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COMMUNICATION DELAY ROBUSTNESS IMPROVEMENT OF LINEAR CONSENSUS PROTOCOL IN UNDIRECTED GRAPHS INSPIRED ON DEFORMED ALGEBRA

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Abstract. *In this paper, we present an application of the notion of q-difference in the definition of consensus protocol control effort in multiagent integrator systems. The consensus problem in multiagent systems is of high relevance in the technology sector; several applications to cooperative control or leader-follower paradigm in robot clusters, or vehicle coordination can be found in the literature. A well-known limit for the maximum admissible time delay of the linear consensus protocol in systems with integrators as agents is that shown by Saber and Murray. By applying a deformed difference inspired by q-algebra, the non-zero average consensus is shown to be attained even for values of time delay greater than that of the linear case. Additionally, the proposed consensus protocol is proved to satisfy the requisites of a non-linear action function so as to attain the average consensus. Some numerical examples are given to illustrate the advantages of the deformed algebra based consensus.*
Keywords: Consensus protocol, Time-delay, Robustness, Deformed algebra

1. **Introduction.** Group concordance, or the consensus problem, is a milestone in the theory and application of multiagent systems. Several topics, such as formation control [1-3], leader following [4,5] and rendezvous [6], can be approached utilizing constructed control laws over the graphs of the agents having intercommunication. Just over a decade ago, Saber and Murray [7] described the main features of the so-called average consensus of undirected, connected graphs with equally weighted edges, by proposing a linear control law for each integrator agent that communicates with others in its neighborhood. Moreover, the maximum uniform delay in the communication of all the agents was characterized as a function of the maximal eigenvalue of the Laplacian matrix of the graph. The maximum uniform delay in the consensus problem, besides the convergence rate to the consensus value, is a relevant merit figure to the performance of consensus, because agents change information through communication networks represented by a given graph topology. Sharing of infrastructure resources and geographical separation are two examples of causes for delay in the communication between the agents. It is known that the classical consensus studied by Saber and Murray [7] has better convergence rates for a denser graph topology, i.e., a topology with more connected agents, such as fully connected graphs. Unfortunately, the maximum communication delay, satisfies an inverse

relationship and decreases as a denser topology is employed. A number of studies since [7] have contributed to the study of the so-called convergence rate and the maximum uniform delay communication. The former is related to speed performance in reaching consensus, and the latter is a very important merit figure for robustness regarding communication delay. In [8], a set-valued approach was proposed to determine delay robustness in linear protocols. In [9], this approach was extended to the case of agents with non-identical dynamics. Regarding the convergence rate and delay robustness in some commonly used fixed topologies, treatment is given to the classical average consensus as a function of the number of agents in [10]. Some recent and relevant contributions to less conservative bounds in delay robustness are discussed in [11], including an analysis of the presence of disturbances and uncertainties in the network topology. In [12], an approach called cluster treatment of characteristic roots was applied to exact computing bounds on delay robustness of linear protocols. In this paper, we approach the average consensus problem using an action function for the graph. We do this by taking an operator inspired by a deformed difference proposed by Borges [13] to achieve higher limits in the maximum communication delay with stability preservation. This deformed difference, named q-difference, belongs to the class of a q-algebra constructed based on the non-extensive entropy of Tsallis [14]. Thus, the main contribution of the present study is a methodology to enlarge the tolerance to delay communication by proposing a novel average consensus protocol. The rest of the paper comprises the preliminary arguments and a brief statement of the problem, followed by a concise description of the proposed approach. Numerical examples are then given to illustrate the merit of the proposal.

2. Preliminaries and Statement of the Problem.

2.1. The average consensus in graphs with integrators.

Let an undirected and connected graph of order n be represented by $\mathcal{G}_n = (v_n, \varepsilon_n)$, consisting of the set of nodes $v = \{v_1, \ldots, v_n\}$ and the set of edges $\varepsilon_n \subseteq v_n \times v_n$, where n is the number of agents. The nodes belong to a finite index set $\Gamma = 1, \ldots, n$. Since the edges of $\mathcal{G}_n$ are denoted by $e_{ij} = (v_i, v_j)$, the set of neighbors of a node v_i is denoted by $\mathcal{N}_i = v_j \in v : (v_i, v_j) \in \varepsilon$. Let $\xi_i \in \mathcal{R}_n$ denote the information state or, simple state, associated with decision group value of the ith agent. Then, we can define $\mathcal{G}_\xi = (\mathcal{G}_n, \xi)$, where $\xi = (\xi_1, \ldots, \xi_n)$, representing a network with full communication topology $\mathcal{G}_n$. Additionally, each agent i is related to node v_i having dynamics and $u_i \in \mathcal{R}_n$ is the input control of the agent. Finally, we define a dynamical system $\mathcal{G}_\xi = (\mathcal{G}_n, \xi)$, in which $\dot{\xi}_i = f(\xi_i, u_i)$. We consider the average consensus problem for first order integrator systems as given in [7]:

$$\dot{\xi}_i(t) = u_i(t), \tag{1}$$

where ξ_i and u_i are, respectively, the state of information and the control effort for the ith agent. The average consensus $\lim_{t\to\infty} \xi_i(t) = \frac{1}{n}\mathbf{1}^T\xi(0)$ can be achieved by applying the control law:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \left[\xi_j(t-\tau) - \xi_i(t-\tau)\right]. \tag{2}$$

Summarily, the set $\mathcal{N}$ just assigns the graph topology, i.e., the communication between the agents; τ is the uniform communication delay for information interchange between any pair of agents in the system. The closed-loop dynamics is given as:

$$\dot{\xi}(t) = -L\xi(t-\tau), \tag{3}$$

where L is the Laplacian matrix of $\mathcal{G}$. In a system with no delay, i.e., $\tau = 0$, due to L being symmetric and positive semidefinite with exactly one null eigenvalue, system (3) will always be Lyapunov stable. The maximum communication delay, τ^*, for which the consensus point $\chi^* = \frac{1}{n}\mathbf{1}^T\xi(0)$ is globally asymptotically stable is given as (see [7]):

$$\tau < \tau^* = \frac{\pi}{2\lambda_L^{\max}}. \tag{4}$$

The linear protocol (2) is only one of a larger class of consensus protocols defined in [7]. The general class of average consensus protocols for undirected, connected graphs is given by the action graphs defined in [7] for the case of single integrator agents:

$$u_i(t) = \sum_{j\in\mathcal{N}} \phi_{ij}\left[\xi_j(t) - \xi_i(t)\right], \tag{5}$$

where $\phi_{ij}\left[\xi_j(t) - \xi_i(t)\right]$ is a function, possibly non-linear, of the information state at the ith node and its neighbor. The protocol given by (2) is an example of the class (5). The conditions on this function such that an action graph reaches average consensus are described in [7].

2.2. **Deformed difference or *q*-difference.** The extensive statistical mechanics based on the Tsallis' entropy is a milestone for physicists, with several models of physical phenomena. It is best explained using the q-entropy proposed by Tsallis [14]. Other applications to physics and engineering were inspired by classes of q-algebras derived from Tsallis' entropy [15,16]. We explore the following deformed difference of two real numbers x and y taken from [13]:

$$x \ominus_q y \equiv \frac{x-y}{1+(1-q)y}, \quad x,\ y \in \mathbb{R}, \quad y \neq \frac{1}{q-1}, \tag{6}$$

with $q \in \mathbb{R}$. Notice that the ordinary algebraic difference is recovered whenever $q \equiv 1$. The q-difference in Equation (6) is nonsymmetrical with respect to the signal of the minuend y and then, later on, to match one of the conditions to construct an action graph, we define a slightly modified q-difference as follows:

$$x \boxminus_q y \equiv \frac{x-y}{1+(1-q)|y|}, \quad x, y \in \mathbb{R}. \tag{7}$$

Notice that there is no constraint on the value of the minuend y compared to that in the original definition. The central problem studied here can be stated as follows.

Problem: Given a multiagent system with an undirected connected graph, to deform the classical linear protocols using an action graph inspired by (7) such that the maximum uniform delay communication delay is higher than (4) in a system with agents of single integrator dynamics.

3. **The Proposed Approach.** Towards a solution to the central problem, the following action graph is then proposed for average consensus:

$$\begin{aligned} u_i(t) &= \sum_{j\in\mathcal{N}_i} \phi_{ji}\left[\xi_j(t) - \xi_i(t), \xi_j(t), \xi_i(t)\right], \\ \phi_{ji}\left[\xi_j(t) - \xi_i(t), \xi_j(t), \xi_i(t)\right] &= \frac{1}{2}\left\{\left[\xi_j(t) \boxminus_q \xi_i(t)\right] - \left[\xi_i(t) \boxminus_q \xi_j(t)\right]\right\}. \end{aligned} \tag{8}$$

Notice that (8) recovers the conventional protocol (4) if $q \equiv 1$. The development of this action graph is used to verify the conditions to reach average consensus in accordance to

[7]. One can then verify that:

$$\begin{aligned}&\frac{1}{2}\left\{[\xi_j(t) \boxminus_q \xi_i(t)] - [\xi_i(t) \boxminus_q \xi_j(t)]\right\}\\ &= \frac{1}{2}\left\{\frac{\xi_j(t)-\xi_i(t)}{1+(1-q)|\xi_i(t)|} - \frac{\xi_i(t)-\xi_j(t)}{1+(1-q)|\xi_j(t)|}\right\}.\end{aligned} \tag{9}$$

By defining $\xi_j(t) - \xi_i(t) \equiv z$, one has:

$$\phi(z) = \frac{z}{2}\left\{\frac{1}{1+(1-q)|\xi_i(t)|} + \frac{1}{1+(1-q)|\xi_j(t)|}\right\}. \tag{10}$$

The deformation parameter q is chosen in the interval $[0, 1]$. Thus, the following conditions can be verified:

i. $\phi(z)$ is continuous and has a unitary Lipschitz constant;
ii. $\phi(0) = 0$;
iii. $\phi(-z) = -\phi(z)$;
iv. $(z_1 - z_2)[\phi(z_1) - \phi(z_2)] = \frac{(z_1-z_2)^2}{2}\left\{\frac{1}{1+(1-q)|\xi_i(t)|} + \frac{1}{1+(1-q)|\xi_j(t)|}\right\} > 0$.

Properties (ii), (iii) and (iv) can be verified in a straightforward manner. Property (i) can be proved by observing that the derivative satisfies

$$\frac{\partial\phi}{\partial z} = \frac{1}{2}\left\{\frac{1}{1+(1-q)|\xi_i(t)|} + \frac{1}{1+(1-q)|\xi_j(t)|}\right\} \leq 1,$$

thus showing that property (i) holds. According to [7], one can conclude that the protocol (8) solves the average consensus problem. To evaluate the improvement in the uniform communication time delay as a function of q, let $\delta_k(t)$, $k = 1, \ldots, n$ be small disagreements around the consensus value such that $\sum_{k=1}^{n} \delta_k(t) = 0$. By replacing $\xi_k(t) = \chi^* + \delta_k(t)$, one obtains the following dynamic vector equations of the small disagreements:

$$\dot{\delta}(t) = -\frac{1}{1+(1-q)|\chi^*|} L\delta(t-\tau), \tag{11}$$

Equation (11) is analogous to (3), and such dynamics will be globally asymptotically stable if and only if:

$$\tau < \tau_q^* = (1 + (1-q)|\chi^*|)\tau^*. \tag{12}$$

Thus, improvements in the maximum uniform communication time delay can be achieved with $q < 1$ and $|\chi^*| \neq 0$.

4. **Numerical Examples.**

In this section, three numerical examples are given to show the effectiveness of the protocol based on the adapted q-difference. The first two examples can be encountered in several applications of flight formation control of unmanned aerial vehicles – UAVs [17], and the third one can reflect flocking formation of UAVs' swarm [18]. The graph topologies Examples 4.1 and 4.2 are illustrated in Figures 1(a) and 1(b), respectively.

4.1. **An undirected graph with three agents.**

The first example is an undirected, connected graph with $n = 3$, as shown in Figure 1(a). The Laplacian matrix and the maximum uniform communication delay for this case are:

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \tau^* = \frac{\pi}{6}.$$

The protocol (8) was applied to the system for values of $q \in \{0, 0.5, 0.9, 1\}$, with initial conditions $\xi(0) = [2 \quad 1 \quad 7]^T$ and $\tau = \tau^*$. The results are presented in Figure 2. One

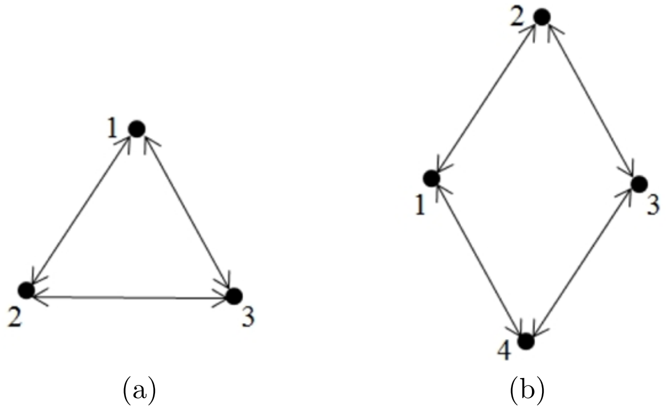

Figure 1. Graphs for Examples 4.1 and 4.2

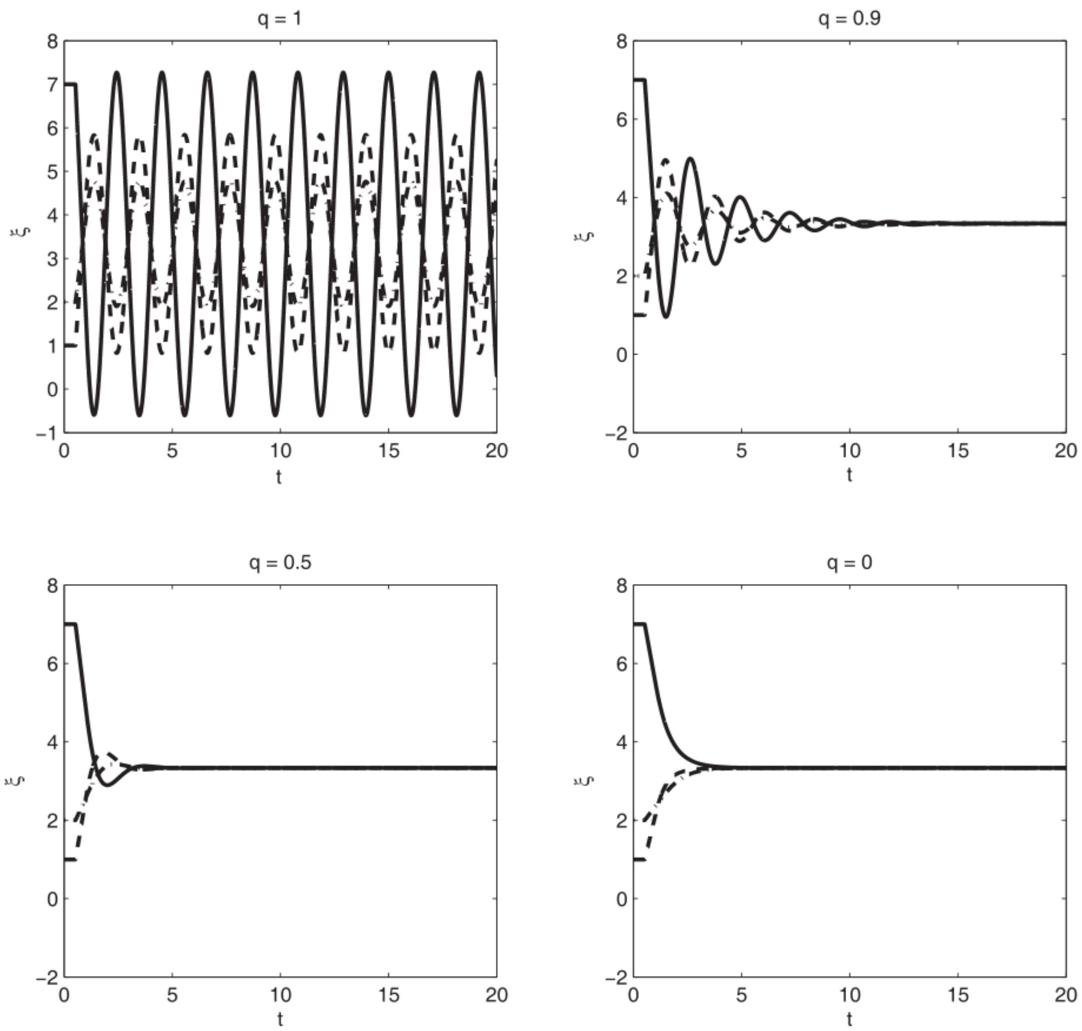

Figure 2. Information state ξ for average consensus protocol (8) in the multiagent graph of Example 4.1

can notice that for $q = 1$, the classical protocol (2) is recovered, with the system in a marginally stable scenario. A better performance is evident for smaller values of the deformation parameter q and, in general, the average consensus is achieved faster for smaller values of q. In Figure 3, the communication delay has been enlarged as in (12), and

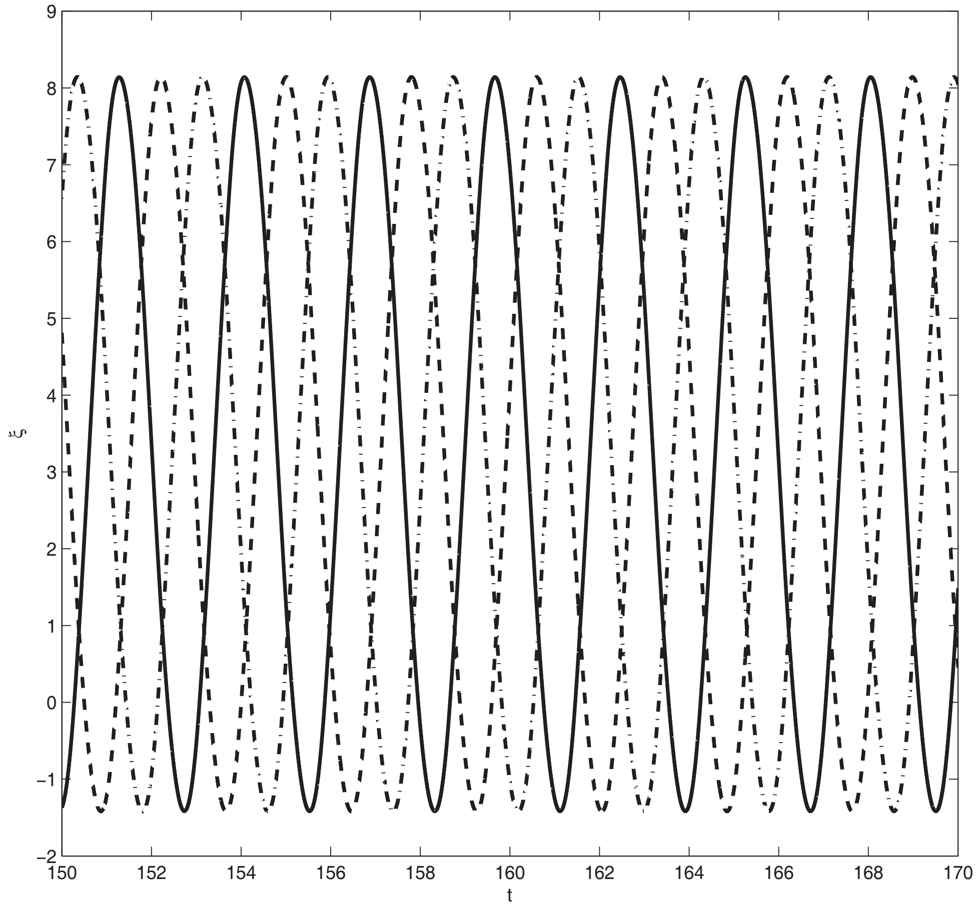

FIGURE 3. Sustainable oscillation of the system in Example 4.1 after enlargement of the uniform communication delay according to (12) and $q = 0.9$

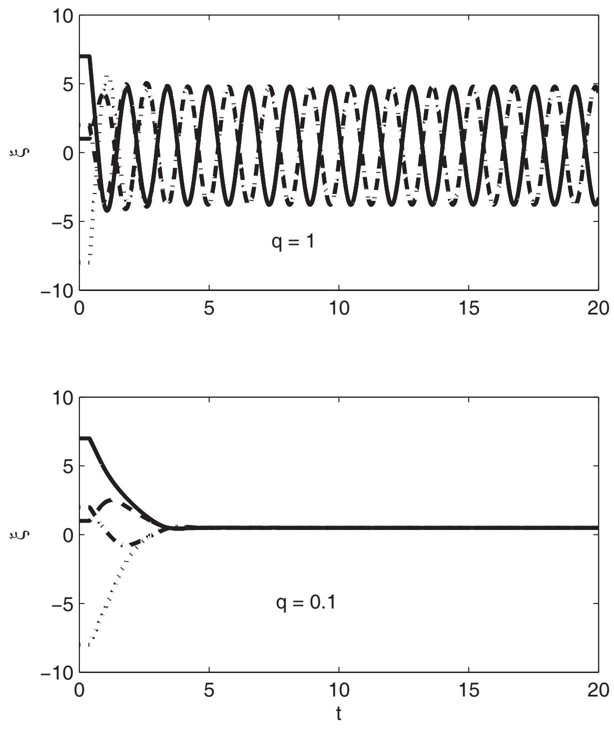

FIGURE 4. Consensus performance for the system in Example 4.2

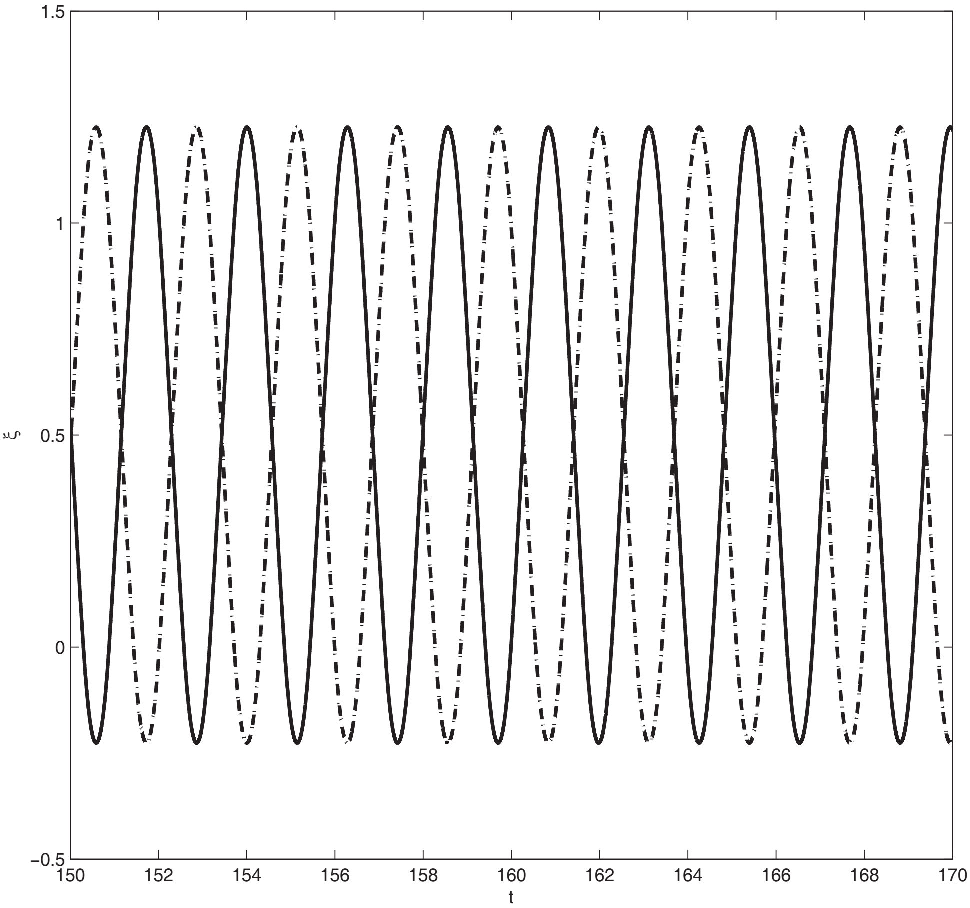

FIGURE 5. Sustainable oscillation of the system in Example 4.2 after enlargement of the uniform communication delay according to (12) and $q = 0.1$

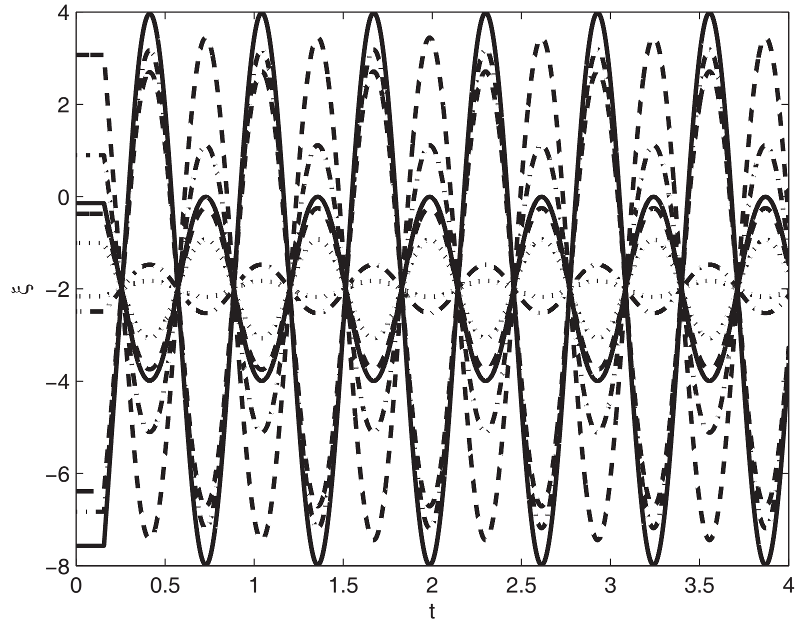

FIGURE 6. Sustainable oscillation of the system in Example 4.3 with maximum tolerable uniform delay communication of protocol (2)

the system presents a sustainable oscillation as predicted in our analysis, thus confirming an improvement in robustness of the system by supporting lager communication delays.

4.2. **An undirected graph with four agents.** In this example, whose communication graph is depicted in Figure 1(b), one has the Laplacian matrix and the maximum uniform communication delay as follows:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad \tau^* = \frac{\pi}{8}.$$

Protocol (8) was simulated for $q = 1$ and $q = 0.1$. Again, the classical linear protocol (2) is recovered by setting $q = 1$. The results of Figure 4 make clear that the best performance is obtained by an application of the protocol deformed by the q-difference, with average consensus being achieved faster when $q = 0.1$. In Figure 5, the communication delay has been enlarged as in (12), and the system presents a sustainable oscillation as predicted in our analysis. As in the first example, the system with deformed protocol supports longer delays and thus the robustness is proved to be better again.

4.3. **An undirected graph with ten agents in full communication or star topology.** This example is given for a comparison with [10], which establishes an upper bound for time delay communications, i.e., delay robustness to some common networked agent graph topology. For the case of undirected full communication, or star topology, of n agents, the maximum uniform communication delay for the protocol (2) can be computed as:

$$\tau^* = \frac{\pi}{2n}. \tag{13}$$

In particular, for a 10-agents undirected graph, one can find the value $\tau^* = \frac{\pi}{20}$s. By simulating the system response to a random initial condition $x(0)$ with mean -2 for this communication delay, the protocol (2) gives the expected sustainable oscillation seen in Figure 6. From this, the deformed protocol (8) achieves a fast convergence for $q = 0.5$ as can be seen in Figure 7. For the sake of comparison, the system is also simulated in a scenario with no delay, i.e., $\tau = 0$. The system response is displayed in Figure 8, and it is clear that the deformed consensus with $q = 0.2$ has a reasonable convergence rate under severe delay conditions when it is compared to that of an ideal no-delay scenario. In all the examples studied so far, a very simple modification in the conventional protocol (2) not only generalizes it but also gives the designer the possibility of tuning back the system capacity to cope with unexpected longer delays. The fact that there is no need modifying the communication topology or inserting dynamic compensators to improve robustness, gives the present methodology a powerful appeal for designers of consensus applications of multi-agent systems.

5. **Concluding Remarks.** In this paper, a deformed consensus protocol based on q-difference was introduced and analyzed in multiagent systems with uniform communication time delays. The proposed consensus was shown to enlarge the maximum uniform communication delay without the necessity of modifying the system topology, except the tuning of a single parameter. Numerical examples were presented to confirm a more robust performance of the proposed protocol compared with the classical one. Future research possibilities involve the study of deformed protocols in second order integrator systems or general linear dynamical systems of the agents.

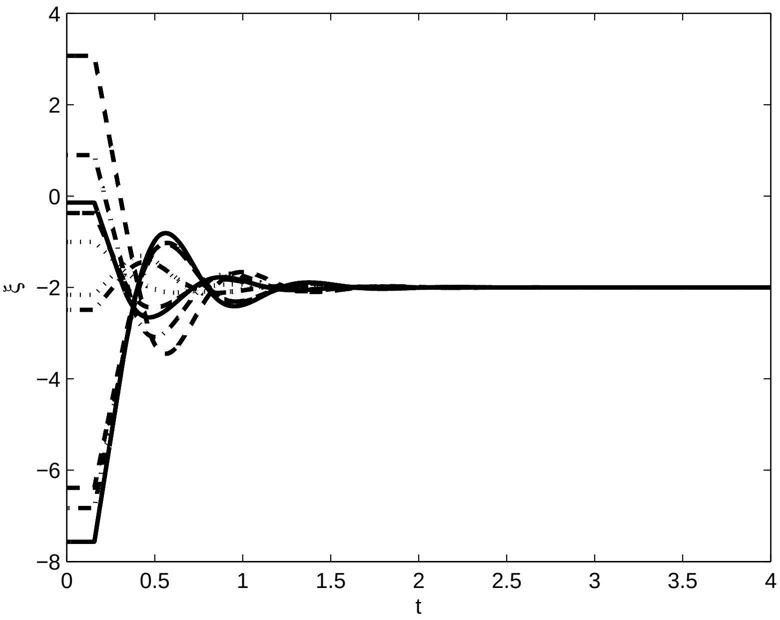

FIGURE 7. Convergence of the deformed protocol (12) with $q = 0.5$ and maximum tolerable uniform delay communication in linear protocol

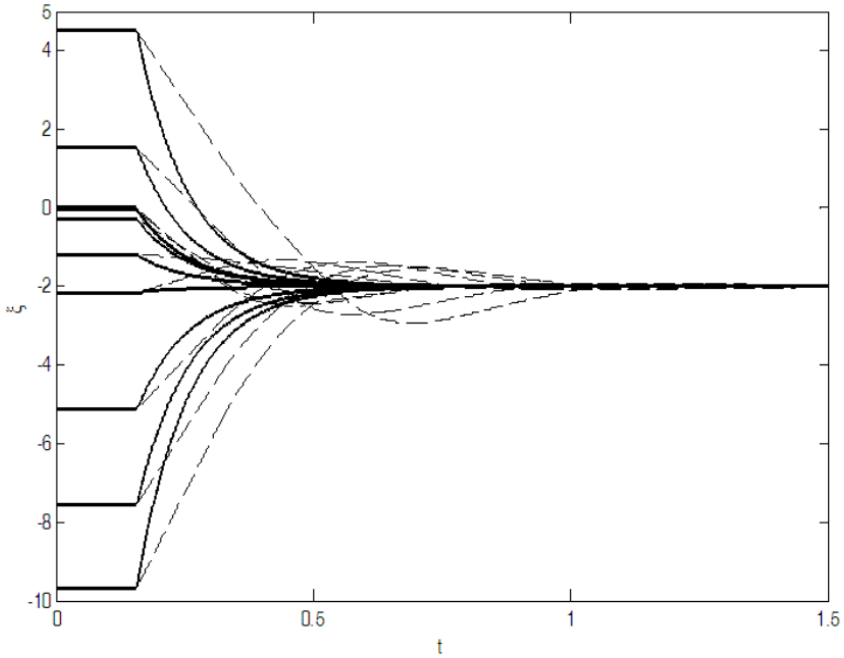

FIGURE 8. Comparison of the convergence rate in Example 4.3 for deformed protocol (8) with $q = 0.2$ with maximum tolerable uniform delay communication in linear protocol (dashed lines), and the protocol (2) with no delay (solid lines)

REFERENCES

[1] W. Ren, Consensus-based formation control strategies for multi-vehicle systems, *Proc. of the American Control Conference*, pp.4237-4242, 2006.

[2] B. Ordonez, U. F. Moreno, J. Cerqueira and L. Almeida, Generation of trajectories using predictive control for tracking consensus with sensing and connectivity constraint, *Studies in Computational Intelligence*, vol.507, pp.19-37, 2014.

[3] H. Huang and Q. Wu, Distributed H_∞ cooperative control of multiple agents to make formations, *International Journal of Innovative Computing, Information and Control*, vol.6, no.11, pp.5221-5236, 2010.

[4] C. Ma, T. Li and J. Zhang, Consensus control for leader-following multi-agent systems with measurement noises, *Journal of Systems Science and Complexity*, vol.23, no.1, pp.35-49, 2013.

[5] W. Liu and J. Huang, Adaptive leader-following consensus for a class of higher-order nonlinear multi-agent systems with directed switching networks, *Automatica*, vol.79, pp.84-92, 2017.

[6] G. Notarstefano and F. Bullo, Distributed consensus on enclosing shapes and minimum time rendezvous, *Proc. of the IEEE Conference on Decision and Control*, pp.4295-4300, 2006.

[7] R. O. Saber and R. M. Murray, Consensus protocols for networks of dynamic agents, *Proc. of the American Control Conference*, vol.2, pp.951-956, 2003.

[8] U. Münz, A. Papachristodoulou and F. Allgöwer, Delay robustness in consensus problems, *Automatica*, vol.46, no.8, pp.1252-1265, 2010.

[9] U. Münz, A. Papachristodoulou and F. Allgöwer, Delay robustness in non-identical multi-agent systems, *IEEE Trans. Automatic Control*, vol.57, no.6, pp.1597-1603, 2012.

[10] B. Yang and F.-H. Zhao, Performance analysis of distributed consensus on regular networks, *Systems Engineering Procedia*, vol.3, pp.312-318, 2012.

[11] Q. Zhang, Z. Jin, Q. Li, J. Tao, Q. Fan and X. Gao, Low conservative criteria for robust consensus of multiagent systems with delays, disturbances, and topologies uncertainties, *Mathematical Problems in Engineering*, vol.2014, 2014.

[12] R. Cepeda-Gomez, Finding the exact delay bound for consensus of linear multi-agent systems, *International Journal of Systems Science*, vol.47, no.11, pp.2598-2606, 2016.

[13] E. P. Borges, A possible deformed algebra and calculus inspired in nonextensive thermostatistics, *Physica A: Statistical Mechanics and Its Applications*, vol.340, nos.1-3, pp.95-101, 2004.

[14] C. Tsallis, A possible generalization of Boltzmann-Gibbs statistics, *Journal of Statistical Physics*, vol.52, nos.1-2, pp.479-487, 1988.

[15] A. C. P. Rosa Jr., J. C. O. de Jesus and M. A. Moret, Nonextensivity and entropy of astrophysical sources, *Physica A: Statistical Mechanics and Its Applications*, vol.392, no.23, pp.6079-6083, 2013.

[16] M. O. de Almeida, E. T. F. Santos and J. M. Araújo, Improved performance phase detector for multiplicative second-order PLL systems using deformed algebra, *Journal of Circuits, Systems and Computers*, vol.23, no.1, 2014.

[17] J. A. Guerrero and R. Lozano (eds.), *Flight Formation Control*, ISTE Ltd., Wiley-ISTE, London, 2012.

[18] B. Li, J. Li and K. Huang, Modeling and flocking consensus analysis for large-scale UAV swarms, *Mathematical Problems in Engineering*, vol.2013, 2013.