{a_1}{a_2}\right\}.$$

The optimal switching time t^* is

$$t^* = \begin{cases} 0, & \dfrac{b_2}{b_1} \leq \min\left\{1 + \dfrac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}, \dfrac{a_1}{a_2}\right\}; \\ T^*, & 1 + \dfrac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1} \leq \min\left\{\dfrac{b_2}{b_1}, \dfrac{a_1}{a_2}\right\}; \\ \infty, & \dfrac{a_1}{a_2} \leq \min\left\{1 + \dfrac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}, \dfrac{b_2}{b_1}\right\}. \end{cases}$$

If we substitute $a_i = 1 - b_i$ for $i = 1, 2$ into Theorem 2.1, then we can get the optimal deterministic competitive ratio for (b_1, b_2, c)-NTOLP, which is the same as that in the study of [22]. This shows our deterministic strategy generalizes the strategy in [22].

3. **Risk-Reward Strategies.** In Section 2, we discussed the deterministic strategy using competitive analysis. However, it is well known that competitive analysis is a kind of worst-case analyses and it is considered to be too pessimistic. And sometimes the decision-maker would take advantage of the risk rather than avoid it completely. Fortunately, Al-Binali [12] put forward a risk-reward framework, by which the decision-maker can benefit from a correct forecast and control the risk within his tolerance when the forecast falls. So in this section we introduce the risk tolerance of the decision-maker and search for the traditional risk-reward strategy for the NTOLP. Besides, we consider the probabilistic forecasts and give a general risk-reward strategy for the NTOLP based on the framework of [15]. Then the decision-maker can take his risk preference and forecasts into account and obtain the optimal strategy according to these two risk-reward strategies when facing the NTOLP.

We assume that the decision-maker's risk tolerance is λ ($\lambda \geq 1$). Then the set of risk tolerant strategies is $I_\lambda = \{S_t | R(t) \leq \lambda R^*\}$. In addition, we assume $1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1} \leq \min\left\{\frac{b_2}{b_1}, \frac{a_1}{a_2}\right\}$ for the moment. The results can be similarly obtained using the same method when $\frac{b_2}{b_1} \leq \min\left\{\frac{a_1}{a_2}, 1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right\}$ and $\frac{a_1}{a_2} \leq \min\left\{\frac{b_2}{b_1}, 1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right\}$. In the calculating process, we just need to remember the corresponding optimal deterministic switching time and competitive ratio. Through simplifying inequality $1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1} \leq \min\left\{\frac{b_2}{b_1}, \frac{a_1}{a_2}\right\}$, we can get $c \leq \min\left\{\frac{a_1 b_2 - a_2 b_1}{a_2}, \frac{(b_2 - b_1)(a_1 b_2 - a_2 b_1)}{a_1 b_1 - a_2 b_1}\right\}$. In this case the optimal deterministic strategy for $(a_1, b_1; a_2, b_2; c)$-NTOLP is S_{T^*}, and its optimal competitive ratio is $R^* = 1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}$. For (b_1, b_2, c)-NTOLP, the assumption turns to $c \leq \min\left\{\frac{b_2 - b_1}{b_1}, \frac{b_2 - b_1}{1 - b_2}\right\}$. And the optimal deterministic strategy is $S_{T^{*\prime}}$, where $T^{*\prime} = 1$, and its optimal competitive ratio is $R^{*\prime} = 1 + c$. In addition, we assume that all the strategies of the decision-maker always use Option 1 first, then switch to Option 2, which indicates $0 < t < \infty$. Then the cost of the online strategy S_t for $(a_1, b_1; a_2, b_2; c)$-NTOLP is exactly Equation (1). The cost of S_t for (b_1, b_2, c)-NTOLP can be obtained similarly by substituting $1 - b_i$ for a_i $(i = 1, 2)$.

3.1. **Traditional risk-reward strategy for $(a_1, b_1; a_2, b_2; c)$-NTOLP.** In this subsection, we determine the optimal risk-reward strategy with a definite forecast by applying Al-Binali's framework to $(a_1, b_1; a_2, b_2; c)$-NTOLP.

Suppose there are two forecasts, one is $F_1 = \{T : T < T^*\}$, and the other is $F_2 = \{T : T \geq T^*\}$. If the forecast F_1 is correct, then in the set I_λ the strategies that switch to Option 2 after time T^* can be used by the online decision-maker. In this case the offline adversary always uses Option 1, so the optimal restricted ratio is $\bar{R}_F^* = 1$. For the forecast F_2, we have the following theorem.

Theorem 3.1. *If the forecast F_2 is correct, then the optimal risk-reward strategy for the online decision-maker is to switch to Option 2 at time t_F^* when the decision-maker's risk tolerance is λ $(1 \le \lambda < \Delta)$ and the parameters in $(a_1, b_1; a_2, b_2; c)$-NTOLP are given, where*

$$\Delta = \frac{(b_1 + c)(a_1 b_2 - a_2 b_1)}{b_1(a_1 b_2 - a_2 b_1) + c b_1(a_1 - a_2)},$$

$$t_F^* = \frac{c(a_1 b_2 - a_2 b_1)}{(\lambda - 1)a_1(a_1 b_2 - a_2 b_1) + \lambda a_1 c(a_1 - a_2)} - \frac{b_1}{a_1}.$$

And the optimal restricted ratio is

$$\bar{R}_F^* = \frac{b_1 + a_1 t_F^* + c + a_2\left(T^* - t_F^*\right)}{b_2 + a_2 T^*}.$$

Proof: Firstly, we compute the set of risk tolerant strategies.

1) When $t < T^*$, through $R(t) = 1 + \frac{c}{b_1 + a_1 t} \le \lambda R^* = \lambda\left(1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right)$ we can obtain

$$t \ge \frac{c(a_1 b_2 - a_2 b_1)}{(\lambda - 1)a_1(a_1 b_2 - a_2 b_1) + \lambda a_1 c(a_1 - a_2)} - \frac{b_1}{a_1} \triangleq t_1.$$

Because $1 \le \lambda < \Delta$, we know that $t_1 > 0$.

2) When $t \ge T^*$, through $R(t) = \frac{b_1 + a_1 t + c}{b_2 + a_2 t} \le \lambda R^* = \lambda\left(1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right)$ we can obtain

$$[(a_1 - \lambda a_2)(a_1 b_2 - a_2 b_1) - \lambda a_2 c(a_1 - a_2)]t \le (\lambda b_2 - b_1 - c)(a_1 b_2 - a_2 b_1) + \lambda b_2 c(a_1 - a_2). \quad (2)$$

Because $R(t)$ increases with respect to t and $R(T^*) = R^* \le \lambda R^*$ holds, it is only to discuss the relationship of $\lim_{t \to \infty} R(t) = \frac{a_1}{a_2}$ and λR^* to solve Inequality (2). We assume

$$W_1 = (a_1 - \lambda a_2)(a_1 b_2 - a_2 b_1) - \lambda a_2 c(a_1 - a_2),$$

$$W_2 = (\lambda b_2 - b_1 - c)(a_1 b_2 - a_2 b_1) + \lambda b_2 c(a_1 - a_2).$$

Then Inequality (2) can be simplified to $W_1 t \le W_2$. Next, we solve this inequality in two cases.

(I) When $\frac{a_1}{a_2} \le \lambda\left(1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right)$, we have $W_1 \le 0$, but

$$\begin{aligned} W_2 &= \lambda b_2[a_1 b_2 - a_2 b_1 + c(a_1 - a_2)] - (b_1 + c)(a_1 b_2 - a_2 b_1) \\ &\ge \frac{a_1 b_2(a_1 b_2 - a_2 b_1)}{a_2} - (b_1 + c)(a_1 b_2 - a_2 b_1) \\ &= \frac{(a_1 b_2 - a_2 b_1 - c a_2)(a_1 b_2 - a_2 b_1)}{a_2} \\ &\ge 0. \end{aligned}$$

In this case, the solution to Inequality (2) is $\{t | t \ge T^*\}$.

(II) When $\frac{a_1}{a_2} > \lambda\left(1 + \frac{c(a_1 - a_2)}{a_1 b_2 - a_2 b_1}\right)$, we have $W_1 > 0$. Through simple computation we get that

$$W_2(a_1 - a_2) - W_1(b_2 - b_1) = (\lambda - 1)(a_1 b_2 - a_2 b_1)[a_1 b_2 - a_2 b_1 + c(a_1 - a_2)] \ge 0,$$

and then we have $W_2 \ge \frac{W_1(b_2 - b_1)}{a_1 - a_2} > 0$. If we define $t_2 = W_2 / W_1$, we obtain the solution to Inequality (2) is $\{t | T^* \le t \le t_2\}$.

Through (I) and (II), we obtain the solution to Inequality (2) is $\{t | t \ge T^*\}$ when $W_1 \le 0$ and $\{t | T^* \le t \le t_2\}$ when $W_1 > 0$.

Hence, considering 1) and 2) we can know that the set of risk tolerant strategies is $I_\lambda = \{S_t | t \ge t_1\}$ when $W_1 \le 0$ and $I_\lambda = \{S_t | t_1 \le t \le t_2\}$ when $W_1 > 0$.

Secondly, we compute the restricted ratio through $\bar{R}_F(t) = \sup\limits_{T\in F_2} \frac{Cost_S(t;T)}{Cost_{OPT}(T)}$. After calculation, we obtain

$$\bar{R}_F(t) = \begin{cases} \dfrac{b_1 + a_1 t + c + a_2(T^* - t)}{b_2 + a_2 T^*}, & t < T^*; \\ \dfrac{b_1 + a_1 t + c}{b_2 + a_2 t}, & t \geq T^*. \end{cases}$$

Obviously, $\bar{R}_F(t)$ increases with respect to t not only when $t < T^*$ but also when $t \geq T^*$.

According to Al-Binali's risk-reward framework, we just need to find a strategy from the set of risk tolerant strategies I_λ such that the restricted ratio reaches its infimum. On the basis of the monotonicity of $\bar{R}_F(t)$, the minimum of $\bar{R}_F(t)$ is $\bar{R}_F(t_1)$ when $t < T^*$ and $\bar{R}_F(T^*)$ when $t \geq T^*$. Then we can get that the switching time of the optimal risk-reward strategy can be chosen between t_1 and T^*. By contrast, we obtain $\bar{R}_F(t_1) < \bar{R}_F(T^*)$. So the optimal switching time is $t_F^* = t_1$, the optimal risk-tolerant online strategy is to switch to Option 2 at time t_F^*, and the optimal restricted ratio is $\bar{R}_F^* = \bar{R}_F(t_F^*)$. Then the theorem is proved.

According to Theorem 3.1 we can find that t_F^* and $\bar{R}_F^*$ decrease with the risk tolerance λ. That is to say, when the forecast F_2 is correct, the bigger the decision-maker's risk tolerance λ, the earlier the optimal switching time t_F^* and the smaller the restricted ratio $\bar{R}_F^*$. Moreover, if we assume $a = a_1 - a_2$ and $b = b_2 - b_1$, we can obtain the expressions of t_F^* and $\bar{R}_F^*$ about a and b. Through simple derivations we can find that t_F^* decreases with a and increases with b, on the contrary, $\bar{R}_F^*$ increases with a and decreases with b.

3.2. **General risk-reward strategy for $(a_1, b_1; a_2, b_2; c)$-NTOLP.** For online problems, the decision-maker does not know the exact duration of using the equipment. However, he can get or presume the probabilities that the duration belongs to some intervals on the basis of his experiences and the past and current market information. Specially, the decision-maker divides the duration into two intervals and has two corresponding forecasts $F_1 = \{T : T < T^*\}$ and $F_2 = \{T : T \geq T^*\}$. According to his experiences and the market information, he can estimate the probabilities P_i that forecast F_i occurs for $i = 1, 2$, where $P_1 + P_2 = 1$. In this subsection, we introduce the risk-reward framework of Dong et al. [15] into the NTOLP. Here, we assume that the decision-maker's risk tolerance is still λ $(1 \leq \lambda < \Delta)$. Then we have the following theorem.

Theorem 3.2. *If the decision-maker's risk tolerance is λ $(1 \leq \lambda < \Delta)$ and probabilistic forecasts are $\{(F_1, P_1), (F_2, P_2)\}$, the optimal risk-reward strategy for $(a_1, b_1; a_2, b_2; c)$-NTOLP is $S_{t_{PF}^*}$, where the optimal switching time is*

$$t_{PF}^* = \begin{cases} t_1, & 0 \leq P_1 < \dfrac{a_1 b_2 - a_2 b_1}{c a_1 + a_1 b_2 - a_2 b_1} \ \& \ \tau < t_1 \ \& \ \bar{R}_{PF}(t_1) < \bar{R}_{PF}(T^*); \\ \tau, & 0 \leq P_1 < \dfrac{a_1 b_2 - a_2 b_1}{c a_1 + a_1 b_2 - a_2 b_1} \ \& \ t_1 \leq \tau \ \& \ \bar{R}_{PF}(\tau) < \bar{R}_{PF}(T^*); \\ T^*, & \text{otherwise}. \end{cases} \tag{3}$$

The expressions of τ and function $\bar{R}_{PF}(\cdot)$ refer to the following proof.

Proof: For the same risk tolerance λ and online leasing problem, the set of risk-tolerant strategies is the same as that in the proof of Theorem 3.1. That is to say, the risk tolerable set is $I_\lambda = \{S_t | t_1 \leq t \leq t_2\}$ when $W_1 > 0$ and $I_\lambda = \{S_t | t \geq t_1\}$ when $W_1 \leq 0$.

Next, we compute the restricted ratio $\bar{R}_{PF}(t) = P_1 \bar{R}_1(t) + P_2 \bar{R}_2(t)$, where $\bar{R}_i(t) = \sup\limits_{T\in F_i} \frac{Cost_S(t;T)}{Cost_{OPT}(T)}$ for $i = 1$ or 2.

When $t < T^*$, we can obtain $\bar{R}_1(t) = \frac{b_1+a_1t+c}{b_1+a_1t}$ and $\bar{R}_2(t) = \sup_{T\in F_2} \frac{b_1+a_1t+c+a_2(T-t)}{b_2+a_2T} = \frac{b_1+a_1t+c+a_2(T^*-t)}{b_2+a_2T^*}$. When $t \geq T^*$, we have $\bar{R}_1(t) = 1$ and $\bar{R}_2(t) = \frac{b_1+a_1t+c}{b_2+a_2t}$. Then, applying the definition of the restricted ratio we have

$$\bar{R}_{PF}(t) = \begin{cases} P_1\dfrac{b_1+a_1t+c}{b_1+a_1t} + P_2\dfrac{b_1+a_1t+c+a_2(T^*-t)}{b_2+a_2T^*}, & t < T^*; \\ P_1 + P_2\dfrac{b_1+a_1t+c}{b_2+a_2t}, & t \geq T^*. \end{cases} \tag{4}$$

By differentiating Equation (4) with respect to t we obtain

$$\frac{d\bar{R}_{PF}(t)}{dt} = \begin{cases} -P_1\dfrac{ca_1}{(b_1+a_1t)^2} + P_2\dfrac{(a_1-a_2)^2}{a_1b_2-a_2b_1}, & t < T^*; \\ P_2\dfrac{a_1b_2-a_2b_1-ca_2}{(b_2+a_2t)^2}, & t \geq T^*. \end{cases}$$

As $c \leq \frac{a_1b_2-a_2b_1}{a_2}$, then $\bar{R}_{PF}(t)$ increases with t when $t \geq T^*$ and $\bar{R}_{PF}(t)$ reaches its minimum $1 + P_2\frac{c(a_1-a_2)}{a_1b_2-a_2b_1}$ at $t = T^*$. For $t < T^*$, we assume that $h(t) = -P_1\frac{ca_1}{(b_1+a_1t)^2} + P_2\frac{(a_1-a_2)^2}{a_1b_2-a_2b_1}$. It is easy to find that $h(t)$ increases with respect to t. If we order $h(t) = 0$, we get $t = -\frac{b_1}{a_1} + \frac{1}{a_1(a_1-a_2)}\sqrt{\frac{ca_1P_1(a_1b_2-a_2b_1)}{1-P_1}} \triangleq \tau$. By comparing τ and T^*, we can obtain the following results. (I) If $\tau \geq T^*$, that is $\frac{a_1b_2-a_2b_1}{ca_1+a_1b_2-a_2b_1} \leq P_1 \leq 1$, then $h(t) \leq 0$ when $t < T^*$. In this case, $\bar{R}_{PF}(t)$ is decreasing and the infimum is $\lim_{t\to T^{*-}} \bar{R}_{PF}(t) = 1+\frac{c(a_1-a_2)}{a_1b_2-a_2b_1}$. (II) If $\tau < T^*$, that is $0 \leq P_1 < \frac{a_1b_2-a_2b_1}{ca_1+a_1b_2-a_2b_1}$, then $h(t) < 0$ when $t < \tau$ and $h(t) \geq 0$ when $\tau \leq t < T^*$. In this case, $\bar{R}_{PF}(t)$ is decreasing when $t < \tau$ and increasing when $\tau \leq t < T^*$. So the minimum for $\bar{R}_{PF}(t)$ is $\bar{R}_{PF}(\tau)$ when $\tau < T^*$. As we must choose a strategy from the risk tolerable set I_λ, the minimum for $\bar{R}_{PF}(t)$ where $t < T^*$ turns to $\bar{R}_{PF}(t_1)$ when $\tau < t_1 < T^*$ and $\bar{R}_{PF}(\tau)$ when $t_1 \leq \tau < T^*$. Above all, through contrasting the local minima we can obtain the optimal switching time as Equation (3). The theorem is thus proved.

According to Theorem 3.2 and Al-Binali's definition of the reward, we find that the optimal switching time is T^* when $P_1 = 1$ and the reward of risk compensation is $R^*/\bar{R}(T^*) = R^*$; the optimal switching time is t_1 when $P_2 = 1$ and the reward of risk compensation is $R^*/\bar{R}_{PF}(t_1)$.

Based on the proof of Theorem 3.2, we obtain the optimal risk-reward algorithm as Algorithm 3.1.

For different input parameters, we can get the corresponding optimal risk-reward strategies by Algorithm 3.1. Besides, when the forecast F_2 is true, that is $P_1 = 0$, Algorithm 3.1 is simplified as the algorithm to obtain the optimal traditional risk-reward strategy.

3.3. **Risk-reward strategies for (b_1, b_2, c)-NTOLP.** In this subsection, we discuss (b_1, b_2, c)-NTOLP in [22] with the assumptions $0 \leq b_1 < b_2 \leq 1$ and $a_i = 1 - b_i$ for $i = 1, 2$, and then we can obtain new optimal switching time and new restricted competitive ratio with less parameters for the traditional and general strategies. In (b_1, b_2, c)-NTOLP, the forecasts for the duration are $F_1' = \{T : T < T^{*\prime}\}$ and $F_2' = \{T : T \geq T^{*\prime}\}$, where $T^{*\prime} = 1$.

Algorithm 3.1. The optimal risk-reward algorithm for $(a_1, b_1; a_2, b_2; c)$-NTOLP

Input $(a_1, b_1; a_2, b_2; c)$ in the NTOLP, the forecast probability P_1 and the risk tolerance λ.

Compute $T^* = \frac{b_2-b_1}{a_1-a_2}$ and $t_1 = \frac{c(a_1b_2-a_2b_1)}{(\lambda-1)a_1(a_1b_2-a_2b_1)+\lambda a_1 c(a_1-a_2)} - \frac{b_1}{a_1}$, simplify the function $\bar{R}_{PF}(t)$ in Equation (4).

1. If $0 \le P_1 < \frac{a_1b_2-a_2b_1}{ca_1+a_1b_2-a_2b_1}$, then
   $\tau := -\frac{b_1}{a_1} + \frac{1}{a_1(a_1-a_2)}\sqrt{\frac{ca_1P_1(a_1b_2-a_2b_1)}{1-P_1}}$.
   (1) If $\tau < t_1$ and $\bar{R}_{PF}(t_1) < \bar{R}_{PF}(T^*)$,
   then $t^*_{PF} := t_1$, $\bar{R}_{PF}(t^*_{PF}) := \bar{R}_{PF}(t_1)$.
   (2) Else if $t_1 \le \tau$ and $\bar{R}_{PF}(\tau) < \bar{R}_{PF}(T^*)$,
   then $t^*_{PF} := \tau$, $\bar{R}_{PF}(t^*_{PF}) := \bar{R}_{PF}(\tau)$.
   (3) Else $t^*_{PF} := T^*$, $\bar{R}_{PF}(t^*_{PF}) := \bar{R}_{PF}(T^*)$.
2. Else $t^*_{PF} := T^*$, $\bar{R}_{PF}(t^*_{PF}) := \bar{R}_{PF}(T^*)$.

Output the optimal switching time t^*_{PF} and the restricted ratio $\bar{R}_{PF}(t^*_{PF})$ of the risk-reward strategy $S_{t^*_{PF}}$.

Theorem 3.3. *For the decision-maker with risk tolerance λ $(1 \le \lambda < \Delta')$, if the forecast F'_2 is true, then the optimal switching time $t^*_{F'}$ and competitive ratio $\bar{R}^*_{F'}$ for (b_1, b_2, c)-NTOLP are*

$$t^*_{F'} = \frac{c}{(1-b_1)(\lambda c + \lambda - 1)} - \frac{b_1}{1-b_1},$$

$$\bar{R}^*_{F'} = \frac{\lambda c(1-b_1) + (\lambda-1)(1-b_2)}{(1-b_1)(\lambda c + \lambda - 1)}(1+c),$$

where $\Delta' = \frac{b_1+c}{b_1(1+c)}$.

Proof: By substituting $a_i = 1 - b_i$ (for $i = 1, 2$) into the expressions of t^*_F and $\bar{R}^*_F$ in Theorem 3.1, we would get $t^*_{F'}$ and $\bar{R}^*_{F'}$.

In this case we can find that $t^*_{F'}$ is independent of b_2. And through simple derivations we can obtain that $t^*_{F'}$ decreases with respect to b_1 and λ, but it increases with respect to c. This shows that bigger b_1 or higher risk tolerance λ can make the optimal risk-reward strategy switch to Option 2 ahead of time; on the contrary, bigger switching cost c can put the optimal switching time off. Similarly, we can get that $\bar{R}^*_{F'}$ decreases with respect to λ and b_2, but increases with c and b_1.

When we consider the decision-maker's probabilistic forecasts $\{(F'_1, P_1), (F'_2, P_2)\}$, we can obtain the following theorem.

Theorem 3.4. *For (b_1, b_2, c)-NTOLP, when the decision-maker's risk tolerance is λ $(1 \le \lambda < \Delta')$ and the probabilistic forecasts are $\{(F'_1, P_1), (F'_2, P_2)\}$, the optimal general risk-reward online strategy is $S_{t^*_{PF'}}$, where*

$$t^*_{PF'} = \begin{cases} t'_1, & \{0 \le P_1 < \phi_3 \ \&\ (\lambda-1)(1+c) \le c\} \\ & \textit{or}\ \{0 \le P_1 < \phi_2 \ \&\ (\lambda-1)(1+c) > c\}; \\ \tau', & \phi_2 \le P_1 < \phi_4 \ \&\ (\lambda-1)(1+c) > c; \\ T^{*\prime}, & \textit{otherwise}, \end{cases} \tag{5}$$

in which the expressions of t'_1, τ', ϕ_2, ϕ_3, ϕ_4 can be found in the following proof.

Proof: We substitute $a_i = 1 - b_i$ into the expressions of t_1, τ, T^* and function $\bar{R}_{PF}(\cdot)$ in Theorem 3.2, and then we can obtain new optimal switching time $t^*_{PF'}$ with less parameters for the optimal general risk-reward strategy, where

$$t^*_{PF'} = \begin{cases} t'_1, & 0 \le P_1 < \dfrac{b_2-b_1}{c(1-b_1)+b_2-b_1} \ \&\ \tau' < t'_1 \ \&\ \bar{R}_{PF'}(t'_1) < \bar{R}_{PF'}(T^{*\prime}); \\ \tau', & 0 \le P_1 < \dfrac{b_2-b_1}{c(1-b_1)+b_2-b_1} \ \&\ t'_1 \le \tau' \ \&\ \bar{R}_{PF'}(\tau') < \bar{R}_{PF'}(T^{*\prime}); \\ T^{*\prime}, & \text{otherwise,} \end{cases} \tag{6}$$

in which $t'_1 = t^*_{F'}$, $T^{*\prime} = 1$, $\tau' = -\frac{b_1}{1-b_1} + \sqrt{\frac{cP_1}{(1-b_1)(b_2-b_1)(1-P_1)}}$, and the function $\bar{R}_{PF'}(\cdot)$ is

$$\bar{R}_{PF'}(t) = \begin{cases} P_1 \dfrac{c}{b_1+(1-b_1)t} + (1-P_1)[(b_2-b_1)t + c + b_1 - b_2] + 1, & t < 1; \\ P_1 + (1-P_1)\dfrac{b_1+(1-b_1)t+c}{b_2+(1-b_2)t}, & t \ge 1. \end{cases} \tag{7}$$

Next, we simplify Equation (6) further.

Define $\phi_1 = \frac{b_2-b_1}{c(1-b_1)+b_2-b_1}$. As

$$\begin{aligned} \tau' < t'_1 &\Leftrightarrow -\frac{b_1}{1-b_1} + \sqrt{\frac{cP_1}{(1-b_1)(b_2-b_1)(1-P_1)}} < \frac{c}{(1-b_1)(\lambda c+\lambda-1)} - \frac{b_1}{1-b_1} \\ &\Leftrightarrow P_1 < \frac{c(b_2-b_1)}{c(b_2-b_1)+(1-b_1)(\lambda c+\lambda-1)^2} \triangleq \phi_2, \end{aligned}$$

and $\phi_1 \ge \phi_2$ because of $\lambda c + \lambda - 1 \ge c$, we can simplify Equation (6) as

$$t^*_{PF'} = \begin{cases} t'_1, & 0 \le P_1 < \phi_2 \ \&\ \bar{R}_{PF'}(t'_1) < \bar{R}_{PF'}(T^{*\prime}); \\ \tau', & \phi_2 \le P_1 < \phi_1 \ \&\ \bar{R}_{PF'}(\tau') < \bar{R}_{PF'}(T^{*\prime}); \\ T^{*\prime}, & \text{otherwise.} \end{cases} \tag{8}$$

On the basis of Equation (7) we can find that $\bar{R}_{PF'}(t'_1) < \bar{R}_{PF'}(T^{*\prime})$ is equivalent to

$$P_1(\lambda c+\lambda-1)+(1-P_1)\left[c + \frac{c(b_2-b_1)-(b_2-b_1)(\lambda c+\lambda-1)}{(1-b_1)(\lambda c+\lambda-1)}\right]+1 < P_1+(1-P_1)(1+c).$$

By solving this inequality we can get that

$$P_1 < \frac{(\lambda-1)(b_2-b_1)(1+c)}{(\lambda-1)(b_2-b_1)(1+c)+(1-b_1)(\lambda c+\lambda-1)^2} \triangleq \phi_3.$$

Then we contrast ϕ_2 and ϕ_3. Since $\phi_3 - \phi_2 = \beta[(\lambda-1)(1+c)-c]$, where β is a function of b_1, b_2, c and λ, and we can prove that $\beta > 0$, then $\phi_3 \le \phi_2$ when $(\lambda-1)(1+c) \le c$ and $\phi_3 > \phi_2$ when $(\lambda-1)(1+c) > c$. So Equation (8) can be simplified further more as

$$t^*_{PF'} = \begin{cases} t'_1, & \{0 \le P_1 < \phi_3 \ \&\ (\lambda-1)(1+c) \le c\} \\ & \text{ or } \{0 \le P_1 < \phi_2 \ \&\ (\lambda-1)(1+c) > c\}; \\ \tau', & \phi_2 \le P_1 < \phi_1 \ \&\ \bar{R}_{PF'}(\tau') < \bar{R}_{PF'}(T^{*\prime}); \\ T^{*\prime}, & \text{otherwise.} \end{cases} \tag{9}$$

Likewise, through $\bar{R}_{PF'}(\tau') < \bar{R}_{PF'}(T^{*\prime})$ we can obtain

$$P_1 < \frac{b_2-b_1}{b_2-b_1+4c(1-b_1)} \triangleq \phi_4.$$

It is easy to find that $\phi_4 \le \phi_1$. Now we compare ϕ_2 and ϕ_4. By simple calculation, we have

$$\phi_4 - \phi_2 = \frac{(1-b_1)(b_2-b_1)(\lambda c+\lambda-1+2c)[(\lambda-1)(1+c)-c]}{[b_2-b_1+4c(1-b_1)][c(b_2-b_1)+(1-b_1)(\lambda c+\lambda-1)^2]}.$$

Then we can know that $\phi_4 > \phi_2$ when $(\lambda-1)(1+c) > c$ and $\phi_4 \leq \phi_2$ when $(\lambda-1)(1+c) \leq c$. However, it is impossible for P_1 to satisfy $\phi_2 \leq P_1 < \phi_1$ and $P_1 < \phi_4$ at the same time when $\phi_4 \leq \phi_2$. Therefore, Equation (9) can be simplified down to Equation (5). Thus, Theorem 3.4 is proved.

According to Theorem 3.4 we can get the optimal restricted ratio $\bar{R}^*_{PF'} = \bar{R}_{PF'}(t^*_{PF'})$ by substituting $t^*_{PF'}$ into Equation (7). We can also obtain the following more simplified conclusion. With regard to the parameters given in Theorem 3.4, if $(\lambda-1)(1+c) \leq c$, then the optimal switching time $t^*_{PF'}$ of the optimal general risk-reward strategy for (b_1, b_2, c)-NTOLP is

$$t^*_{PF'} = \begin{cases} t'_1, & 0 \leq P_1 < \phi_3; \\ T^{*\prime}, & \text{otherwise}, \end{cases}$$

and if $(\lambda-1)(1+c) > c$, then it is

$$t^*_{PF'} = \begin{cases} t'_1, & 0 \leq P_1 < \phi_2; \\ \tau', & \phi_2 \leq P_1 < \phi_4; \\ T^{*\prime}, & \text{otherwise}. \end{cases}$$

Then the decision-maker can switch to Option 2 at the matching time $t^*_{PF'}$ according to the above conclusion.

Besides, we can find that the conclusion in Theorem 3.4 coincides with that in Theorem 3.3 when $P_1 = 0$. This means that our general risk-reward strategy extends Al-Binali's risk-reward strategy.

Remark 3.1. *To make $1 \leq \lambda < \Delta$ (or $1 \leq \lambda < \Delta'$) is to have $t_1 > 0$ (or $t'_1 > 0$). Since the decision-makers always have limited risk tolerance, the restriction on λ is reasonable. When $\lambda \geq \Delta$ (or Δ'), we can obtain the corresponding optimal switching time for the risk-reward strategies in the same way. And the optimal switching time is similar to ours. The only difference is that t^*_F, $t^*_{F'}$, t_1 and t'_1 are replaced by an arbitrarily small number $\varepsilon > 0$. In this paper we do not make a specific analysis of this case for the moment.*

4. **Numerical Analysis.** In this section, we analyze the influence of the decision-maker's risk tolerance, parameters in the NTOLP, and forecast probabilities on the optimal traditional and general risk-reward strategies. For simplicity, we only discuss the NTOLP with the assumptions in [22], that is (b_1, b_2, c)-NTOLP. For $(a_1, b_1; a_2, b_2; c)$-NTOLP, the influence can be analyzed similarly. In addition, we use $b_1 = 0.2$, $b_2 = 0.7$, $c = 1$ and $\lambda = 1.4$ as the benchmarks. In the following examples we will analyze the risk-reward strategies based on these benchmarks. Additionally, we take $P_1 = 0.1$ when we discuss the general risk-reward strategy. The problem with other benchmarks and forecast probabilities can be studied in the same way.

Firstly, we discuss the influence of the parameters on the optimal switching time. By functions $t^*_{F'}$ and $t^*_{PF'}$, we obtain Figure 1.

According to Figure 1 we can find that for fixed b_2 and λ, the optimal switching time $t^*_{F'}$ and $t^*_{PF'}$ is non-increasing with respect to b_1 and increasing with respect to c. For fixed b_1 and c, $t^*_{PF'}$ is non-increasing with respect to b_2. However, $t^*_{F'}$ does not change along with b_2, which agrees with the theoretical results. Moreover, larger risk tolerance can bring earlier optimal switching time. Generally, both $t^*_{F'}$ and $t^*_{PF'}$ are not larger than $T^{*\prime} = 1$, which means that the risk-reward strategies usually switch to Option 2 before the deterministic optimal strategy.

Next, we discuss the influence of the parameters on the optimal restricted ratios. According to the functions of $\bar{R}^*_{F'}$ and $\bar{R}^*_{PF'}$, we can obtain Figure 2.

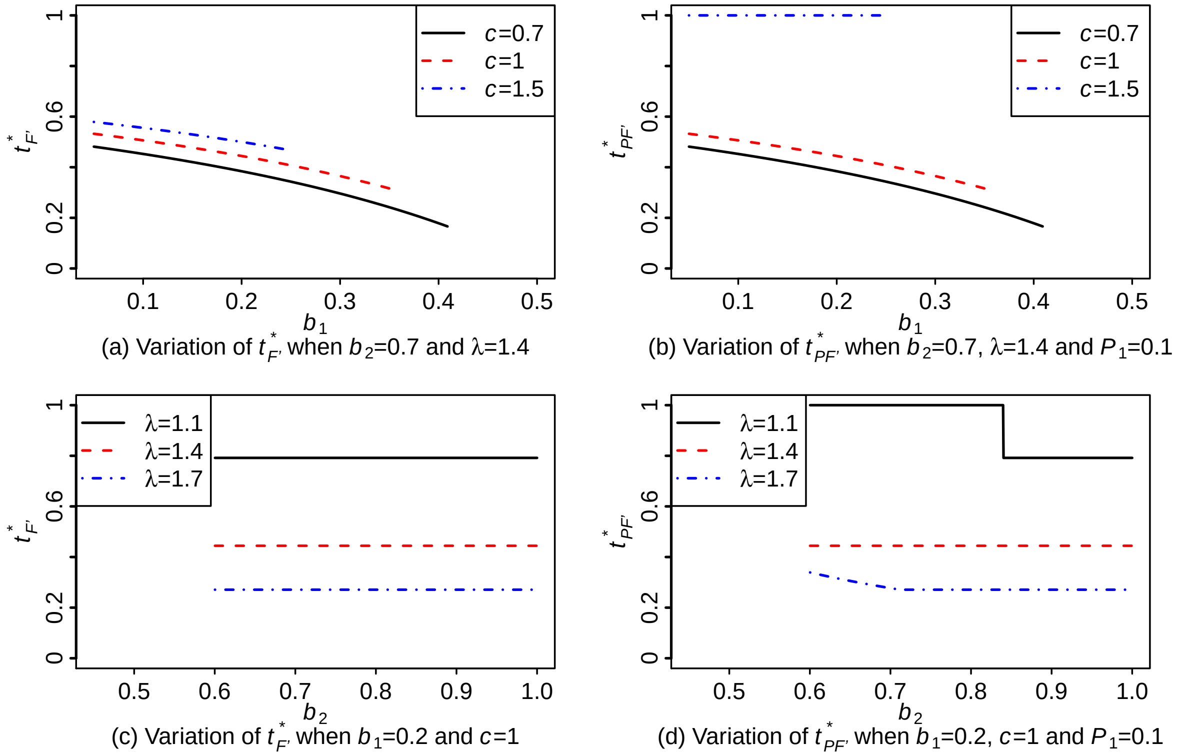

FIGURE 1. Variation of optimal switching time with different parameters

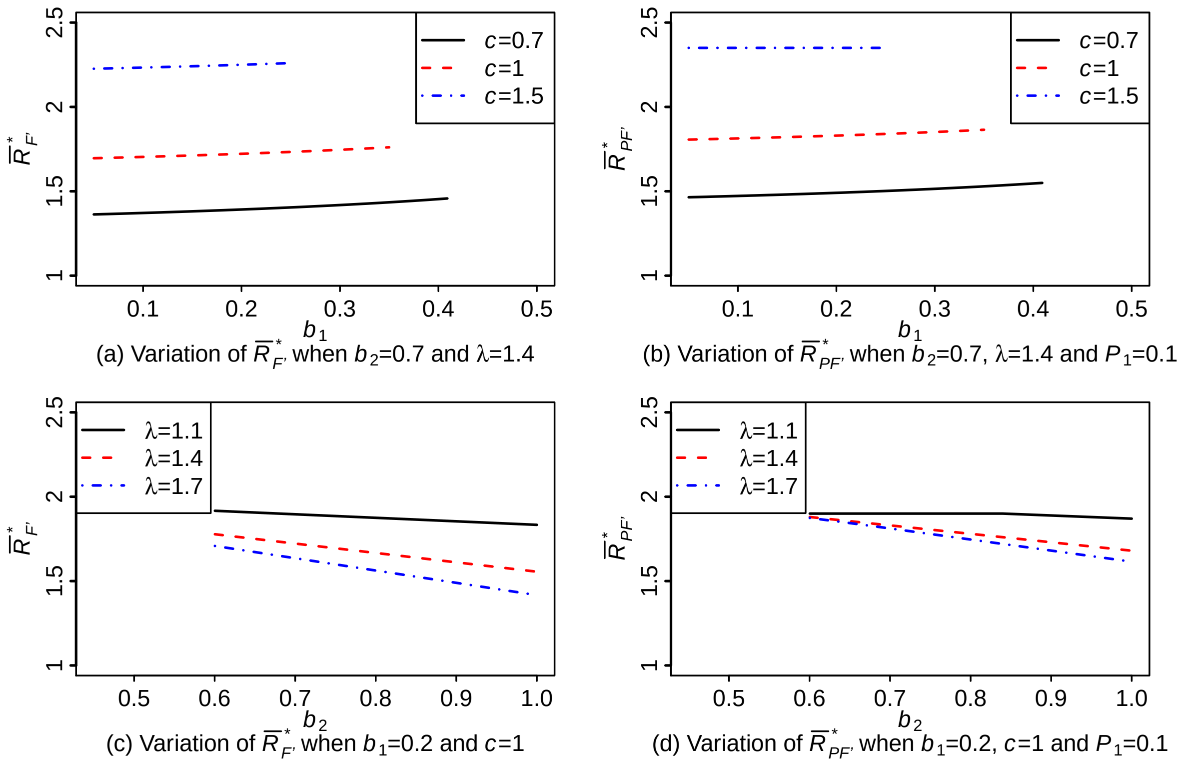

FIGURE 2. Variation of optimal restricted ratio with different parameters

According to Figure 2 we discover that the monotonicity of $\bar{R}^*_{PF'}$ with respect to b_1, b_2, c and λ is similar to that of $\bar{R}^*_{F'}$. When $b_2 = 0.7$ and $\lambda = 1.4$, $\bar{R}^*_{F'}$ and $\bar{R}^*_{PF'}$ increase with respect to b_1 for fixed c, and they also increase with respect to c for fixed b_1. In addition, when $b_1 = 0.2$ and $c = 1$, the larger the decision-maker's risk tolerance is, the smaller the restricted ratios are. And they decrease with respect ro b_2 when λ is fixed. Because $\bar{R}^{*\prime} = 1 + c$, from Figure 2 we can find that the restricted ratios are smaller than $\bar{R}^{*\prime}$, which means that both of the risk-reward strategies are superior to the deterministic optimal strategy.

Because the largest improvement is $R^{*\prime} - 1$, we refer to the studies of [13, 16] and take $imp = \frac{R^{*\prime} - \bar{R}^*_{F'}}{R^{*\prime} - 1}$ as the improvement measurement of the risk-reward strategy over the deterministic competitive ratio $R^{*\prime}$. Here we only take the traditional risk-reward strategy for example. The analysis of the improvement is similar for the general one.

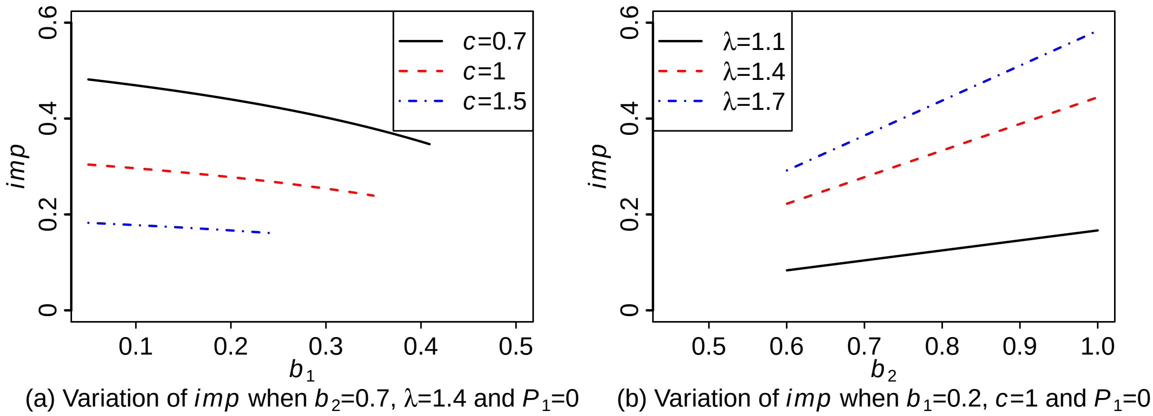

FIGURE 3. Variation of the improvement with different parameters

From Figure 3 we can find that no matter how the parameters change, the risk-reward strategy can improve the online strategy to different degrees. Especially, larger risk tolerance can bring higher improvement when other parameters are given.

Finally, we analyze the sensitivities of the optimal switching time and restricted ratio to the parameters for the traditional risk-reward strategy. The analysis for the general risk-reward strategy is similar. The decision-maker's risk tolerance does not usually change a lot in the short term, so we assume $\lambda = 1.4$ for the moment. Then according to the functions $t^*_{F'}$ and $\bar{R}^*_{F'}$ we can compute the corresponding values when b_1, b_2 and c vary. We mark the values of $t^*_{F'}$ and $\bar{R}^*_{F'}$ as the reference values when $b_1 = 0.2$, $b_2 = 0.7$ and $c = 1$, which are the benchmarks. Then the relative deviations of $t^*_{F'}$ and $\bar{R}^*_{F'}$ are $\frac{\Delta t^*_{F'}}{t^*_{F'}}$ and $\frac{\Delta \bar{R}^*_{F'}}{\bar{R}^*_{F'}}$, respectively. And we give the relative deviations in percentage when the sensitivity parameters range between -20% and 20% of the values of the benchmarks in Figure 4.

From Figure 4 it is easy to find that both $t^*_{F'}$ and $\bar{R}^*_{F'}$ are very sensitive to c. And $t^*_{F'}$ is also very sensitive to b_1. However, it is not sensitive to b_2, which fits the fact that $t^*_{F'}$ is independent of b_2. In contrast, $\bar{R}^*_{F'}$ is sensitive to b_2, but it is less sensitive to b_1. In general, b_1 and c have a great influence on the optimal risk-reward strategy. However, in comparison to b_1 and b_2, the transition cost c influences the performance of the optimal risk-reward strategy to a greater extent.

5. **Conclusion.**

In this paper, we give the optimal deterministic competitive strategy for the NTOLP. As the traditional competitive analysis does not take any information about

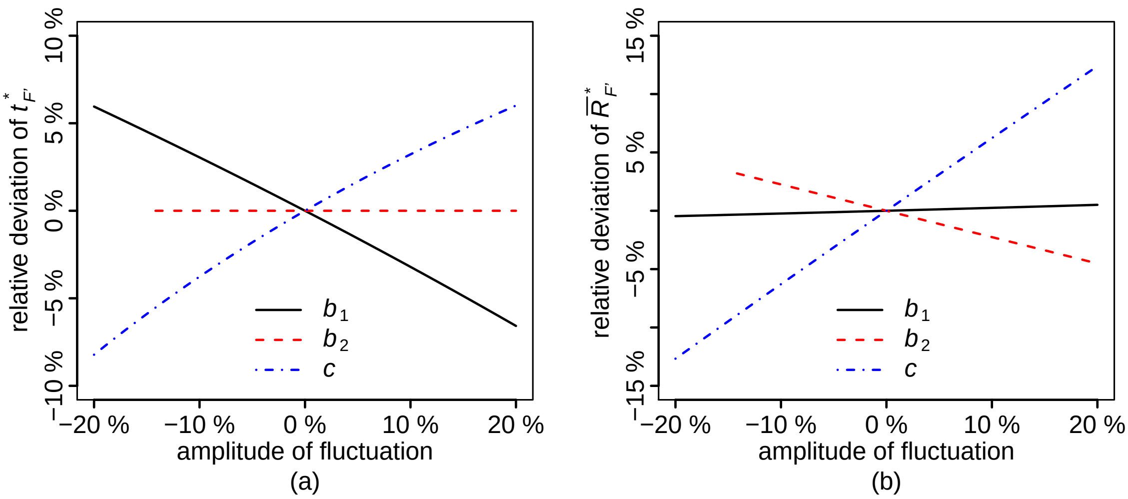

FIGURE 4. Relative deviation of $t^*_{F'}$ and $\bar{R}^*_{F'}$ for sensitivity parameters b_1, b_2 and c with $\lambda = 1.4$

the market and decision-maker's risk preference into account, we consider the decision-maker's risk tolerance and forecasts, and obtain the optimal traditional and general risk-reward strategies for $(a_1, b_1; a_2, b_2; c)$-NTOLP and (b_1, b_2, c)-NTOLP. Thus, we can know the optimal strategy and its performance according to the parameters in the market. We also analyze the influence of the parameters on the optimal risk-reward strategies by numerical analysis. And we obtain the sensitivity of the traditional risk-reward strategy to the sensitivity parameters.

We hope that our results will help the decision-maker who faces the NTOLP to make a good decision and the researchers to do further research on the NTOLP as references. In the risk-reward models, we give two special forecasts where the critical value of duration is just the critical time of the OPT algorithm. An interesting direction for future research is to consider different forecasts, for example, the decision-maker can have the forecasts with different critical time points or they can have more than two forecasts.

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