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ON THE GLOBAL EXPONENTIAL STABILIZATION OF TAKAGI-SUGENO FUZZY UNCERTAIN SYSTEMS

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Abstract. *In this paper, the design problem of feedback controller for Takagi-Sugeno fuzzy models is considered. Some new sufficient conditions will be given to ensure the exponential stability of the fuzzy control uncertain systems. A simulation of an interconnected tanks will be given to illustrate the applicability of the main result.*
Keywords: Fuzzy systems, Uncertainties, Exponential stabilization

1. Introduction.

Over the past thirty years, Takagi-Sugeno (T-S) fuzzy model has attracted great attention, since it is proved to be a very good representation of a certain class of nonlinear dynamic systems. The common practice is as follows. First, this fuzzy model is described by a family of fuzzy if-then rules which represents local linear input-output relations of the systems. The overall fuzzy model of the system is achieved by a smooth blending of these local linear models through the membership functions. Then, based on this fuzzy model, the control design is worked out by taking full advantage of the strength of modern linear control theory. Moreover, it has been proved that a linear T-S fuzzy model is a universal approximator of any smooth nonlinear system on a compact set where the stability and controller design issues on T-S fuzzy systems have been discussed in the extensive literature, and Takagi-Sugeno fuzzy models [2, 3, 7, 9, 10, 11, 12, 13, 14] are nonlinear systems described by a set of if-then rules which gives local linear representations of an underlying system. Such models can represent exactly a wide class of nonlinear systems. Hence, it is important to study their stability or the synthesis of stabilizing controllers. The stability analysis of nonlinear systems has received considerable attention [1, 4, 6, 15, 16]. We are interested in studying the global uniform exponential stability for a class of uncertain Takagi-Sugeno fuzzy systems when the origin is an equilibrium point.

In this paper, a new approach for the stability analysis is proposed. This approach allows the computation of the bound which characterizes the exponential rate of convergence of the solutions under some assumptions on the perturbed term. The common quadratic Lyapunov function and parallel distributed compensation controller are used to show the exponential stability of solutions of the uncertain T-S fuzzy systems, provided that the uncertainties are supposed uniformly bounded by a known function.

In this work, we give some new sufficient conditions to ensure the asymptotic stability of a class of uncertain fuzzy systems. Furthermore, an illustrative numerical example is given. The remainder of this paper is organized as follows. Section 2 reviews the conventional T-S fuzzy model and issues about stability. Section 3 presents the global uniform exponential stability for T-S fuzzy uncertain systems at the origin where some controllers are constructed to ensure the exponential stability. Also new LMIs are presented in order to handle the uncertainties. Section 4 presents the example of the interconnected tanks. Finally, Section 5 draws the conclusions.

2. **Takagi-Sugeno Fuzzy Control System.** Exact mathematical models of most physical systems are difficult to obtain because of the existence of complexities and uncertainties. However, the dynamics of these systems may include linear or nonlinear behaviors for small range motion. Lyapunov's linearization method is often implemented to deal with the local dynamics of nonlinear systems and to formulate local linearized approximation. So, the complex system can be divided into a set of local mathematical models. Takagi and Sugeno have proposed an effective means of aggregating these models by using the fuzzy inferences to construct the system.

Design of fuzzy control system. The T-S fuzzy model is given by:

$$\text{Rule } l: \text{ If } z_1(t) \text{ is } F_{l1} \text{ and } z_2(t) \text{ is } F_{i2} \dots \text{ and } z_p(t) \text{ is } F_{lp}, \text{ then}$$
$$\dot{x} = A_l x + B_l u + f_l(t,x), \quad l = 1, \dots, r \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $A_l(n,n)$ constant matrix, $B_l(n,m)$ matrix control input, the functions f_l represent the uncertainties of each fuzzy subsystem and are time-varying for $l = 1, \dots, r$. F_{lk} is the fuzzy set $(k = 1, 2, \dots, p)$, $z(t) = (z_1(t), \dots, z_p(t))^T$ is the premise variable vector associated with the system states and inputs and r is the number of fuzzy rules. The center of gravity defuzzification yields the output of fuzzy system:

$$\dot{x} = \frac{\sum_{l=1}^r w_l(z)(A_l x(t) + B_l u(t) + f_l(t,x))}{\sum_{l=1}^r w_l(z)}$$

where $w_l(z) = \prod_{i=1}^p F_{li}(z_i)$ and $F_{li}(z_i)$ denoted the grade of the membership function F_{li}, corresponding to $z_i(t)$.

Let $\mu_l(z)$ be defined as:

$$\mu_l(z) = \frac{w_l(z)}{\sum_{l=1}^r w_l(z)}.$$

Then, the fuzzy system has the state-space form:

$$\dot{x} = \sum_{l=1}^r \mu_l(z)(A_l x(t) + B_l u(t) + f_l(t,x)).$$

Clearly, $\sum_{l=1}^r \mu_l(z) = 1$ and $\mu_l(z) \geq 0$ for $l = 1, \dots, r$.

The following assumption is made regarding the T-S fuzzy system (1). The pairs (A_l, B_l), $l = 1, \dots, r$ are controllable. That is, the nominal fuzzy system is locally controllable.

Based on this assumption, a state feedback control gain K_l can be obtained by pole placement design or Ackerman's formula, such that each local dynamics is stably controlled. The representation of the global control input matrix, denoted by B, is in the form:

$$B = \sum_{l=1}^r \mu_l B_l.$$

This means that the global control input matrix dominates the control performance. The design of the fuzzy controller can be taken as a linear state feedback control for the system (1) which can be defined as:

$$\text{Rule } l: \text{ If } z_1(t) \text{ is } F_{l1} \text{ and } z_2(t) \text{ is } F_{i2} \dots \text{ and } z_p(t) \text{ is } F_{lp}, \text{ then}$$
$$u(t) = K_l x(t), \quad l = 1, 2, \dots, r,$$

where K_l is the local state feedback gain. Consequently, the defuzzified result is:

$$u(t) = \sum_{l=1}^{r} \mu_l(z) K_l x(t).$$

As the first step, we need to recall what is meant by uniform global exponential stability of dynamic systems (see [1, 3, 6]). Consider a system described by

$$\dot{x} = F(t, x) \tag{S}$$

with $t \in \mathbb{R}_+$ being the time and $x \in \mathbb{R}^n$ being the state.

Definition 2.1. *The system (S) is said to be globally uniformly exponentially stable, if for all $x(t_0) \in \mathbb{R}^n$, we have:*

$$\|x(t)\| \leq \gamma \|x(t_0)\| e^{-\upsilon(t-t_0)}, \quad \textit{for all } t \geq t_0,$$

with $\gamma > 0$, $\upsilon > 0$.

The goal of this work is to find some sufficient conditions such that the fuzzy system (1) is globally uniformly exponentially stable.

3. **Control of Fuzzy Systems with Uncertainties.** Consider the T-S fuzzy model (1):

$$\text{Rule } l: \text{ If } z_1(t) \text{ is } F_{l1} \text{ and } z_2(t) \text{ is } F_{i2} \dots \text{ and } z_p(t) \text{ is } F_{lp}, \text{ then}$$
$$\dot{x} = A_l x + B_l u + f_l(t, x), \quad l = 1, \dots, r$$

The functions f_l represent the uncertainties of each fuzzy subsystem and are time-varying satisfying the following assumption:
$(\mathcal{H})$ For all $l = 1, \dots, r$,

$$\|f_l(t, x)\| \leq \rho_l(x) \|x\|, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^n,$$

where ρ_l are some nonnegative continuous functions, such that $\rho(0) = 0$.

The representation of the global nonlinearities is denoted by the following bound positive continuous function, $\rho : \mathbb{R}^n \to \mathbb{R}_+$ such that $\rho(0) = 0$ which has the form

$$\rho(x) = \left[\sum_{l=1}^{r} \rho_l^2(x) \right]^{\frac{1}{2}}.$$

We will use the following fuzzy controller:

$$u(t) = \sum_{j=1}^{r} \mu_j(z) K_j x(t). \tag{2}$$

The closed-loop system is given by

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z) \mu_j(z) [A_i + B_i K_j] x(t) + \sum_{i=1}^{r} \mu_i f_i(t, x) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i^2 G_{ii} x(t) + 2 \sum_{i<j}^{r} \mu_i \mu_j G_{ii} x(t) + \sum_{i=1}^{r} \mu_i f_i(t, x)
\end{aligned}$$

where

$$G_{ii} = A_i + B_i K_i$$
$$G_{ij} = \frac{1}{2}(A_i + B_i K_j + A_j + B_j K_i).$$

The controller synthesis initially considers the stability of the local fuzzy dynamics. That is, the stable feedback gains are determined for every subsystem. Suppose that there exists a symmetric and positive definite matrix P, and some matrices K_i, $i = 1, \dots, r$, such that the following stability conditions held:

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < -Q_i, \quad i = 1, \dots, r, \tag{3}$$

where Q_i is a positive define matrix.

Based on this assumption, each subsystem is locally controllable and a stable feedback gain is obtained. Let us consider $V(x) = x^T P x$ as a Lyapunov candidate function. The derivative of $V(x)$ with respect to time is,

$$\dot{V}(x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + P G_{ii}) x + 2 \sum_{i<j}^{r} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t, x).$$

Regarding each matrix $(G_{ii}^T P + P G_{ii})$, one has

$$\lambda_{\min}(G_{ii}^T P + P G_{ii}) \|x\|^2 \le x^T (G_{ii}^T P + P G_{ii}) x \le \lambda_{\max}(G_{ii}^T P + P G_{ii})$$

$\lambda_{\min}(.)$ (resp. $\lambda_{\max}(.)$) denotes the smallest (resp. the largest) eigenvalue of the matrix.

Define

$$\alpha = \max_{i,j} \lambda_{\max}(G_{ii}^T P + P G_{ii})$$

for $1 \le i < j \le r$. A relaxed condition concerning the coupling effect is expressed as:

$$\sum_{i<j}^{r} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x \le k \|x\|^2$$

where $k = \frac{r(r-1)}{2}\alpha$. Indeed, one has

$$\sum_{i<j}^{r} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x \le \sum_{i<j}^{r} \mu_i \mu_j \lambda_{\max}(G_{ij}^T P + P G_{ij}) \|x\|^2.$$

It follows that,

$$\sum_{i<j}^{r} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x \le \sum_{i<j}^{r} \mu_i \mu_j \max_{i,j} \lambda_{\max}(G_{ij}^T P + P G_{ij}) \|x\|^2.$$

Hence,

$$\sum_{i<j}^{r} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x \le \alpha \sum_{i<j}^{r} \mu_i \mu_j \|x\|^2 = \alpha \frac{r(r-1)}{2} \|x\|^2.$$

Then one can state the following theorem:

Theorem 3.1. *If the assumption $(\mathcal{H})$ is satisfied regarding the fuzzy system (1) and there exists a common positive define matrix P and some feedback gain matrices K_i, $i = 1, \dots, r$ such that the reduced stability conditions (3) are satisfied, then the fuzzy closed-loop system is guaranteed to be globally uniformly exponentially stable with the control law (2) provided that ρ satisfies:*

$$\rho(x) < \frac{1}{2\|P\|} \frac{1}{(\sum_{i=1}^{r} \mu_i^2)^{\frac{1}{2}}} \left(\inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 - 2k - l \right), \tag{4}$$

with $l > 0$ *and*

$$k < \frac{1}{2}\left(\inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 - l\right),$$

$l > 0$ *can be chosen such that*

$$\inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 > l.$$

Note that, condition (4) implies that,

$$2\|P\|\rho(x)\left(\sum_{i=1}^{r} \mu_i^2\right)^{\frac{1}{2}} < \inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 - 2k - l, \quad l > 0$$

and so,

$$-\inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 + 2k + 2\|P\|\rho(x)\left(\sum_{i=1}^{r} \mu_i^2\right)^{\frac{1}{2}} < -l, \quad l > 0.$$

Proof: Using the Lyapunov function

$$V(x) = x^T P x.$$

The derivative of $V(x)$ along the trajectories of (1) in closed-loop with (2) with respect to time is given by,

$$\dot{V}(x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + P G_{ii}) x + 2\sum_{i<j} \mu_i \mu_j x^T (G_{ij}^T P + P G_{ij}) x + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x).$$

Thus,

$$\dot{V}(x) \le -\sum_{i=1}^{r} \mu_i^2 \lambda_{\min}(Q_i)\|x\|^2 + 2k\|x\|^2 + 2\|P\|\sum_{i=1}^{r} \mu_i \rho_i(x)\|x\|^2.$$

Using Cauchy-Schwartz inequality

$$\dot{V}(x) \le -\sum_{i=1}^{r} \mu_i^2 \lambda_{\min}(Q_i)\|x\|^2 + 2k\|x\|^2 + 2\|P\|\left[\sum_{i=1}^{r} \mu_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{r} \rho_i(x)^2\right]^{\frac{1}{2}}\|x\|^2.$$

It follows that,

$$\dot{V}(x) \le \left(-\inf_{i=1,\dots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 + 2k + 2\|P\|\rho(x)\left(\sum_{i=1}^{r} \mu_i^2\right)^{\frac{1}{2}}\right)\|x\|^2.$$

Since $\rho(x)$ satisfies (4), then $\dot{V}(x)$ is negative define function and one can obtain an estimation as follows:

$$\dot{V}(x) \le -l\|x\|^2, \quad l > 0.$$

Taking into account the fact that

$$V(x) = x^T P x \le \lambda_{\max}(P)\|x\|^2$$

we obtain

$$\dot{V}(x) \le -\frac{l}{\lambda_{\max}(P)} V(x), \quad l > 0,$$

then all trajectories satisfy the following estimation for all $t \ge t_0 \ge 0$ and initial condition $x(t_0) \in \Re^n$:

$$\|x(t)\| \le \gamma\|x(t_0)\| e^{-v(t-t_0)},$$

γ and υ are positive constants.

Hence, the fuzzy system (1) in closed-loop with (2) is globally uniformly exponentially stable.

Now, let us consider the following uncertain fuzzy model:

$$\text{Rule } l: \text{ If } z_1(t) \text{ is } F_{l1} \text{ and } z_2(t) \text{ is } F_{i2} \ldots \text{ and } z_p(t) \text{ is } F_{lp}, \text{ then}$$

$$\dot{x} = A_l x + B_l u + B_l f_l(t,x), \quad l = 1, \ldots, r \tag{5}$$

The functions

$$f_l : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$$

represent the uncertainties of each fuzzy subsystem and are time varying satisfying the following assumption for $l = 1, \ldots, r$:

$(\mathcal{H}')$

$$\text{For all } l = 1, \ldots, r, \ \|f_l(t,x)\| \le \tilde{\rho}_l(x) \le \tilde{\rho}(x), \ \forall t \ge 0, \ \forall x \in \mathbb{R}^n,$$

where $\tilde{\rho}_l$ are nonnegative continuous functions, such that $\tilde{\rho}_l(x) \le \tilde{\rho}(x)$ with $\tilde{\rho}$ being a positive continuous function, such that $\tilde{\rho}(0) = 0$.

We will use the following composite fuzzy controller:

$$u(t) = \sum_{j=1}^{r} \mu_j \tilde{K}_j x + \tilde{u}, \tag{6}$$

$\tilde{u}$ is related to the uncertainties, which is chosen in the following form:

$$\tilde{u}(x(t)) = \begin{cases} -\dfrac{B^T P x \tilde{\rho}(x)}{\|B^T P x\| + \varepsilon(x)\|x\|^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for a certain positive function $\varepsilon(x) > 0$.

Theorem 3.2. *If the assumption $(\mathcal{H}')$ is satisfied regarding the fuzzy system (5) and there exists a common positive define matrix P and some feedback gain matrices K_i, $i = 1, \ldots, r$ such that the reduced stability conditions (3) are satisfied, then the fuzzy closed-loop system is guaranteed to be globally uniformly exponentially stable with the control law (6) provided that $\tilde{\rho}$ satisfies*

$$\tilde{\rho}(x) \le \frac{1}{2\varepsilon(x)} \left(- \inf_{i=1,\ldots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2 + 2k \right) \tag{7}$$

with $\varepsilon(x) > 0$ and

$$k < \frac{1}{2} \inf_{i=1,\ldots,r} \lambda_{\min}(Q_i) \sum_{i=1}^{r} \mu_i^2.$$

Note that, given $\tilde{\rho}(x)$, $\varepsilon(x)$ can be chosen small enough in such a way (7) holds.

Proof: Taking into account the fact that

$$B = \sum_{l=1}^{r} \mu_l B_l,$$

the derivative of

$$V(x) = x^T P x$$

is given by:

$$\dot{V}(x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + PG_{ii})x + 2\sum_{i<j}^{r} \mu_i\mu_j x^T (G_{ij}^T P + PG_{ij})x + 2x^T \tilde{P} B\tilde{u}$$
$$+ 2x^T P \sum_{i=1}^{r} \mu_i B_i f_i(t,x).$$

Thus, for $x \neq 0$, one gets

$$\dot{V}(x) \leq -\sum_{i=1}^{r} \mu_i^2 \lambda_{\min}(Q_i)\|x\|^2 + 2k\|x\|^2 + 2\|x^T PB\|\tilde{\rho}(x) - 2\frac{x^T PBB^T Px\tilde{\rho}(x)}{\|B^T Px\| + \varepsilon(x)\|x\|^2}.$$

It follows that

$$\dot{V}(x) \leq -\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2\|x\|^2 + 2k\|x\|^2 + 2\frac{\|B^T Px\|\tilde{\rho}(x)\varepsilon(x)\|x\|^2}{\|B^T Px\| + \varepsilon(x)\|x\|^2}.$$

Hence,

$$\dot{V}(x) \leq -\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2\|x\|^2 + 2k\|x\|^2 + 2\tilde{\rho}(x)\varepsilon(x)\|x\|^2$$

and so,

$$\dot{V}(x) \leq \left[-\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2 + 2k + 2\tilde{\rho}(x)\varepsilon(x)\right]\|x\|^2.$$

Since $\tilde{\rho}(x)$ satisfies (7), then for a suitable choice of $\varepsilon(x)$,

$$-\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2 + 2k + 2\tilde{\rho}(x)\varepsilon(x) < 0$$

and then one can obtain an estimation on $\dot{V}(x)$ as

$$\dot{V}(x) \leq -l\|x\|^2, \quad l > 0.$$

Taking into account the fact that

$$V(x) = x^T Px \leq \lambda_{\max}(P)\|x\|^2$$

we obtain

$$\dot{V}(x) \leq -\frac{l}{\lambda_{\max}(P)}V(x), \quad l > 0,$$

then all trajectories satisfy the following estimation for all $t \geq t_0 \geq 0$ and initial condition $x(t_0) \in \Re^n$:

$$\|x(t)\| \leq \gamma_1\|x(t_0)\|e^{-\upsilon_1(t-t_0)},$$

γ_1 and υ_1 are positive constants.

Hence, the fuzzy system (5) in closed-loop with (6) is globally uniformly exponentially stable.

Note that, given $\tilde{\rho}(x)$ we can choose $\varepsilon(x) > 0$ small enough such that (7) holds and then

$$-\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2 + 2k + 2\tilde{\rho}(x)\varepsilon(x) < -l, \quad l > 0,$$

provided that

$$k < \frac{1}{2}\inf_{i=1,\ldots,r} \lambda_{\min}(Q_i)\sum_{i=1}^{r} \mu_i^2.$$

According to the above analysis, the design procedure for T-S fuzzy systems is summarized as follows:

Step 1: Verify that assumptions (A_l, B_l) are controllable for $l = 1, \dots, r$.

Step 2: Verify that assumption $(\mathcal{H})$ (resp. $(\mathcal{H}')$) is satisfied.

Step 3: Solve the Lyapunov Equation (3) to obtain P, K_i, Q_i, $i = 1, \dots, r$.

Note that, for simplicity one can choose $Q_i = I$.

Step 4: By using the control toolbox, execute the nonlinear program based on equations

$$(G_{ii}^T P + P G_{ii}), \quad 1 \le i < j \le r,$$

to determine k.

The nonlinear programming is expressed in the following way:
Determine

$$\lambda_{\max}(G_{ii}^T P + P G_{ii})$$

and then

$$\alpha = \max_{i,j} \lambda_{\max}(G_{ii}^T P + P G_{ii})$$

for $1 \le i < j \le r$.

Step 5: Construct the fuzzy controller (2) for (1) (respectively (6) for (5)).

Step 6: Verify the condition (4) imposed on $\rho(x)$ for a suitable choice of $l > 0$, (respectively for (7)) for a suitable choice of $\varepsilon(x)$.

4. **Numerical Example.** We will take the example used in [7]. The plant consists of three interconnected tanks used in chemical, pharmaceutical and agroalimentary industries.

The goal is to obtain a mixture of two liquids in the third tank exiting with a previously fixed concentration of every liquid in the mixture. The first contains a liquid A and the second a liquid B. These two tanks flow into tank 3 that supplies consumers. The pumps supply the tanks respectively by variable flows. We suppose that these flows are proportional to the voltages applied to the Moto-pumps. The dynamic of these actioners are neglected.

Both pumps convey an amount of liquid $q_{11}(t)$ and $q_{22}(t)$ that are proportional to the applied controls $u_1(t)$ and $u_2(t)$ respectively.

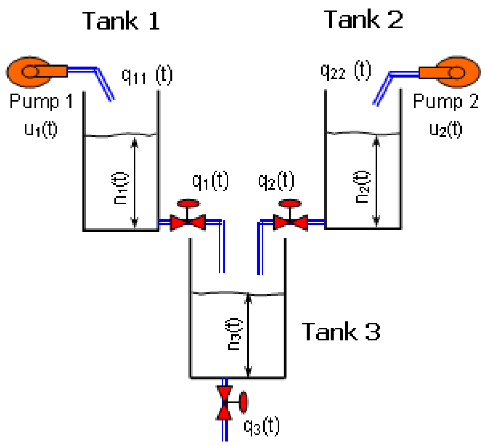

FIGURE 1. Interconnected tank system

Nonlinear model:

The level of each tank depends on the difference of the liquid flowing into the tank and the liquid flowing off.

Depending on the section of the opening that can be considered by the constant p_i, the amount of liquid flowing off by an outlet valve according to Torricelli's law is:

$$q_i(t) = p_i\sqrt{n_i(t)}, \quad i \in \{1,2,3\}.$$

The amount of water $q_{ii}(t)$ flowing into tanks 1 and 2 can be described as:

$$q_{ii}(t) = p_{ii}(t)u_i(t), \quad i \in \{1,2\}$$

p_{11} and p_{22} are constants,

$$p_i = \rho S_i\sqrt{2g}, \quad i \in \{1,2,3\},$$

g is the gravitational constant, ρ is the density of liquid and S_i is the section of valve.

Based on these physical relations one immediately gets the equations to describe the nonlinear dynamic behavior of the plant:

$$\begin{cases} \dot{n}_1(t) = -p_1\sqrt{n_1(t)} + p_{11}(t)u_1(t) \\ \dot{n}_2(t) = -p_2\sqrt{n_2(t)} + p_{22}(t)u_2(t) \\ \dot{n}_3(t) = p_1\sqrt{n_1(t)} + p_2\sqrt{n_2(t)} - p_3\sqrt{n(t)} \end{cases}.$$

T-S fuzzy model:

A T-S model can be designed to represent exactly, in a compact set of the state variables, a nonlinear system (see [2, 4]). The following property can be used for every bounded function $f(x) \in [\underline{f}\bar{f}]$.

$$f(x) = \frac{f(x) - \underline{f}}{\bar{f} - \underline{f}}\,\bar{f} + \frac{\bar{f} - f(x)}{\bar{f} - \underline{f}}\,\underline{f}.$$

The non-linearities are:

$$f_i(t) = \frac{1}{\sqrt{n_i}(t)} \in [\underline{f}_i\bar{f}_i], \quad i \in \{1,2,3\}.$$

Defining

$$w_i(x) = \frac{f_i(x) - \underline{f}_i}{\bar{f}_i - \underline{f}_i}.$$

We obtain eight rules with:

$$h_1 = w_1w_2w_3,$$
$$h_2 = w_1w_2(1 - w_3)$$

and so on. With

$$x = (n_1, n_2, n_3)^T$$

the state vector, a continuous T-S fuzzy model is given by:

$$\dot{x} = \sum_{i=1}^{s} \mu_i(z(t))(A_ix(t) + B_iu(t)) + D_id(t). \tag{8}$$

Then the fuzzy system has the state-space form:

$$\dot{x} = \sum_{i=1}^{s} \mu_i(z)(A_ix(t) + B_iu(t) + f_i(t,x)).$$

This means that the global control input matrix dominates the control performance. The design of the fuzzy controller can be taken as a linear state feedback control for the system (1) which can be defined as:

Rule i : If $z_1(t)$ is F_{i1} and $z_2(t)$ is F_{j2} ... and $z_p(t)$ is F_{ip}, then

$$u(t) = K_i x(t), \quad i = 1, 2, \ldots, s,$$

where K_i is the local state feedback gain. Consequently, the defuzzified result is:

$$u(t) = \sum_{i=1}^{s} \mu_i(z) K_i x(t).$$

We take for an example,

$$A_1 = \begin{bmatrix} \frac{-p_1}{\sqrt{n_{1\max}}} & 0 & 0 \\ 0 & \frac{-p_2}{\sqrt{n_{2\max}}} & 0 \\ \frac{p_1}{\sqrt{n_{1\max}}} & \frac{p_2}{\sqrt{n_{2\max}}} & \frac{-p_3}{\sqrt{n_{3\max}}} \end{bmatrix}$$

and

$$\forall i \in \{1, \ldots, 8\}, \quad D_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \\ 0 & 0 \end{bmatrix}.$$

The premises of the 8 models (8) are based on the measurable variables n_1, n_2 and n_3.

The bounds are:

$$n_{1\min} = n_{2\min} = n_{3\min} = \underline{f}_1 = \underline{f}_2 = \underline{f}_3 = 0.0001,$$

$$n_{1\max} = n_{2\max} = \bar{f}_1 = \bar{f}_2 = 5.5,$$

$$n_{3\max} = \bar{f}_3 = 10,$$

and $\alpha = 2$.

We realize that control gains are very similar. That is not surprising, and we find well the property indicated in [2] when the matrices B_i are constants.

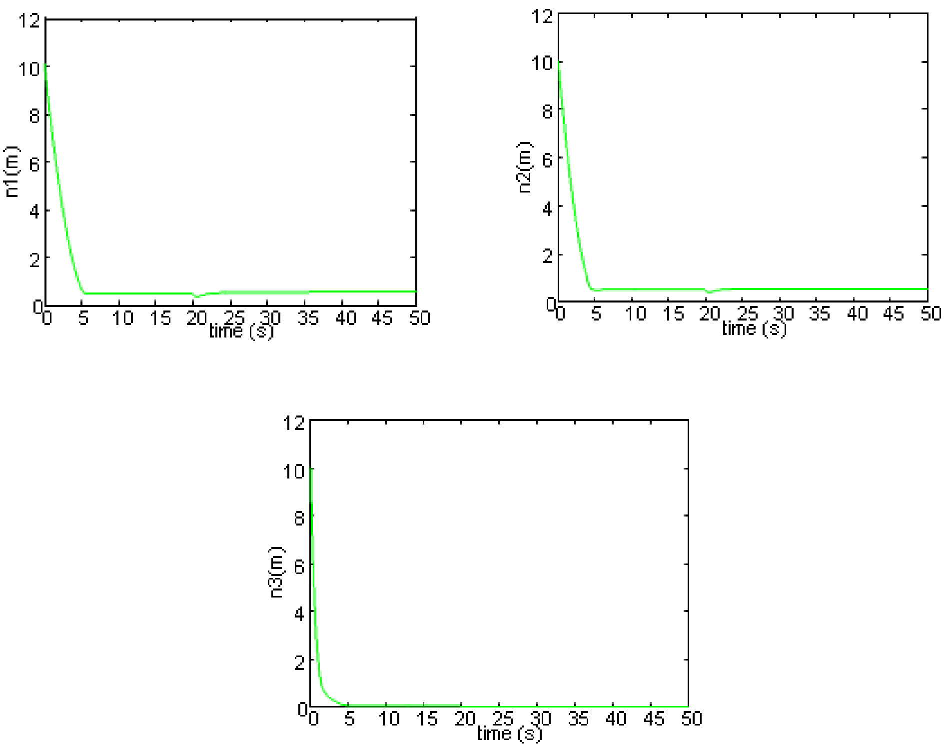

FIGURE 2. Levels in tanks 1, 2 and 3

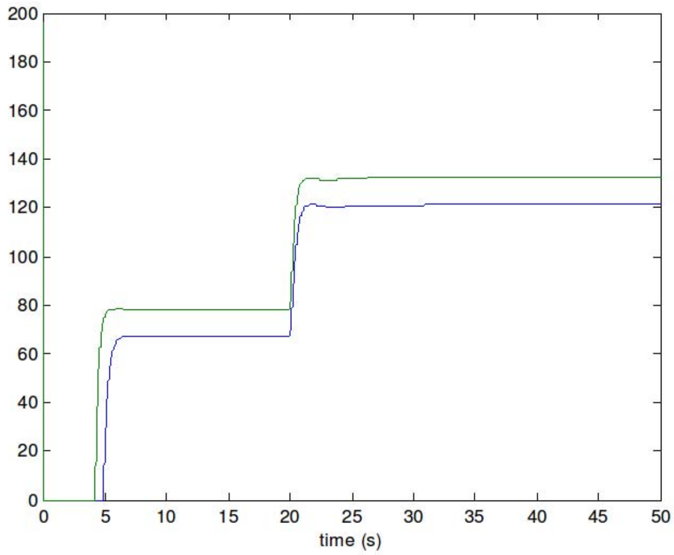

FIGURE 3. Control variables

For example we have:

$$K_1 = \begin{bmatrix} 6.2 & -0.3 & -0.6 & 0.3 \\ 0.1 & 6 & 1.9 & 1.7 \\ -0.1 & 1.5 & 6.9 & 2.5 \end{bmatrix}, \quad K_8 = \begin{bmatrix} 3.3 & -0.3 & 2.3 & 0.3 \\ 0.1 & -5 & 12 & 1.7 \\ -0.1 & 1.5 & -29 & 2.5 \end{bmatrix}.$$

We choose the first output, $q_3 = 0.9\text{m}^3/\text{s}$ from 0s until 50s and $1.3\text{m}^3/\text{s}$ after. To guarantee a fixed relative concentration, the second output must be equal to 0. A two dimension-disturbance was introduced at $t = 20$s.

In Figure 2, we show the levels of liquid in the three tanks, and Figure 3 shows the control variables.

Note that, if we introduce a two dimension disturbance on the state variables these errors tend to zero.

Now, we compare this method with a classic method of linearization. For example, we use the Taylor linearization [8].

Taylor linearization:

We recall that with $\dot{n}_i = f(n_i, u_i)$. Taylor series can be written as:

$$f(n_i, u_i) = f(n_{si}, u_{si}) + \left.\frac{\partial f}{\partial n_i}\right|_{(n_{si}-u_{si})} (n_i, n_{si}) + \left.\frac{\partial f}{\partial u_i}\right|_{(n_{si},u_{si})} (u_i - u_{si}),$$

n_{si} and u_{si} are the set points. After linearisation and with:

$$\tilde{n}_i = (n_i - n_{si}),$$

$$\tilde{u}_i = (u_i - u_{si}),$$

we obtain:

$$\begin{cases} \dot{\tilde{n}}_i(t) = -\frac{p_i}{2\sqrt{n_{si}(t)}}\tilde{n}_i + r_i\tilde{u}_i(t), \ i = 1, 2 \\ \dot{\tilde{n}}_k(t) = \displaystyle\sum_{i=1}^{k-1} \frac{p_i}{2\sqrt{n_{si}(t)}}(n_i - n_{si}) - \frac{p_k}{2\sqrt{n_{sk}(t)}}(n_k - n_{sk}), \ k = 3 \end{cases}.$$

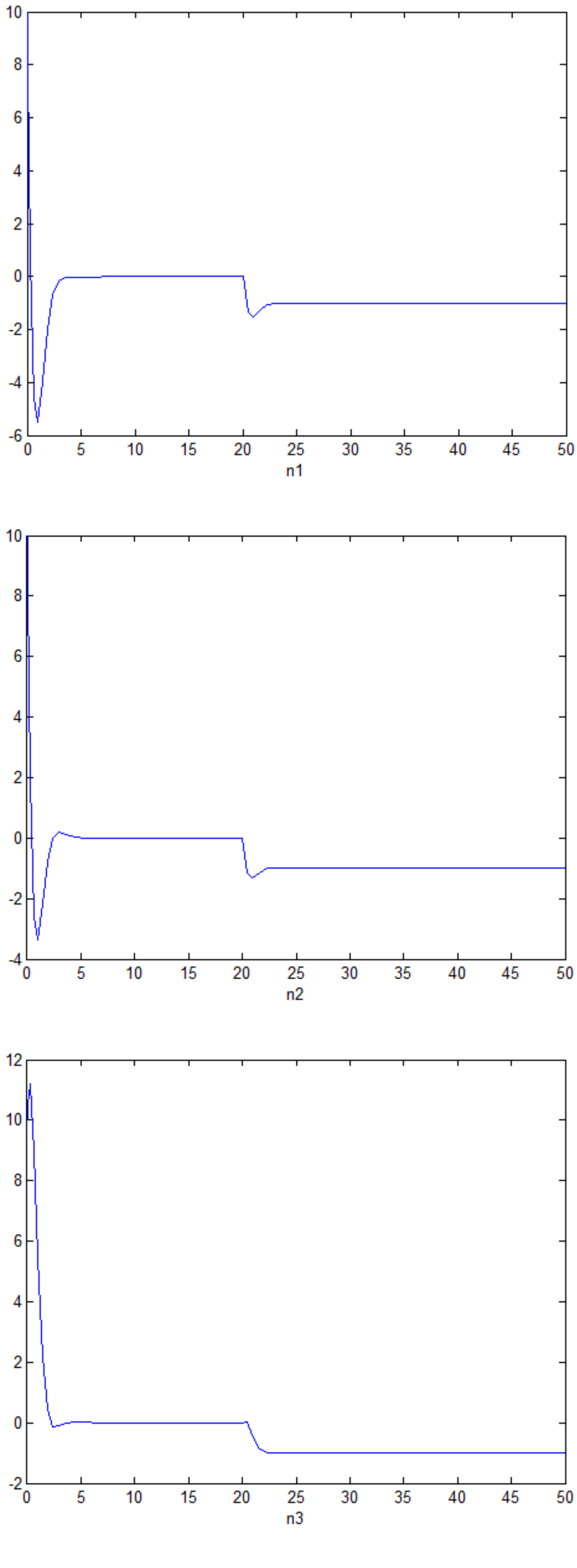

FIGURE 4. Levels in tanks 1, 2 and 3

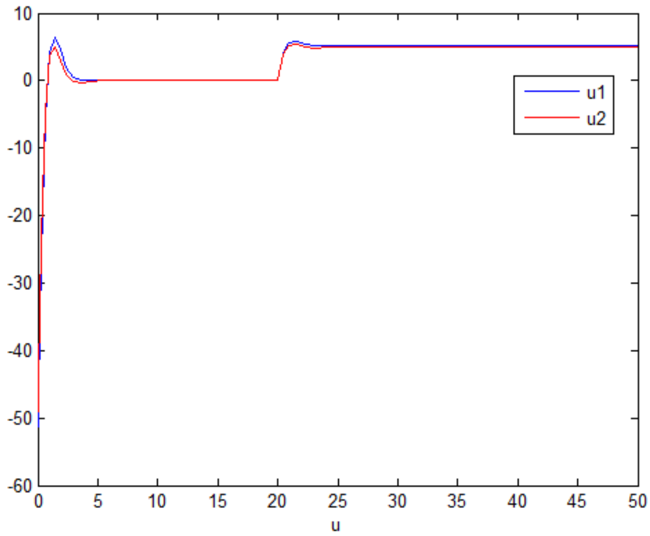

FIGURE 5. Control variables

From this equation the linear model is:

$$\begin{cases} \dot{\tilde{n}}_i(t) = \tilde{n} + B\tilde{u}(t);\ \tilde{n} = (n - n_s),\ \tilde{u} = (u - u_s) \\ \tilde{u}(t) = \tilde{n},\ F \text{ are control gain} \end{cases}.$$

Matrices of the system are:

$$A = \begin{bmatrix} \frac{-p_1}{2\sqrt{n_{s1}}} & 0 & 0 \\ 0 & \frac{-p_2}{2\sqrt{n_{s2}}} & 0 \\ \frac{p_1}{2\sqrt{n_{s1}}} & \frac{p_2}{2\sqrt{n_{s2}}} & \frac{-p_3}{2\sqrt{n_{s3}}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The comparison with Taylor linearization shows that the latter does not give satisfied result: in Figure 4, n_1, n_2 and n_3 are not better than in our approach, particulary when we introduce disturbance.

5. **Conclusions.** In this paper, we have proposed new sufficient conditions for the stability of the fuzzy systems with uncertainties. These conditions guarantee the asymptotic stability of such systems. The application of this result has been done on an interconnected tank plant.

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