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CONTROLLABILITY AND OBSERVABILITY OF MULTI-RATE NETWORKED CONTROL SYSTEMS WITH BOTH TIME DELAY AND PACKET DROPOUT

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Abstract. *The issue of controllability and observability of multi-rate networked control systems with both time delay and packet dropout is discussed in this paper. In multi-rate networked control systems, the sampling periods of the sensors, controllers and actuators are not the same. That is to say, there are more than one sampling rates in the systems. Based on approaches of switched systems and communicate sequence, the conditions of controllability and observability for multi-rate networked control systems with both time delay and packet dropout are given in this paper. Illustrative examples are provided to demonstrate the effectiveness of the proposed methods.*
Keywords: Multi-rate networked control systems, Controllability, Observability, Time delay, Packet dropout

1. Introduction.

Recently, networked control systems (NCS) more and more are being used in industrial fields, such as auto industry, aerospace science and technology, factory automation. NCS have been attracted much attention in the control research field as the problem is not only an academically challenging, but also practical importance. Compared with traditional communication architecture networked control systems have many advantages such as reduces installation, reconfiguration and maintenance time and costs. In the new communication architecture, sensors, actuators and controllers are connect to common bus. However, signals of sensors and controllers may be lost in communication, and time delays are inevitable. Meanwhile, the control loops of NCS make it more complex to analyze.

For the issue of the time delay of NCS, many good achievements had been made. The stochastic control of integrated communication and control systems is discussed in [1], the stability of NCS is investigated in [2], and the controller and the effect of the controller to the stability of the NCS is dealt with in [3]. For the problem of packet dropout, the state feedback stabilizing controller is provided in [4], the output feedback stabilization of networked control systems with packet dropouts is presented in [5-7].

The model of NCS is more complex when the time delay and packet dropout are considered together. The arbitrary switched system model is introduced in [8], the sampled date system model is discussed in [9], the model of NCS considering both networked-induced delay and packet dropout is developed in [10]. Unfortunately, most aforementioned conclusions under the following assumption: the sampling rates of each node in NCSs are the same. This brings convenience for the theoretical research of NCSs; however, the sampling rate of each node is not identical in practical application. For the multi-rate network, the rates are not the same, the sampling period of sensor is T_s, sampling period of the controller is T_c, $T_s \neq T_c$. In recent years, the investigations of multi-rate control system have made a great progress [14-28]. The NCSs are modeled as switched systems by using multi-rate method, when sensor, controller and actuator are all event-driven, in [26], and the stability was analyzed. The exponential stability of multi-rate NCSs was analyzed including three cases of perfect transmission, delayed transmission and time-varying transmission, in [27]. In [28], the condition of stabilizing controller of multi-rate NCSs was discussed by the way of using a V-K iteration algorithm.

The controllability and observability play a fundamental role in modern control theory and parameters optimization. These properties of multi-rate NCS are the basis of the design of observer for multi-rate NCS. Due to the importance of controllability and observability in theory or applications, many prior results only about single rate NCS is established in [7,9,11-15]. What is more, most of the aforementioned conclusions are hard to be applied in multi-rates systems. Some results without packet dropout are presented in [16,17]. It is regrettable that there are not any papers put forward results on the controllability and observability in multi-rate NCS with both time delay and packet dropout so far. However, the research of multi-rate sampling networked control system is inevitable.

The main contributions of this paper are as follows. (1) The model of multi-rates networked control system with both time delay and packet dropout of multi-rate of NCS is carried out. (2) Different from the general analysis methods of NCS, we introduce a new approach by combining linear switched control system (LSCS) with communicate sequence. (3) The conditions of controllability and observability of multi-rate NCS with both time delay and packet dropout is proposed.

For conciseness, we simply use the term "system" instead of "multi-rate networked control system". The organization of this paper is as follows. Section 2 presents the formulations of system. In Section 3, the controllability of the multi-rate systems is discussed. In Section 4, the conditions of observability are given. Two illustrative examples are provided in Section 5. Section 6 concludes the paper.

2. **Problem Formulations.** It is assumed that the controlled process is a linear time-invariant system, which can be expressed as:

$$\begin{aligned} \dot{x}(t) &= A^c x(t) + B^c u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and A^c, B^c, C are matrices of appropriate sizes, $v(t)$ is white noise with zero mean. The sampling period of the sensor is noted as T_s, the sampling periods of controller and actuator are the same and noted as T_c. $T_c = T_s/N$ and N is a positive integer not less than 2. That is to say, the controller and actuator have a higher sampling frequency than the sensor. In order to facilitate the discussion, in this paper, the delay of the system is considered to be a constant short delay.

For convenience of investigation, we make the following rational assumptions.

A1: The sensor and the actuator are all time driven, and sensor to controller delay is noted as τ_{sc}, the controller to actuator delay is denoted by τ_{ca}. The delay of the system is $\tau = \tau_{sc} + \tau_{ca} = hT_c$, $h \le N$, h is a positive integer.
A2: The number of successive packet dropouts is upper bounded, and the bound is denoted a know constant.
A3: The system adopts the Zero Order Hold (ZOH) strategy.

3. Controllability of Multi-rate NCS both with Time Delay and Packet Dropout.

3.1. Modelling of multi-rate NCS with networked induced delay. In this section, the packet dropout model of multi-rate NCS is not considered. Because of the sampling rate of the controller is higher than sensor, the control signal within a sampling period has to be divided into subintervals corresponding to the controller's sampling period. Within each subinterval, the control signal is a constant. The continuous time plant may be discrete as follows.

$$
\begin{aligned}
&x[(k+1)T_s] \\
&= Ax(kT_s) + \left(\int_{kT_s}^{kT_s+hT_c} e^{A^c[(k+1)T_s-\eta]}B^c d\eta\right) u[(k-1)T_s] \\
&\quad + \left(\int_{kT_s+hT_c}^{(k+1)T_s} e^{A^c[(k+1)T_s-\eta]}B^c d\eta\right) u(kT_s) \\
&= Ax(kT_s) + \left(\int_0^{(N-h)T_c} e^{A^c\eta_1}B^c d\eta_1\right) u(kT_s) + \left(\int_{(N-h)T_c}^{NT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-1)T_s] \\
&= Ax(kT_s) + \left(-\int_{T_s-hT_c}^{0} e^{A^c\eta_1}B^c d\eta_1\right) u(kT_s) + \left(-\int_{T_s}^{T_s-hT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-1)T_s] \\
&= Ax(kT_s) + \left(\int_0^{T_c} e^{A^c\eta_1}B^c d\eta_1 + \int_{T_c}^{2T_c} e^{A^c\eta_1}B_c d\eta_1 + \ldots + \int_{(N-h-1)T_c}^{(N-h)T_c} e^{A^c\eta_1}B^c d\eta_1\right) u(kT_s) \\
&\quad + \left(\int_{(N-h)T_c}^{(N-h+1)T_c} e^{A^c\eta_1}B^c d\eta_1 + \int_{(N-h+1)T_c}^{(N-h+2)T_c} e^{A^c\eta_1}B^c d\eta_1 + \ldots + \int_{(N-h)T_c}^{(N-h+1)T_c} e^{A^c\eta_1}B^c d\eta_1\right. \\
&\quad \left. + \int_{(N-h+1)T_c}^{(N-h+2)T_c} e^{A^c\eta_1}B^c d\eta_1 + \ldots + \int_{(N-1)T_c}^{NT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-1)T_s] \\
&= Ax(kT_s) + (B_1 + B_2 + \ldots + B_{N-h})u(kT_s) \\
&\quad + (B_{N-h+1} + B_{N-h+2} + \ldots + B_N)u[(k-1)T_s]
\end{aligned}
$$

where $A = e^{A^cT_s}$, $B_1 = \int_0^{T_c} e^{A^c\eta_1}B^c d\eta_1$, $B_k = A^{k-1}B_1$, $1 \le k \le N$.

3.2. Modelling of multi-rate NCS with both time delay and packet dropout. When packet dropout occurs, the control signal is lost. The timing sequence diagram of the signal in system is illustrated in Figure 1. The controller will adopt the latest input signal, since the system adopted ZOH strategy. Such as in the interval $[(k+2)T_s, (k+3)T_s]$, packet dropout and time delay exist simultaneously, the latest effective input will be used for the system. For packet dropout, there are three different cases as follows.

Case $S_i^{(0)}$: There is no packet dropout in the current sampling interval and the previous one. This case is discussed in the previous section. Such as in the interval $[(k+5)T_s, (k+6)T_s]$.

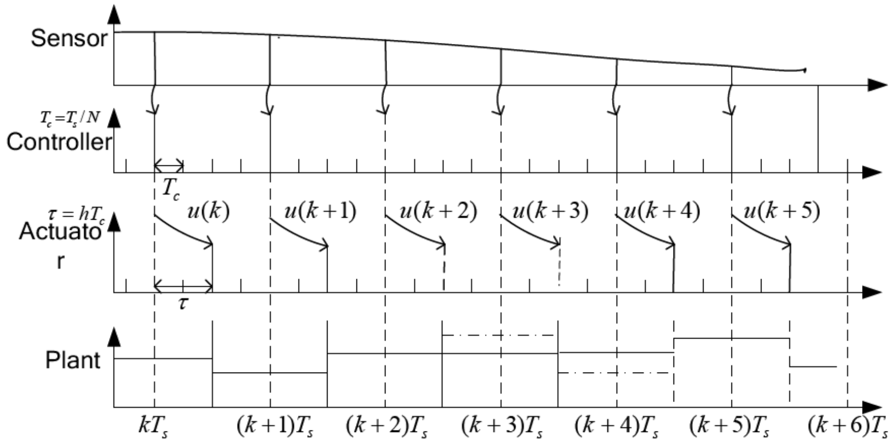

FIGURE 1. The timing sequence diagram of the signal in multi-rate NCS

Case $S_i^{(1)}$: There are i successive packet dropouts in the current sampling interval. Such as in interval $[(k+2)T_s, (k+3)T_s]$ to interval $[(k+3)T_s, (k+4)T_s]$. Packet dropouts exist in 2 successive sampling periods, the inputs of actuator in this period are the latest effective inputs.

$$\begin{aligned}
x[(k+1)T_s] &= Ax(kT_s) + \left(\int_{kT_s}^{kT_s+T_s} e^{A^c[(k+1)T_s-\eta]}B^c d\eta\right) u[(k-i)T_s] \\
&= Ax(kT_s) + \left(-\int_{T_s}^{0} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-i)T_s] \\
&= Ax(kT_s) + \left(\int_{0}^{NT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-i)T_s] \\
&= Ax(kT_s) + \left(\int_{0}^{T_c} e^{A^c\eta_1}B^c d\eta_1 + \int_{T_c}^{2T_c} e^{A^c\eta_1}B^c d\eta_1 + \ldots\right. \\
&\quad \left. + \int_{(N-1)T_c}^{NT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-i)T_s] \\
&= Ax(kT_s) + (B_1 + B_2 + \ldots + B_N)u[(k-i)T_s]
\end{aligned}$$

where $A = e^{A^cT_s}$, $B_1 = \int_0^{T_c} e^{A^c\eta_1}B^c d\eta_1$, $B_k = A^{k-1}B_1$, $1 \le k \le N$.

Case $S_i^{(2)}$: There is no packet dropout within the current sampling interval, there are i successive packet dropouts in the last period. Such as in the interval $[(k+4)T_s, (k+5)T_s]$ packet dropout does not happen, but in the interval before it happens.

$$\begin{aligned}
x[(k+1)T_s] &= Ax(kT_s) + \left(\int_{kT_s}^{kT_s+hT_c} e^{A^c[(k+1)T_s-\eta]}B^c d\eta\right) u[(k-i-1)T_s] \\
&\quad + \left(\int_{kT_s+hT_c}^{kT_s+T_s} e^{A^c[(k+1)T_s-\eta]}B^c d\eta\right) u(kT_s) \\
&= Ax(kT_s) + \left(-\int_{T_s-hT_c}^{0} e^{A^c\eta_1}B^c d\eta_1\right) u(kT_s) \\
&\quad + \left(-\int_{T_s}^{T_s-hT_c} e^{A^c\eta_1}B^c d\eta_1\right) u[(k-i-1)T_s]
\end{aligned}$$

$$
\begin{aligned}
&= Ax(kT_s) + \left(\int_0^{(N-h)T_c} e^{A^c\eta_1} B^c d\eta_1\right) u(kT_s) + \left(\int_{(N-h)T_c}^{NT_c} e^{A\eta_1} B^c d\eta_1\right) u[(k-i-1)T_s] \\
&= Ax(kT_s) + \left(\int_0^{T_c} e^{A^c\eta_1} B^c d\eta_1 + \int_{T_c}^{2T_c} e^{A^c\eta_1} B^c d\eta_1 + \ldots + \int_{(N-h-1)T_c}^{(N-h)T_c} e^{A^c\eta_1} B^c d\eta_1\right) u(kT_s) \\
&\quad + \left(\int_{(N-h)T_c}^{(N-h+1)T_c} e^{A^c\eta_1} B^c d\eta_1 + \int_{(N-h+1)T_c}^{(N-h+2)T_c} e^{A^c\eta_1} B^c d\eta_1 + \ldots \right. \\
&\quad \left. + \int_{(N-1)T_c}^{NT_c} e^{A^c\eta_1} B^c d\eta_1\right) u[(k-i-1)T_s] \\
&= Ax(kT_s) + (B_1 + B_2 + \ldots + B_{N-h}) u(k)T_s \\
&\quad + (B_{N-h+1} + B_{N-h+2} + \ldots + B_N) u[(k-i-1)T_s]
\end{aligned}
$$

where $A = e^{A^cT_s}$, $B_1 = \int_0^{T_c} e^{A^c\eta_1} B^c d\eta_1$, $B_k = A^{k-1}B_1$, $1 \le k \le N$.

Three cases can be summarized as follows, by ignoring T_s.

$$
\begin{array}{l}
S_i^{(0)} : x(k+1) = Ax(k) + \Gamma_0 u(k) + \Gamma_1 u(k-1) \\
S_i^{(1)} : x(k+1) = Ax(k) + \Gamma_2 u(k-i) \\
S_i^{(2)} : x(k+1) = Ax(k) + \Gamma_0 u(k) + \Gamma_1 u(k-1-i)
\end{array}
\tag{2}
$$

where $\Gamma_0 = B_1 + B_2 + \cdots + B_{N-h}$, $\Gamma_1 = B_{N-h+1} + B_{N-h+2} + \cdots + B_N$, $\Gamma_2 = \Gamma_0 + \Gamma_1$.

3.3. **Controllability of multi-rate NCS both with time delay and packet dropout.** In multi-rate NCS, the packet dropout phenomenon of each sampling period can be described by a sequence of binary. In a multi-rate NCS, let $s(k)$ denote $[T_s, (k+1)T_s]$. If packet dropout occurs in this interval in the system, letting $s(k) = 0$, otherwise letting $s(k) = 1$. For example, the system has 6 sampling periods $s(0)$, $s(1)$, $s(2)$, $s(3)$, $s(4)$, $s(5)$, where packet dropout occurs in $[0, T_s]$, $[T_s, 2T_s]$ and $[3T_s, 4T_s]$.

Then $s(0) = 0$, $s(1) = 0$, $s(2) = 1$, $s(3) = 0$, $s(4) = 1$, $s(5) = 1$, $\pi_p = \{001011\}$.

$S_i^{(0)}$ is the model of system without dropout, $S_i^{(1)}$ is the model of system with dropout, as the number of successive packet dropouts is upper bounded, so there must be a data can be received, and at this time, the model of system is $S_i^{(2)}$. The three cases can be in series to look as a basic unit. And many basic units can be in series to represent all cases of system. For example, 1 means no dropout, and 0 means dropout.

The sequence 1111111111111111111111 means no dropout in systems, the number of $S_i^{(0)}$ is 21, the number of $S_i^{(1)}$ is 0, and the number of $S_i^{(2)}$is 1.

The sequence 1111110000001 means there are dropouts in systems, the number of $S_i^{(0)}$ is 6, the number of $S_i^{(1)}$ is 6, and the number of $S_i^{(2)}$ is 1.

The sequence 111111000000.... does not exist, because the number of successive packet-dropouts is upper bounded.

If the sequence of 0 or 1 with the same color is denoted as a unit, then the sequence 1111110000001000111 is made of three units.

No matter how does 0 and 1 arrange irregularly, the sequence can be made of these three units. For example, the sequence 11110101001101000000000011011111111111101 is made of these three units.

That is to say, the system can be in series by these three units, which will be convenient to analyze the controllability of the system. Firstly, we introduce the simple case, and then extended to the complex cases. By this way, we can analyze the controllability of NCS with any case.

A system has p_1 sampling intervals, $S_i^{(0)}$ i_1 times, $S_i^{(1)}$ j_1 times, and $S_i^{(2)}$ only one time. Let $\pi_{p_1} = \{s(0), s(1), \ldots, s(i_1)..s(p_1-1)\} = (\underbrace{\overbrace{111....1}^{i_1}\overbrace{000.....0}^{j_1}1}_{p_1}) = \{p_1, i_1, j_1\}$ denote the system, $p_1 = i_1 + j_1 + 1$. ($s(0) = s(1) = s(2) = \ldots = s(i_1) = 1$, $s(i_1+1) = s(i_1+2) = \ldots s(i_1+j_1) = 0$).

Then,

$$\begin{aligned}
x(1) &= Ax(0) + \Gamma_0 u(0) \\
x(2) &= A^2x(0) + (A\Gamma_0 + \Gamma_1)u(0) + \Gamma_0 u(1) \\
&\ldots \\
x(i_1) &= A^{i_1}x(0) + (A^{i_1-1}\Gamma_0 + A^{i_1-2}\Gamma_1)u(0) + (A^{i_1-2}\Gamma_0 + A^{i_1-3}\Gamma_1)u(1) + \ldots \\
&\quad + \Gamma_1 u(i_1-1) \\
x(i_1+1) &= Ax(i_1) + \Gamma_2 u(i_1-1) \\
&= A^{i_1+1}x(0) + (A^{i_1}\Gamma_0 + A^{i_1-1}\Gamma_1)u(0) + (A^{i_1-1}\Gamma_0 + A^{i_1-2}\Gamma_1)u(1) + \ldots \\
&\quad + A^2\Gamma_1 u(i_1-2) + (A\Gamma_1 + \Gamma_2)u(i_1-1) \\
&\ldots \\
x(i_1+j_1) &= Ax(i_1+j_1-1) + \Gamma_2 u(i_1-1) \\
&= A^{i_1+j_1}x(0) + (A^{i_1+j_1-1}\Gamma_0 + A^{i_1+j_1-2}\Gamma_1)u(0) + (A^{i_1+j_1-2}\Gamma_0 + A^{i_1+j_1-3}\Gamma_1)u(1) \\
&\quad + \ldots + A^{j_1+1}\Gamma_1 u(i_1-2) + (A^{j_1}\Gamma_1 + A^{j_1-1}\Gamma_2 + \ldots + \Gamma_2)u(i_1-1) \\
x(p_1) &= Ax(i_1+j_1) + \Gamma_1 u(i_1-1) + \Gamma_0 u(p_1-1) \\
&= A^{i_1+j_1+1}x(0) + (A^{i_1+j_1}\Gamma_0 + A^{i_1+j_1-1}\Gamma_1)u(0) + (A^{i_1+j_1-1}\Gamma_0 + A^{i_1+j_1-2}\Gamma_1)u(1) \\
&\quad + \ldots + A^{j_1+1}\Gamma_1 u(i_1-2) + (A^{j_1+1}\Gamma_1 + A^{j_1}\Gamma_2 + A^{j_1-1}\Gamma_2 + \ldots + A\Gamma_2 + \Gamma_1) \\
&\quad u(i_1-1) + \Gamma_0 u(p_1-1)
\end{aligned}$$

Denote $b_0, b_1, b_2, b_3, \ldots, b_{p_1-1}$ are the coefficient matrices of $u(0), u(1), .u(i-2).....u(p_1-1)$.

Hence,

$$x(p_1) = A^{p_1}x(0) + \sum_{k=1}^{p_1-1} b_{k-1}u(k-1) \tag{3}$$

where $b_{i_1} = b_{i_1+1} = \cdots b_{i_1+j_1+1} = 0$.

Theorem 3.1. *The sufficient conditions for controllability of system* $\pi_{p_1} = \{p_1, i_1, j_1\}$ *is* $rank(b_0, b_1, b_2, b_3, \ldots, b_{p_1-1}) = n$.

Proof: If initial state of the system is $x(0)$, then the solution

$$x(p_1) = A^{p_1}x(0) + \sum_{k=0}^{i_1} b_k u(k) + b_{p_1-1}u(p_1-1)$$

can be determined when the system is controllable and $k = n$, $x(k = n) = 0$. The equation can be described as

$$b_0u(0) + b_1u(1) + b_2u(2) + b_3u(3) + \ldots + b_{i_1}u(i_1) + b_{p_1-1}u(p_1-1) = -A^nx(0)$$

The equation can be solved, when $rank(b_0, b_1, b_2, b_3, \ldots, b_{p_1-1}) = n$.

For any networked control systems, packet dropout is an uncertain process that it is difficult to be described. What is more, previous works do not solve this question.

However, the complex phenomenon can be described by combining some simple binary sequences which described above.

$$\pi_p = (\underbrace{\overbrace{111....1}^{i_1}\overbrace{000.....0}^{j_1}1}_{p_1})(\underbrace{\overbrace{11...111}^{i_2}\overbrace{0......0}^{j_2}1}_{p_2})(\underbrace{\overbrace{11...111}^{i_3}\overbrace{0.....00000...0}^{j_3}1}_{p_3})$$

$$.........(\underbrace{\overbrace{11...111}^{i_v}\overbrace{0.....00000...0}^{j_v}1}_{p_v})$$

$$= \pi_{p_1}\Lambda\pi_{p_2}\Lambda\pi_{p_3}\Lambda\pi_{p_4}\Lambda........\pi_{p_v}$$

(1 represents transmit successfully, 0 represents packet dropout), it has i_1 times successfully transmitting periods; j_1 times successive packet dropout; i_2 times successfully transmitting periods and so on.

$$\pi_p = \{p, i, j\}, \quad p = p_1 + p_2 + \ldots + p_v$$

Firstly, we introduce the system combined only by two simple sequences.

$$\pi_p = \pi_{p_1}\Lambda\pi_{p_2} = (\underbrace{\overbrace{111....1}^{i_1}\overbrace{000.....0}^{j_1}1}_{p_1})(\underbrace{\overbrace{11....1}^{i_2}\overbrace{0......0}^{j_2}1}_{p_2})$$

$$\begin{aligned}
x(p_1) &= Ax(i_1+j_1) + \Gamma_1 u(i_1-1) + \Gamma_0 u(p_1-1) \\
&= A^{i_1+j_1+1}x(0) + (A^{i_1+j_1}\Gamma_0 + A^{i_1+j_1-1}\Gamma_1)u(0) + (A^{i_1+j_1-1}\Gamma_0 + A^{i_1+j_1-2}\Gamma_1)u(1) \\
&\quad + \ldots + A^{j_1+1}\Gamma_1 u(i_1-2) + (A^{j_1+1}\Gamma_1 + A^{j_1}\Gamma_2 + A^{j_1-1}\Gamma_2 + \ldots + A\Gamma_2 + \Gamma_1) \\
&\quad u(i_1-1) + \Gamma_0 u(p_1-1) \\
x(p_1+p_2) &= Ax(i_2+j_2+p_1) + \Gamma_1 u(p_1+i_2-1) + \Gamma_0 u(p_1+p_2-1) \\
&= A^{i_2+j_2+1}x(p_1) + (A^{i_2+j_2}\Gamma_0 + A^{i_2+j_2}\Gamma_1)u(p_1) + (A^{i_2+j_2}\Gamma_0 + A^{i_2+j_2}\Gamma_1)u(p_1+1) \\
&\quad + \ldots + A^{j_2+1}\Gamma_1 u(p_1+i_2-2) \\
&\quad + (A^{j_2+1}\Gamma_1 + A^{j_2}\Gamma_2 + A^{j_2-1}\Gamma_2 + \ldots + A\Gamma_2 + \Gamma_1)u(p_1+i_2-1) \\
&\quad + \Gamma_0 u(p_1+p_2-1) \\
&= A^{i_2+j_2+i_1+j_1+2}x(0) + (A^{i_2+j_2+i_1+j_1+1}\Gamma_0 + A^{i_2+j_2+i_1+j_1}\Gamma_1)u(0) \\
&\quad + (A^{i_2+j_2+i_1+j_1}\Gamma_0 + A^{i_2+j_2+i_1+j_1-1}\Gamma_1)u(1) + \ldots \\
&\quad (A^{j_1+1}\Gamma_1 + A^{j_1}\Gamma_2 + A^{j_1-1}\Gamma_2 + \ldots + A\Gamma_2 + \Gamma_1)u(i_1-1) \\
&\quad + A^{i_2+j_2+1}\Gamma_0 u(p_1-1) + (A^{i_2+j_2}\Gamma_0 + A^{i_2+j_2}\Gamma_1)u(p_1) \\
&\quad + (A^{i_2+j_2}\Gamma_0 + A^{i_2+j_2}\Gamma_1)u(p_1+1) + \ldots + A^{j_2+1}\Gamma_1 u(p_1+i_2-2) \\
&\quad + (A^{j_2+1}\Gamma_1 + A^{j_2}\Gamma_2 + A^{j_2-1}\Gamma_2 + \ldots + A\Gamma_2 + \Gamma_1)u(p_1+i_2-1) \\
&\quad + \Gamma_0 u(p_1+p_2-1)
\end{aligned} \tag{4}$$

Denote $b_0, b_{1,}b_2, b_3, \ldots, b_{p_1+p_2-1}$ are the coefficient matrices of $u(0), u(1), .u(i-2).....u(p_1+p_2-1)$

$$x(p_1+p_2) = A^{p_1+p_2}x(0) + \sum_{k=0}^{p_1+p_2-1} b(k)u(k) \tag{5}$$

where $\sum\limits_{n=1}^{2}\sum\limits_{k=i_n}^{i_n+j_n-1}\|b_k\| = 0.$

Corollary 3.1. *$rank(b_0, b_1, b_2, b_3, \ldots, b_{p_1+p_2-1}) = n$ is the necessary and sufficient conditions for controllability of system $\pi_p = \pi_{p_1}\Lambda\pi_{p_2}$.*

Corollary 3.2. *If system π_{p_1} is controllable, the system $\pi_{p_1}\Lambda\pi_{p_2}$ is controllable.*

Proof: If system π_{p_1} is controllable, that is to say $rank(\pi_{p_1}) = n$.
$rank(\pi_p) = rank(\pi_{p_1}\Lambda\pi_{p_2}) = n$, system $\pi_p = \pi_{p_1}\Lambda\pi_{p_2}$ is controllable.

$$\pi_p = (\underbrace{\overbrace{111....1}^{i_1}\overbrace{000.....0}^{j_1}1}_{p_1})(\underbrace{\overbrace{11...111}^{i_2}\overbrace{0......0}^{j_2}1}_{p_2})(\underbrace{\overbrace{11...111}^{i_3}\overbrace{0....000...0}^{j_3}1}_{p_3})$$
$$.........(\underbrace{\overbrace{11...111}^{i_v}\overbrace{0.....000...0}^{j_v}1}_{p_v})$$
$$= \pi_{p_1}\Lambda\pi_{p_2}\Lambda\pi_{p_3}\Lambda\pi_{p_4}\Lambda........\pi_{p_v}$$

Similar to (4), we can get

$$x\left(\sum_1^v p_j\right) = A^{\sum_1^v p_j}x(0) + \sum_{k=0}^{\sum_1^v p_j - 1} b(k)u(k) \tag{6}$$

where $b_0, b_1, b_2, b_3, \ldots, b_{\sum_1^v p_j - 1}$ are the coefficient matrices of $u(0), u(1), .u(i-2).....u\left(\sum_1^v p_j -1\right)$, $\sum_{n=1}^{v}\sum_{k=i_n}^{i_n+j_n-1}\|b_k\| = 0$, then we can obtain the following conclusion.

Corollary 3.3. *$rank\left(b_0, b_1, b_2, b_3, \ldots, b_{\sum_1^v p_i - 1}\right) = n$ is the necessary and sufficient condition for controllability of system $\pi_p = \pi_{p_1}\Lambda\pi_{p_2}\Lambda\pi_{p_3}\Lambda\pi_{p_4}\Lambda........\pi_{p_v}$.*

4. Observability of Multi-rate NCS both with Time Delay and Packet Dropout.

Lemma 4.1. [19] *Considering the linear discrete constant systems Σ_1 and Σ_2, Σ_1 can be observed which is the necessary and sufficient condition for observability of Σ_2, where*

$$\Sigma_1 \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{7}$$

$$\Sigma_2 \begin{cases} x(k+1) = A'x(k) + B'u(k) \\ y(k) = C'x(k) \end{cases} \tag{8}$$

There is no time delay in the system Σ_1, and the time delay of the system Σ_2 is τ.

Theorem 4.1. *If $rank(C, CA, CA^2, CA^3, \ldots, CA^n) = n$, system $\pi_p = \{p, i, j\}$ is observable.*

Proof: When analysis the observability of NCS, $y(t)$ only depends on $x(t)$ and $u(t)$ can be considered as initial states, which is omitted here. The model of observability can be got with omitting inputs, which described as the following.

$$\begin{aligned} x(k+1) &= Ax(k) \\ y(k) &= Cx(k) \end{aligned} \tag{9}$$

From Lemma 4.1, we know that the observability of NCS has nothing to do with time delay. Based on this point, the time delay can be omitted when analyze the observability of NCS both with packet dropout and time delay. Therefore, the condition for observability of system (2) is that system (9) can be observed. And we know, if $rank(C, CA, CA^2, CA^3, \ldots, CA^n) = n$, system (9) is observable. So the proof is completed.

5. **Examples.**

5.1. **Example 1.** Consider the continuous-time integrator with disturbances as follows

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} x(t)$$

Assume that the sampling period of the sensor is 0.1 second. The controller reads the receiving buffer every 0.05s, that is to say $T_c = 0.05$s, $T_s = 0.1$s and $T_c = T_s/4 = 0.025$s, $\tau = 3T_c$.

Case 1: The sequence of the system $\pi_{p_1} = \{p_1, i, j\}$ is $\{111001\}$.

Then, we have

$$A = e^{A^c T_s} = \begin{pmatrix} 1.1052 & 1.0000 & 1.0000 \\ 1.0000 & 1.2214 & 0.8187 \\ 0.9048 & 1.1052 & 1.0000 \end{pmatrix},$$
$$B_1 = \int_0^{0.05} e^{A^c(T_s-s)} B^c ds = \begin{pmatrix} 0.0513, & 0.0526, & 0 \end{pmatrix}^T$$
$$B_2 = \int_{0.05}^{1.00} e^{A^c(T_s-s)} B^c ds = \begin{pmatrix} 0.0539, & 0.0508, & 0 \end{pmatrix}^T$$
$$B_0 = \int_0^{1.00} e^{A^c(T_s-s)} B^c ds = \begin{pmatrix} 0.1052, & 0.1107, & 0 \end{pmatrix}^T$$
$$\Gamma_0 = B_1, \quad \Gamma_1 = B_2, \quad \Gamma_2 = B_1 + B_2$$

As $x(p_1) = A^{p_1} x(0) + \sum_{k=1}^{i} b_k u(k-1) + b_{p_1-1} u(p_1 - 1)$

$rank(b_0, b_1, b_2) = rank \begin{pmatrix} 0.2632 & 0.2265 & 0.1949 \\ 0.4652 & 0.3447 & 0.2553 \\ 0.6518 & 0.208 & 0.0077 \end{pmatrix} = 3$, the system is controllable.

$rank(C, CA, CA^2, CA^3) = rank \begin{pmatrix} 2.1366 & -0.7298 & -1 \\ 1.6997 & -0.4111 & -1 \\ 1.3237 & -0.1749 & -1 \\ 1 & 0 & -1 \end{pmatrix} = 3$, the system is observable.

Case 2: The sequence of the system is $\{1110010001\}$.

The system can be described as $\pi_{p_1} = \pi_{p_1} \Lambda \pi_{p_2}, \pi_{p_1} = \{111001\}, \pi_{p_2} = \{0001\}$

$rank\,(b_0, b_1, b_2) = 3$. So: $rank\,(b_0, b_1, b_2) = rank(\pi_{p_1}) = 3$.

$rank(C; CA; CA^2; CA^3) = rank(C; CA; CA^2; CA^3; CA^4) = 3$. The system can be observed and controlled.

5.2. **Example 2.** Consider the continuous-time integrator with disturbances as follows

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} x(t)$$

Assume that the sampling period of the sensor is 0.1 second. The controller reads the receiving buffer every 0.25s, that is to say $T_c = 0.025$s, $T_s = 0.1$s and $N = T_s/T_c = 4$, $\tau = 2T_c$. The sequence of the system $\pi_{p_1} = \{P_1, i, j\}$ is $\{100001\}$.

Then, we have

$$A = e^{AT_s} = \begin{bmatrix} 1.1052 & 0 & 0 \\ 0 & 1.1052 & 0 \\ 0.1105 & 0 & 1.1052 \end{bmatrix}, \quad B_1 = \int_0^{T_c} e^{A^c\eta} B^c d\eta = \begin{bmatrix} 0.0253 \\ 0 \\ 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0553 \\ 0 \\ 0.0017 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.0567 \\ 0 \\ 0.0032 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0.0581 \\ 0 \\ 0.0047 \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 0.0806 \\ 0 \\ 0.0017 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.1148 \\ 0 \\ 0.0079 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.1954 \\ 0 \\ 0.0096 \end{bmatrix}$$

$$x(p_1) = A^{P_1}x(0) + \sum_{k=1}^{i} b_{k-1}u(k-1) + b_{p_1-1}u(p_1-1)$$

$rank(b_0, b_5) < 3$ can be obtained, the system cannot be controlled.

Using Proposition 1 in [29],

$$\Gamma_{0i_1} = \begin{bmatrix} 0.0806 \\ 0 \\ 0.0017 \end{bmatrix}, \quad \Gamma_{ji_1} = \begin{bmatrix} 0.1148 \\ 0 \\ 0.0079 \end{bmatrix}; \quad \Gamma_{0i_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Gamma_{ji_2} = \begin{bmatrix} 0.1954 \\ 0 \\ 0.0096 \end{bmatrix}$$

$W_2 =< A|[\Gamma_{0i_1} + \Gamma_{ji_1}] > + < A|[\Gamma_{0i_2} + \Gamma_{ji_2}] >= R^2$. The result suggests system is controllable; however, the truth is not.

These two examples showing, the theory in this paper can cure the problem in controllability and observability of multi-rate networked control systems with both time-delay and packet-dropout, rather than [29].

6. **Conclusions.** For multi-rate networked control systems, the sampling periods of the sensor, the controller and the actuator in networked control systems are not the same, and that is to say there are more than one sampling rates in networked control systems. The model of multi-rate of NCS with both time delay and packet dropout is given in this paper when the sensor and controller are all time driven. Based on approaches of switched systems and communicate sequence, the conditions of controllability and observability for multi-rate networked control systems with both time delay and packet dropout are given in this paper.

The stabilization of multi-rate NCS has not fully been investigated. The approaches in [31,33] will be used in the future work.

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