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LINEAR-QUADRATIC STOCHASTIC DIFFERENTIAL GAMES WITH MARKOV JUMPS AND MULTIPLICATIVE NOISE: INFINITE-TIME CASE

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Abstract. *This paper discusses the linear quadratic (LQ) differential games for stochastic systems with Markov jumps and multiplicative noise in infinite-time case. Firstly, we consider zero-sum games for stochastic systems with multiplicative noise. Here the state weighting matrix is allowed to be indefinite, and an important theorem is gained. Further, we discuss the LQ differential games for stochastic systems with Markov jumps and multiplicative noise. We introduce the important definition of stochastic detectability, which has close relation to Lyapunov equation. Based on Lyapunov equation, we obtain four-coupled generalized algebraic Riccati equations (GAREs), which are essential on finding the optimal strategies (Nash equilibrium strategies) and the optimal cost values for infinite stochastic differential games. Finally, the corresponding simulation examples are presented to illustrate the main results.*
Keywords: Stochastic differential games, Nash equilibrium, Markov jumps, Stochastic detectability, Exact detectability, Generalized algebraic Riccati equations

1. **Introduction.** Stochastic control theory has made great progress in the engineering and scientific fields for many years. Particularly, the stochastic control problems governed by Itô-type linear differential equations can describe many practical systems. In recent years, the study for stochastic Itô-type controlled linear systems has become one popular research field of modern control theory, see [1-5].

To deal with military problems such as pursuit, battle and aiming games, Isaacs firstly pioneered differential games [6], who extended the notions of the cost value, optimal strategies, saddle point equilibrium, etc. In fact, many situations in industry, economies, management and elsewhere are characterized by multiple decision makers and enduring consequences of decisions which can be treated as dynamic games [7, 8]. Meanwhile, stochastic differential games have attracted considerable research interest [9, 10]. There is a special class of dynamic games, where the process can be modeled by a set of linear differential equations and the performance index is formalized by quadratic cost functions. They are the so-called LQ differential games. By solving the LQ control problems, players can avoid most of the additional cost incurred by this perturbation [11]. Nonzero-sum and zero-sum linear differential games with quadratic cost functions in deterministic case have been also widely investigated in many literatures [11-14]. They mainly dealt with the optimization behavior according to different performance criteria. Reference [11] not only presented the theory on finding the Nash equilibria but also gave the solutions based on the feedback Nash equilibrium algorithms. Moreover, an iterative algorithm to solve a kind of state-perturbed stochastic algebraic Riccati equation in LQ zero-sum game problems was proposed in [15]. In our previous work [16], we have considered nonzero-sum stochastic differential games in infinite-time horizon, where the diffusion term in dynamics depends on both of the state and the control variables. By introducing stochastic exact observability and stochastic exact detectability, the optimal strategies (Nash equilibrium strategies) and the optimal cost values have been given.

In the paper, we further investigate zero-sum stochastic differential games with state and controls dependent noises in infinite-time horizon. Here players do not cooperate. The reason why the players do not cooperate may be caused by individual motivations or for physical reasons. It is reasonable that all players will try to play actions individually which are optimal for themselves [11]. For researching the work in an infinite-time case, restraining the players to constant strategies is reasonable and prescribing linearity is also natural in the linear-quadratic context. It should be pointed out that the state weighting matrix can be indefinite in the zero-sum case, and this leads to an indefinite game problem and brings considerable complications, but we believe that this generality is natural, see [13]. We deal with zero-sum stochastic differential games in infinite-time horizon in Section 2, which allow the state weighting matrix to be indefinite. A theorem is obatined to show the optimal controls and the optimal cost values corresponding to the zero-sum case.

As is well known, there are some significant applications for linear stochastic systems with Markov jumps. In practice, a lot of physical systems have variable structures subject to random changes. These changes may result from abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. As one of the most basic dynamics models, systems with Markov jumps can be used to represent these random failure processes in manufacturing and some investment portfolio models. The stochastic systems with Markov jumps are special class of hybrid systems and have been widely investigated, see [17-26]. Researchers mainly focus on the analysis and synthesis of Markovian jump system, including stability analysis, observability, detectability, state feedback and output feedback controller design, filter design, etc. Detectability and observability are crucial concepts in linear deterministic control theory and have been extended to stochastic case. The relevant issues are introduced in [18-21], respectively. Spectral technique, stability, exact observability and exact detectability are mainly discussed for stochastic Markov jump systems in [18]. Some criteria and interesting properties for both W-observability and W-detectability are obtained in [21]. Recently, filtering design makes new progress in [22-25]. [22] considers the exponential H_∞ filter design for a class of continuous-time singular Markovian jump systems with mixed mode-dependent time-varying delay. In [23], the problem of robust fuzzy H_∞ filtering is investigated for a class of uncertain nonlinear discrete-time Markov jump systems with nonhomogeneous jump transition probabilities. [24] is concerned with the H_∞ filtering problem for a class of discrete-time systems with stochastic incomplete measurement and mixed random delays. And sufficient conditions for the existence of the admissible filter are derived. In [25], it discusses the problem of asynchronous l_2-l_∞ filtering for discrete-time stochastic Markov jump systems with sensor nonlinearity. A sufficient condition is first given such that the resultant filtering error system and the existence criterion of the desired asynchronous filter with piecewise homogeneous Markov chain is proposed.

In [3], based on stabilization and exact observability or exact detectability, it is indicated that the optimal control law and the optimal cost value exist for the problem of stochastic linear quadratic regulator. In the meanwhile, the solutions of the GAREs are also considered. However, up to now, there have been few attentions on stochastic differential games for infinite horizon LQ stochastic systems with Markov jumps and multiplicative noise. The problems of stochastic differential games with Markov jumps and state and controls dependent noises are mainly researched in the nonzero-sum cases. Unlike most previous researchers, stochastic differential games in the nonzero-sum cases are dealt by means of the stochastic detectability in Section 3, which extends Theorem 4.1 in [16] to the stochastic version with Markov jumps and is more in-depth study of the conference article [26]. Finally, some important theorems that have extensive application value in the economy, military and intelligent robots are obtained and an simulation example shows its efficiency.

For convenience, we adopt the following notations. It uses $\mathbb{R}^n$ to denote the linear space of all n-dimensional real vectors. $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ matrices. $\mathbb{S}^n$ denotes the set of all $n \times n$ symmetric matrices. A' represents the transpose of matrix A. $\chi_{\{A\}}$ is the indicator function of a set A. $P \geq (>)\ 0$ means P is a semi-positive (positive) definite symmetric matrix. $\mathcal{L}_2^k(0,\infty)$ is the space of the R^k-valued functions that are quadratically integrable on $(0,\infty)$. $E(\cdot)$ represents the mathematical expectation and $\mathbb{C}$ denotes the complex plane.

2. **Infinite-time Stochastic Differential Games with Multiplicative Noise.**

2.1. **Main results.** In many applications, state changes that are beneficial to one player may be harmful to another player. In the section, we consider the problem of infinite-time zero-sum stochastic differential games and we only treat with (x, v)-dependent noise for the sake of simplicity. And we allow the state weighting matrix to be indefinite, allowing it brings considerable technical complications, but we believe this generality is natural. Consider the problem of infinite-time zero-sum stochastic differential games described by the following linear stochastic differential equation:

$$\begin{cases} dx(t) = [A_1x(t) + B_1u(t) + C_1v(t)]dt + [A_2x(t) + C_2v(t)]dw(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0. \end{cases} \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the system state. $u(t) \in \mathcal{L}_2^{n_u}(0,\infty)$ and $v(t) \in \mathcal{L}_2^{n_v}(0,\infty)$ represent the system control inputs. A_1, A_2, B_1, C_1, C_2 are constant matrices with appropriate dimension. System (1) is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$. $w(t)$ is a one-dimensional Wiener process with $w(0) = 0$. $\mathcal{F}_t$ denotes the smallest σ-algebra generated by process $w(s)$, $0 \leq s \leq t$, i.e., $\mathcal{F}_t = \sigma\{w(s) \mid 0 \leq s \leq t\} \subset \mathcal{F}$.

Throughout this section, the cost function is determined by

$$J(u, v) = E\int_0^\infty (x'Qx + u'Ru - v'Sv)dt. \tag{2}$$

Here $R \in \mathbb{S}^n$ and $S \in \mathbb{S}^n$. R and S are positive definite and the matrix Q can be indefinite. It is obvious that strategy u is to be chosen to minimize and v is chosen to maximize the function. Hence, the problem is to find the optimal strategies $u^*(t)$ and $v^*(t)$ such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*). \tag{3}$$

We restrain the admissible control set to be the constant linear feedback strategies, so we take the controls as $u(t) = K_1x(t)$, and $v(t) = K_2x(t)$, where K_1, K_2 are constant matrices with appropriate dimension, and (K_1, K_2) belong to the admissible set

$$\mathcal{K} := \{K = (K_1, K_2) \mid \textit{system } (1) \textit{ can be stabilized with } u(t) = K_1x(t),\ v(t) = K_2x(t)\}.$$

The stabilization constrain is imposed to ensure the finiteness of the infinite-time cost function integrals.

In what follows, we focus on finding the optimal strategies $(u^*,\ v^*)$, which are called the saddle point strategies. In order to guarantee the unique global game solutions in (3), both the players are only allowed to take constant feedback controls.

Definition 2.1. [1] *The following stochastic system*

$$dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dw(t), \ \ x(0) = x_0,$$

is called stabilizable (in the mean square sense), if there exists a feedback control $u(t) = Kx(t)$*, such that the closed-loop system*

$$dx(t) = (A + BK)x(t)dt + (C + DK)x(t)dw(t), \ x(0) = x_0,$$

is asymptotically mean-square stable for any $x_0 \in \mathbb{R}^n$*, i.e.,* $\lim_{t\to\infty} E \parallel x(t) \parallel^2 = 0$*, where* K *is a constant matrix.*

Theorem 2.1. *Assume* $u(t)$ *and* $v(t)$ *are both the stabilizing controls such that system (1) is stabilizable, i.e.,* $(K_1, K_2) \in \mathcal{K}$*. Suppose the following GARE*

$$\begin{cases} A_1'P + PA_1 + A_2'PA_2 - PB_1R^{-1}B_1'P + (C_1'P + C_2'PA_2)'\mathcal{M}^{-1}(C_1'P + C_2'PA_2) + Q = 0, \\ \mathcal{M} = S - C_2'PC_2 > 0, \end{cases}$$

has a solution $P \geq 0$*. Furthermore, assume that there exists a real symmetric matrix* Y *satisfying the following inequality*

$$A_1'Y + YA_1 + A_2'YA_2 - YB_1R^{-1}B_1'Y + Q \geq 0. \tag{4}$$

Then, for zero-sum stochastic differential games problem (3), the saddle point $(u^*(t), v^*(t))$ *is determined by*

$$u^*(t) = K_1^*x(t) = -R^{-1}B_1'Px(t), \ \ v^*(t) = \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\tilde{x}(t),$$

where $\tilde{x}(t)$ *is generated by*

$$d\tilde{x}(t) = [(A_1 - B_1R^{-1}B_1'P)\tilde{x}(t) + C_1v(t)]dt + (A_2\tilde{x}(t) + C_2v(t))dw(t), \ \ \tilde{x}(0) = x_0.$$

Moreover, the optimal cost value $J(u^*, v^*) = x_0'Px_0$*.*

Proof: According to the assumption on $u(t)$ and $v(t)$, they make system (1) stabilizable. And by Definition 2.1, we have $\lim_{t\to\infty} E\|x'(t)\|^2 = \lim_{t\to\infty} E[x'(t)Px(t)] = 0$. Thus, $E\int_0^\infty d[x'(t)Px(t)] = \lim_{t\to\infty} E[x'(t)Px(t)] - x'(0)Px(0) = -x'(0)Px(0)$. Setting $u(t) = K_1x(t)$, where K_1 is a matrix with appropriate dimension, then taking a scalar function $x'(t)Px(t)$ and completing the squares, we have

$$\begin{aligned} J(u, v) &= E\int_0^\infty (x'Qx + u'Ru - v'Sv)dt + E\int_0^\infty d[x'(t)Px(t)] + x'(0)Px(0) \\ &= x_0'Px_0 + E\int_0^\infty x'[(K_1 + R^{-1}B_1'P)'R(K_1 + R^{-1}B_1'P)]xdt \\ &\quad -E\int_0^\infty [v - \mathcal{M}^{-1}(C_1'P + C_2'PA_2)x]'\mathcal{M}[v - \mathcal{M}^{-1}(C_1'P + C_2'PA_2)x]dt. \end{aligned}$$

So it follows that $u^*(t) = -R^{-1}B_1'Px(t)$ denoted by $u^*(t) = K_1^*x(t)$ and

$$\begin{aligned} J(u^*, v) &= x_0'Px_0 - E\int_0^\infty [v - \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\tilde{x}]'\mathcal{M}[v - \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\tilde{x}]dt \\ &\leq x_0'Px_0. \end{aligned}$$

Hence, $v^*(t) = \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\tilde{x}(t)$, $J(u^*, v^*) = x_0'Px_0$, and $J(u^*, v) \leq J(u^*, v^*)$ for all $v(t)$.

Next, we show $J(u^*, v^*) \leq J(u, v^*)$. Let $\hat{x}(t)$ and $\bar{x}(t)$ be generated by

$$d\hat{x}(t) = [(A_1 + B_1K_1)\hat{x}(t) + C_1v^*(t)]dt + [A_2\hat{x}(t) + C_2v^*(t)]dw(t), \ \hat{x}(0) = x_0,$$

and

$$d\bar{x}(t) = [(A_1 + B_1K_1^*)\bar{x}(t) + C_1v^*(t)]dt + [A_2\bar{x}(t) + C_2v^*(t)]dw(t), \ \bar{x}(0) = x_0,$$

respectively. Defining

$$\vartheta(t) = (K_1^* - K_1)\hat{x}(t), \quad \eta(t) = v^*(t) - \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\hat{x}(t),$$

we have

$$J(u, v^*) - J(u^*, v^*) = E\int_0^\infty (\vartheta' R\vartheta - \eta'\mathcal{M}\eta)dt.$$

Introducing $\xi(t) = \bar{x}(t) - \hat{x}(t)$, then $d\xi(t) = [(A_1 + B_1K_1^*)\xi(t) + B_1\vartheta(t)]dt + A_2\xi(t)dw(t)$ with $\xi(0) = 0$ and $\eta(t) = \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\xi(t)$. Again, from the assumption on $u(t)$ and $v(t)$, we have $E\int_0^\infty d[\xi'(t)P\xi(t)] = \lim_{t\to\infty}[\xi'(t)P\xi(t)] - \xi'(0)P\xi(0) = 0$. Hence,

$$\begin{aligned}
&J(u, v^*) - J(u^*, v^*)\\
&= E\int_0^\infty (\vartheta' R_1\vartheta - \eta'\mathcal{M}\eta)dt + E\int_0^\infty d[\xi'(t)P\xi(t)]\\
&= E\int_0^\infty \{(\vartheta - R^{-1}B_1'P\xi)'R(\vartheta - R^{-1}B_1'P\xi) - \xi'[A_1'P + PA_1 + A_2'PA_2\\
&\quad -PB_1R^{-1}B_1'P + (C_1'P + C_2'PA_2)'\mathcal{M}^{-1}(C_1'P + C_2'PA_2)]\xi\}dt\\
&= E\int_0^\infty (\vartheta + K_1^*\xi)'R(\vartheta' + K_1^*\xi) + \xi'Q\xi dt. \qquad (5)
\end{aligned}$$

Next, defining $\theta(t) = \vartheta(t) + K_1^*\xi(t) = K_1^*\bar{x}(t) - K_1\hat{x}(t)$, we have $d\xi(t) = [A_1\xi(t) + B_1\theta(t)]dt + A_2\xi(t)dw(t)$. Since $\xi(0) = 0$ and $\lim_{t\to\infty}[\xi'(t)Y\xi(t)] = 0$, we also have $E\int_0^\infty d[\xi'(t)Y\xi(t)] = 0$. Hence,

$$\begin{aligned}
J(u, v^*) - J(u^*, v^*) &= E\int_0^\infty (\theta' R\theta + \xi'Q\xi)dt + E\int_0^\infty d[\xi'(t)Y\xi(t)]\\
&= E\int_0^\infty (\theta + R^{-1}B_1'Y\xi)'R(\theta + R^{-1}B_1'Y\xi)dt\\
&\quad +E\int_0^\infty \xi'(A_1'Y + YA_1 + A_2'YA_2 - YB_1R^{-1}B_1'Y + Q)\xi dt.
\end{aligned}$$

Since (4) holds, we have $J(u, v^*) \geq J(u^*, v^*)$ which ends the proof of Theorem 2.1.

Remark 2.1. *If the state weighting matrix Q is assumed to be positive semidefinite, then Theorem 2.1 still holds without the assumption on (4) which is consistent with (5). Theorem 2.1 extends Theorem 3.1 in* [13] *and Theorem 9.8 in* [11] *from the deterministic case to a stochastic version.*

2.2. **Simulation.** Next, a numerical example is given to show the efficiency of our main results. Theorem 2.1 indicates that once the condition is met, it is easy to obtain the saddle point $(u^*(t),\ v^*(t))$ for zero-sum stochastic differential games problem. We take that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix},\ C_1 = \begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & -10 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix},\ R = 1,\ S = 16.9402$$

in(1) and (2). By calculating, there are some matrices Y satisfying the inequality (4). We only choice one of them, which is allowed:

$$P = \begin{bmatrix} 14.8421 & 0 \\ 0 & 0.7495 \end{bmatrix},\ Y = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

And the optimal control inputs and optimal cost value are as follows:

$$\begin{cases} u^*(t) = K_1^* x(t) = -R^{-1}B_1'Px(t) = [-11.8737 \quad -0.3748]x(t), \\ v^*(t) = \mathcal{M}^{-1}(C_1'P + C_2'PA_2)\tilde{x}(t) = [1.9789 \quad 0.1764]\tilde{x}(t), \\ J(u^*, v^*) = x_0'Px_0 = 26.8341. \end{cases}$$

3. Infinite-time Stochastic Differential Games with Markov Jumps and Multiplicative Noise.

3.1. Models. In the section, we further consider the problem of infinite-time nonzero-sum stochastic differential games described by the following linear stochastic differential equation with Markov jumps and multiplicative noise:

$$\begin{cases} dx(t) = [A^1(r_t)x(t) + B^1(r_t)u(t) + C^1(r_t)v(t)]dt \\ \qquad\qquad + [A^2(r_t)x(t) + B^2(r_t)u(t) + C^2(r_t)v(t)]dw(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0. \end{cases} \tag{6}$$

Here, for each player, the measurement outputs are $y^\tau(k) = Q^\tau(r_t)x(k)$, $\tau = 1, 2$. And $y(k) \in \mathbb{R}^m$. $x(0) = x_0$ is a deterministic vector. System (6) is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$. There exists a right continuous homogeneous Markov chain $\{r_t;\ t \geq 0\}$ with state space $D = \{1, 2, \cdots, N\}$. $w(t)$ is a one-dimensional Wiener process with $w(0) = 0$. $\mathcal{F}_t$ stands for the smallest σ-algebra generated by process $w(s)$ and r_s, $0 \leq s \leq t$, i.e., $\mathcal{F}_t = \sigma\{w(s),\ r_s \mid 0 \leq s \leq t\} \subset \mathcal{F}$. We assume that r_t is independent of $w(t)$ and has the following transition probability:

$$P\{r_t + \Delta_t = j / r_t = i\} = \begin{cases} q_{ij}\Delta_t + o(\Delta_t), & i \neq j, \\ 1 + q_{ii}\Delta_t + o(\Delta_t), & i = j, \end{cases}$$

where $\Lambda = [q_{ij}]$ is the stationary transition rate matrix of r_t with $q_{ij} \geq 0$ for $i \neq j$. The notation $o(\Delta_t)$ denotes an infinitesimal of higher order than Δ_t.

The coefficients matrices of system (6) are constant real matrices with appropriate dimension, $A^\tau(r_t)$, $B^\tau(r_t)$, $C^\tau(r_t)$ are assigned as $A^\tau(r_t){:=}A_i^\tau$, $B^\tau(r_t){:=}B_i^\tau$, $C^\tau(r_t){:=}C_i^\tau$, for $r_t = i$, $i \in D$, $\tau = 1, 2$, and they are associated to "ith" mode.

Throughout this section, each player has a quadratic cost function, respectively,

$$J^\tau(u, v) = E\int_0^\infty [x'Q^\tau(r_t)'Q^\tau(r_t)x + u'R^\tau(r_t)u + v'S^\tau(r_t)v]dt, \quad \tau = 1, 2. \tag{7}$$

$Q^\tau(r_t) = Q_i^\tau \geq 0$. $R^\tau(r_t) = R_i^\tau \in \mathbb{S}^n$ and $S^\tau(r_t) = S_i^\tau \in \mathbb{S}^n$ are real positive definite matrices.

Let the optimality be defined by the inequalities:

$$J^1(u^*, v^*) \leq J^1(u^*, v), \quad J^2(u^*, v^*) \leq J^2(u, v^*), \tag{8}$$

where $u^*(t) \in \mathcal{L}_2^{n_u}(0, \infty)$, $v^*(t) \in \mathcal{L}_2^{n_v}(0, \infty)$.

The optimal strategies u^* and v^* determined by (8) are also called the Nash equilibrium strategies $(u^*,\ v^*)$. In order to guarantee the unique global Nash equilibrium solutions in (8), both the players are only allowed to take constant feedback controls. So we take the controls as $u(t) = T_i^1 x(t)$, and $v(t) = T_i^2 x(t)$, where T_i^1, T_i^2 are constant matrices with appropriate dimension, and (T_i^1, T_i^2) belong to

$$\mathcal{T} := \{T = (T_i^1, T_i^2) \mid \textit{system (6) can be stabilized} \\ \textit{with}\ \ u(t) = T_i^1 x(t),\ v(t) = T_i^2 x(t)\}.$$

In what follows, we focus on finding the Nash equilibrium strategies $(u^*,\ v^*)$.

3.2. **Definitions and preliminaries.** For the purpose of finding the optimal strategies, we give some useful definitions and lemmas as follows:

Definition 3.1. [18] *The following stochastic system*

$$dx(t) = [A^1(r_t)x(t) + B^1(r_t)u(t)]dt \\ +[A^2(r_t)x(t) + B^2(r_t)u(t)]dw(t),\ x(0) = x_0, \tag{9}$$

is called stochastic stabilizable (in mean-square sense), if there exists a feedback control $u(t) = \sum_{i=1}^{l} K(i)x(t)\chi_{\{r_t=i\}}(t)$ *with* $K(1), K(2), \cdots, K(l)$ *being constant matrices, such that for any initial state* $x(0) = x_0$, $r_0 = i$, *the closed-loop system (9) is asymptotically mean-square stable, i.e.,* $\lim_{t\to\infty} E[x(t)x'(t)] = 0$.

Here, system output equation is $y(t) = Q_i x(t)$. We briefly denote system (9) as $[(A_i^1, B_i^1), (A_i^2, B_i^2) \mid Q_i]$. If $u(t) \equiv 0$, this system is denoted as $[A_i^1, A_i^2 \mid Q_i]$.

Definition 3.2. *System* $[A_i^1, A_i^2 \mid Q_i]$ *is said to be exactly detectable if*

$$y(t) \equiv 0,\ a.s.,\ t \in [0, T],\ \forall T \geq 0 \Rightarrow \lim_{t\to\infty} E \parallel x(t) \parallel^2 = 0.$$

Definition 3.3. [18] $[A_i^1, A_i^2 \mid Q_i]$ *is called stochastically detectable, if there exists a set of gain matrices* H_i *which is constant for each value of* $r_t = i \in D$, *such that* $[A_i^1 + H_iQ_i, A_i^2]$ *is mean-square stable, i.e., for any* $x(0) = x_0$, $r_0 = i$, $\lim_{t\to\infty} E \parallel x(t) \parallel^2 = 0$.

The following lemma generalizes PBH Criterion from complete detectability of deterministic linear systems to exact detectability of $[A_i^1, A_i^2 \mid Q_i]$.

Lemma 3.1. [20] $[A_i^1, A_i^2 \mid Q_i]$ *is exactly detectable if and only if*

$$(Q_1X_1, Q_2X_2, \cdots, Q_NX_N) \neq 0$$

for every eigenvector X_i *of the linear operator* $\mathcal{L}$, $i \in D$, *corresponding to some eigenvalue* λ *with* $Re(\lambda) \geq 0$.

Meanwhile, according to the reference [20], we can know that if $[A_i^1, A_i^2 \mid Q_i]$ is stochastically detectable, then it is also exactly detectable, which indicates that exact detectability is weaker than stochastic detectability.

Define

$$\begin{cases} \Phi_i(t) = E(x_t x_t' I_{\{r_t=i\}}),\ i \in D, \\ \Psi(t) = (\Phi_1(t), \Phi_2(t), \cdots, \Phi_N(t)), \end{cases}$$

and the linear operators:

$$\begin{cases} \mathcal{L}_i(\Psi) = A_i^1\Phi_i + \Phi_i A_i^{1'} + A_i^2\Phi_i A_i^{2'} + \sum_{j=1}^{N} q_{ji}\Phi_j, \\ \mathcal{L}(\Psi) = (\mathcal{L}_1(\Psi), \mathcal{L}_2(\Psi), \cdots, \mathcal{L}_N(\Psi)). \end{cases} \tag{10}$$

The spectrum of $\mathcal{L}$ is the set defined by $\sigma(\mathcal{L}_i) = \{\lambda \in \mathbb{C} | \mathcal{L}_i(\Psi) = \lambda\Phi_i, \Phi_i \in \mathbb{S}^n, \Phi_i \neq 0, i \in D\}$. It is easily seen that $\mathcal{L}$ are a bounded linear operator on the Hilbert space with the inner product defined as $< A,\ B >= \sum_i^N Tr(A_iB_i)$, and its adjoint operators are

$$\begin{cases} \mathcal{L}_i^*(\Psi) = \Phi_i A_i^1 + A_i^{1'}\Phi_i + A_i^{2'}\Phi_i A_i^2 + \sum_{j=1}^{N} q_{ij}\Phi_j, \\ \mathcal{L}^*(\Psi) = (\mathcal{L}_1^*(\Psi), \mathcal{L}_2^*(\Psi), \cdots, \mathcal{L}_N^*(\Psi)). \end{cases}$$

We refer the reader to [20], and it is easy to know that:

$$\frac{d\Psi(t)}{dt} = \mathcal{L}(\Psi(t)).$$

Next, some results relevant to the exact detectability and stochastic detectability are presented as follows.

For convenience, we write the system (6) in another way because it is not brief enough to convey some definitions and lemmas. The form can be described as follows:

$$\begin{cases} dx(t) = M_i x(t)dt + N_i x(t)dw(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{11}$$

where M_i stands for $A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2$, N_i stands for $A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2$.

Proposition 3.1. *If $[A_i^1, A_i^2 \mid Q_i]$ is exactly detectable, then so is $[M_i, N_i \mid \hat{\sum}_i]$, where $\hat{\sum}_i = Q_i' Q_i + T_i^{1'} R_i T_i^1 + T_i^{2'} S_i T_i^2$.*

Proof: Suppose $[M_i, N_i \mid \hat{\sum}_i]$ is not exactly detectable, then according to PBH criterion in Lemma 3.1, there exists an eigenvector X_i of $\mathcal{L}$ such that

$$\hat{\sum}_i X_i = \left(\hat{\sum}_1 X_1, \hat{\sum}_2 X_2, \cdots, \hat{\sum}_N X_N \right) = 0, \ i \in D.$$

So, by pre-multiplying $X_i{}'$ on the both sides of above equation, we can get that

$$X_i'(Q_i' Q_i + T_i^{1'} R_i T_i^1 + T_i^{2'} S_i T_i^2) X_i = 0, \quad i \in D.$$

Because $Q_i \geq 0$, $R_i > 0$, $S_i > 0$ and $X_i \neq 0$, it follows that

$$T_i^1 X_i = 0, \quad T_i^2 X_i = 0, \quad Q_i X_i = 0, \quad i \in D.$$

Thus for any $i = 1, 2, \cdots, N$, according to (10) and the spectrum of $\mathcal{L}$ we get

$$\begin{aligned} \lambda X_i &= (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2) X_i + X_i (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2)' \\ &\quad + (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2) X_i (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2)' + \sum_{j=1}^{N} q_{ji} X_j \\ &= A_i^1 X_i + X_i (A_i^1)' + A_i^2 X_i (A_i^2)' + \sum_{j=1}^{N} q_{ji} X_j. \end{aligned}$$

Therefore, $[A_i^1, A_i^2 \mid Q_i]$ is not exactly detectable, which contradicts our assumption.

In order to obtain the main results, we give the very useful lemmas as follows.

Lemma 3.2. *For system $[A_i^1, A_i^2 \mid Q_i]$, if it is stochastically detectable, then so is $[M_i, N_i \mid \hat{\sum}_i]$, where $\hat{\sum}_i = Q_i' Q_i + T_i^{1'} R_i T_i^1 + T_i^{2'} S_i T_i^2$.*

Proof: Since system $[A_i^1, A_i^2 \mid Q_i]$ is stochastically detectable, then by Definition 3.3, we know that Q_i is of full row-rank. And due to $Q_i \geq 0, R_i > 0, S_i > 0$, consequently, $\hat{\sum}_i = Q_i' Q_i + T_i^{1'} R_i T_i^1 + T_i^{2'} S_i T_i^2$ is of full row-rank. Therefore, we can obtain the conclusion that $[M_i, N_i \mid \hat{\sum}_i]$ is stochastically detectable.

Lemma 3.3. [18] *If system $[A_i^1, A_i^2 \mid Q_i]$ is stochastically detectable, then $[A_i^1, A_i^2]$ is asymptotically mean-square stable if and only if the following Lyapunov-type equation:*

$$\mathcal{L}_i^*(P) := P_i A_i^1 + A_i^{1'} P_i + A_i^{2'} P_i A_i^2 + \sum_{j=1}^{N} q_{ij} P_j + Q_i' Q_i = 0$$

has a unique set of positive semi-definite solution $P_i \geq 0 \in \mathbb{S}^n$, $i \in D$.

3.3. **Main results.** Under the condition of stochastic detectability, we give the following theorems, which generalize the results of Theorem 4.1 in [16] to stochastic system with Markov jumps.

Theorem 3.1. *For system (6) or (11), assume the following coupled equations*

$$\begin{cases} (A_i^1 + B_i^1 T_i^1)' P_i^1 + P_i^1 (A_i^1 + B_i^1 T_i^1) + (A_i^2 + B_i^2 T_i^1)' P_i^1 (A_i^2 + B_i^2 T_i^1) \\ + T_i^{1'} R_i^1 T_i^1 + Q_i^{1'} Q_i^1 - [C_i^{1'} P_i^1 + C_i^{2'} P_i^1 (A_i^2 + B_i^2 T_i^1)]' \\ \times (S_i^1 + C_i^{2'} P_i^1 C_i^2)^{-1} [C_i^{1'} P_i^1 + C_i^{2'} P_i^1 (A_i^2 + B_i^2 T_i^1)] + \sum_{j=1}^N q_{ij} P_j^1 = 0, \\ S_i^1 + C_i^{2'} P_i^1 C_i^2 > 0, \end{cases} \tag{12}$$

$$T_i^1 = -(R_i^1 + B_i^{2'} P_i^2 B_i^2)^{-1} [B_i^{1'} P_i^2 + B_i^{2'} P_i^2 (A_i^2 + C_i^2 T_i^2)], \tag{13}$$

$$\begin{cases} (A_i^1 + C_i^1 T_i^2)' P_i^2 + P_i^2 (A_i^1 + C_i^1 T_i^2) + (A_i^2 + C_i^2 T_i^2)' P_i^2 (A_i^2 + C_i^2 T_i^2) \\ + T_i^{2'} S_i^2 T_i^2 + Q_i^{2'} Q_i^2 - [B_i^{1'} P_i^2 + B_i^{2'} P_i^2 (A_i^2 + C_i^2 T_i^2)]' \\ \times (R_i^2 + B_i^{2'} P_i^2 B_i^2)^{-1} [B_i^{1'} P_i^2 + B_i^{2'} P_i^2 (A_i^2 + C_i^2 T_i^2)] + \sum_{j=1}^N q_{ij} P_j^2 = 0, \\ R_i^2 + B_i^{2'} P_i^2 B_i^2 > 0, \end{cases} \tag{14}$$

$$T_i^2 = -(S_i^1 + C_i^{2'} P_i^1 C_i^2)^{-1} [C_i^{1'} P_i^1 + C_i^{2'} P_i^1 (A_i^2 + B_i^2 T_i^1)], \tag{15}$$

admit the solution $(P_i^1, P_i^2; T_i^1, T_i^2)$ *with* $P_i^1 \geq 0$, $P_i^2 \geq 0$. *If* $[A_i^1, A_i^2 \mid Q_i^\tau]$ $(\tau = 1, 2)$ *is stochastically detectable, then*
(i) $(T_i^1, T_i^2) \in \mathcal{T}$.
(ii) The problem of infinite-time stochastic differential games admits a pair of solutions $(u^*(t), v^*(t))$ *with* $u^*(t) = T_i^1 x(t)$, $v^*(t) = T_i^2 x(t)$.
(iii) The optimal cost functions incurred by playing strategies $(u^*(t), v^*(t))$ *are* $J^\tau(u^*, v^*) = x_0' P_i^\tau x_0$ $(\tau = 1, 2)$.

Proof: For system (11), we give the Lyapunov-type equations as follows:

$$\begin{cases} P_i^1 (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2) + (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2)' P_i^1 \\ + Q_i^{1'} Q_i^1 + T_i^{1'} R_i^1 T_i^1 + T_i^{2'} S_i^1 T_i^2 + \sum_{j=1}^N q_{ij} P_j^1 \\ + (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2)' P_i^1 (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2) = 0, \\ S_i^1 + C_i^{2'} P_i^1 C_i^2 > 0, \end{cases} \tag{16}$$

$$\begin{cases} P_i^2 (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2) + (A_i^1 + B_i^1 T_i^1 + C_i^1 T_i^2)' P_i^2 \\ + Q_i^{2'} Q_i^2 + T_i^{1'} R_i^2 T_i^1 + T_i^{2'} S_i^2 T_i^2 + \sum_{j=1}^N q_{ij} P_j^2 \\ + (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2)' P_i^2 (A_i^2 + B_i^2 T_i^1 + C_i^2 T_i^2) = 0, \\ R_i^2 + B_i^{2'} P_i^2 B_i^2 > 0. \end{cases} \tag{17}$$

Then for (16), we can get

$$\begin{cases} P_i^1 (A_i^1 + B_i^1 T_i^1) + P_i^1 C_i^1 T_i^2 + (A_i^1 + B_i^1 T_i^1)' P_i^1 \\ + (C_i^1 T_i^2)' P_i^1 + Q_i^{1'} Q_i^1 + T_i^{1'} R_i^1 T_i^1 + T_i^{2'} S_i^1 T_i^2 \\ + (A_i^2 + B_i^2 T_i^1)' P_i^1 (A_i^2 + B_i^2 T_i^1) + (A_i^2 + B_i^2 T_i^1)' P_i^1 C_i^2 T_i^2 \\ + (C_i^2 T_i^2)' P_i^1 (A_i^2 + B_i^2 T_i^1) + (C_i^2 T_i^2)' P_i^1 (C_i^2 T_i^2) + \sum_{j=1}^N q_{ij} P_j^1 = 0, \\ S_i^1 + C_i^{2'} P_i^1 C_i^2 > 0. \end{cases}$$

Next, we can obtain the following equation:

$$\begin{aligned}
&P_i^1C_i^1T_i^2+(C_i^1T_i^2)'P_i^1+T_i^{2'}S_i^1T_i^2+(C_i^2T_i^2)'P_i^1C_i^2T_i^2\\
&+(A_i^2+B_i^2T_i^1)'P_i^1C_i^2T_i^2+(C_i^2T_i^2)'P_i^1(A_i^2+B_i^2T_i^1)\\
=\ &T_i^{2'}(S_i^1+C_i^{2'}P_i^1C_i^2)T_i^2+T_i^{2'}[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2\\
&+B_i^2T_i^1)]+[P_i^1C_i^1+(A_i^2+B_i^2T_i^1)'P_i^1C_i^2]T_i^2\\
=\ &\{T_i^2+(S_i^1+C_i^{2'}P_i^1C_i^2)^{-1}[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)]\}'(S_i^1+C_i^{2'}P_i^1C_i^2)\\
&\times\{T_i^2+(S_i^1+C_i^{2'}P_i^1C_i^2)^{-1}[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)]\}\\
&-[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)]'(S_i^1+C_i^{2'}P_i^1C_i^2)^{-1}[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)].
\end{aligned}$$

Let

$$T_i^2=-(S_i^1+C_i^{2'}P_i^1C_i^2)^{-1}[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)].$$

As a result, we can get (12) and (15). Similarly, by rearranging (17), let

$$T_i^1=-(R_i^1+B_i^{2'}P_i^2B_i^2)^{-1}[B_i^{1'}P_i^2+B_i^{2'}P_i^2(A_i^2+C_i^2T_i^2)],$$

we can get (13) and (14). If $[A_i^1,A_i^2\mid Q_i^\tau]$ $(\tau=1,2)$ is stochastically detectable, then we can get $[M_i,N_i\mid\hat{\sum}_i^\tau]$ is also stochastically detectable by Lemma 3.2. And we assume that (12)-(15) admit the solution $(P_i^1,P_i^2;T_i^1,T_i^2)$ with $P_i^1\geq 0$, $P_i^2\geq 0$, i.e., (16) and (17) have the solution $P_i^1\geq 0$ and $P_i^2\geq 0$, respectively. Then, the system $[M_i,N_i\mid\hat{\sum}_i^\tau]$ satisfies Lemma 3.3. Therefore, system (11) is asymptotically mean-square stable, i.e., system (11) can be stabilized with $u(t)=T_i^1x(t)$, $v(t)=T_i^2x(t)$. Since we have found the feedback control $(T_i^1,T_i^2)\in\mathcal{T}$, then (i) is proved.

To prove (ii) and (iii), we note that $u^*(t)=T_i^1x(t)$, and by substituting $u^*(t)$ into (6), we can obtain the following system:

$$\begin{cases}dx(t)=[(A_i^1+B_i^1T_i^1)x(t)+C_i^1v(t)]dt+[(A_i^2+B_i^2T_i^1)x(t)+C_i^2v(t)]dw(t),\\ x(0)=x_0\in\mathbb{R}^n.\end{cases}$$

Then considering the scalar function $V(t,x(t)){:=}x'(t)P_i^1x(t)$, and by applying $It\hat{o}$ formula to $V(t,x(t))$, we have

$$\begin{aligned}
&d[V(t,x(t))]\\
=\ &d(x'P_i^1x)\\
=\ &\{x'[(A_i^1+B_i^1T_i^1)'P_i^1+P_i^1(A_i^1+B_i^1T_i^1)+(A_i^2+B_i^2T_i^1)'P_i^1(A_i^2+B_i^2T_i^1)]x\\
&+v'[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)]x\\
&+x'[C_i^{1'}P_i^1+C_i^{2'}P_i^1(A_i^2+B_i^2T_i^1)]'v+v'C_i^{2'}P_i^1C_i^2v\}dt\\
&+\{x'[(A_i^2+B_i^2T_i^1)'P_i^1+P_i^1(A_i^2+B_i^2T_i^1)]x+v'C_i^{2'}P_i^1x+x'P_i^1C_i^2v\}dw(t).
\end{aligned}$$

Due to

$$E\int_0^\infty d(x'P_i^1x)=\lim_{t\to\infty}x'(t)P_i^1x(t)-x_0'P_i^1x_0=-x_0'P_i^1x_0,$$

and by (12) and completing the squares, (7) can be derived that

$$\begin{aligned}
J^1(u^*,v)=\ &E\int_0^\infty[x'(Q_i^{1'}Q_i^1+T_i^{1'}R_i^1T_i^1)x+v'S_i^1v]dt\\
&+E\int_0^\infty d(x'P_i^1x)+x_0'P_i^1x_0
\end{aligned}$$

$$
\begin{aligned}
&= E\int_0^\infty (v - T_i^2 x)'(S_i^1 + C_i^{2'} P_i^1 C_i^2)(v - T_i^2 x)dt + x_0' P_i^1 x_0 \\
&\geq x_0' P_i^1 x_0.
\end{aligned}
$$

Then by (8), it follows that $v^*(t) = T_i^2 x(t)$ and $J^1(u^*, v^*) = x_0' P_i^1 x_0 \leq J^1(u^*, v)$. Next, by substituting $v^*(t) = T_i^2 x(t)$ into (6) and following the same procedures as before, we have $u^*(t) = T_i^1 x(t)$ and $J^2(u^*, v^*) = x_0' P_i^2 x_0 \leq J^2(u, v^*)$. So this ends the proof.

Since Lemma 3.3 presents a necessary and sufficient condition of asymptotically mean-square stability under the condition of stochastic detectability, the following result still holds.

Theorem 3.2. *For system (6) or (11), assume (12)-(15) admit the solution* $(P_i^1, P_i^2; T_i^1, T_i^2)$ *with* $(T_i^1, T_i^2) \in \mathcal{T}$*, if* $[A_i^1, A_i^2 \mid Q_i^\tau]$ $(\tau = 1, 2)$ *is stochastically detectable, then*
(i) $P_i^1 \geq 0$, $P_i^2 \geq 0$.
(ii) The problem of infinite-time stochastic differential games admits a pair of solutions $(u^*(t), v^*(t))$ *with* $u^*(t) = T_i^1 x(t)$, $v^*(t) = T_i^2 x(t)$.
(iii) The optimal cost functions incurred by playing strategies $(u^*(t), v^*(t))$ *are* $J^\tau(u^*, v^*) = x_0' P_i^\tau x_0$ $(\tau = 1, 2)$.

Proof: Theorem 3.2 can be proved following the similar way in the proof of Theorem 3.1 and thus the proof is omitted.

3.4. **Simulation.** A numerical example is presented to demonstrate the efficiency of our main results in the section. Theorem 3.1 shows that the infinite-time Nash equilibrium can be obtained by solving the coupled equations. It is hard to solve (12)-(15), so we take system (6) as one dimensional system. For convenience, let $A_i^1 = A_i^2 = Q_i^1 = Q_i^2 = R_i^1 = R_i^2 = S_i^1 = S_i^2 = 1$, $B_i^1 = C_i^1 = -1$, $B_i^2 = C_i^2 = 0$, $r_t = i \in D = \{1, 2\}$, $q_{11} = q_{12} = q_{21} = q_{22} = 0.5$, $x(0) = x_0 = 2$, respectively. $w(t)$ is Gaussian white noise. When $i = 1$ and 2, by (12)-(15), the coupled equations are obtained respectively as follows:

$$
\begin{cases}
2P_1^1(1 - T_1^1) + P_1^1 + (T_1^1)^2 - (P_1^1)^2 + 0.5P_1^1 + 0.5P_2^1 + 1 = 0, \\
T_1^1 = P_1^2, \\
2P_1^2(1 - T_1^2) + P_1^2 + (T_1^2)^2 - (P_1^2)^2 + 0.5P_1^2 + 0.5P_2^2 + 1 = 0, \\
T_1^2 = P_1^1,
\end{cases}
$$

and

$$
\begin{cases}
2P_2^1(1 - T_2^1) + P_2^1 + (T_2^1)^2 - (P_2^1)^2 + 0.5P_1^1 + 0.5P_2^1 + 1 = 0, \\
T_2^1 = P_2^2, \\
2P_2^2(1 - T_2^2) + P_2^2 + (T_2^2)^2 - (P_2^2)^2 + 0.5P_1^2 + 0.5P_2^2 + 1 = 0, \\
T_2^2 = P_2^1.
\end{cases}
$$

By calculations, we have $P_1^1 = P_1^2 = P_2^1 = P_2^2 = 2.2247 > 0$, $T_1^1 = T_1^2 = T_2^1 = T_2^2 = 2.2247$, $x^*(t) = 2e^{-3.9494t + w(t)}$, $u^*(t) = 4.4494e^{-3.9494t + w(t)}$, $v^*(t) = 4.4494e^{-3.9494t + w(t)}$. And it is easy to verify system (6) is stochastically detectable, then $(T_i^1, T_i^2) \in \mathcal{T}$. $J^\tau(u^*, v^*) = P_1^1 x_0^2 = 8.8988$, $\tau = 1, 2$. System state and control inputs optimal trajectory are shown in Figure 1. The simulation results are reasonable and effective.

4. **Conclusions.** In this paper, we have dealt with zero-sum and nonzero-sum LQ stochastic differential games in infinite-time case, which is the further research of our previous results [16, 26]. Firstly, we have focused on zero-sum LQ stochastic differential games with multiplicative noise, where the state weighting matrix is allowed to be indefinite. An interesting result is obtained, which extends the results in the deterministic

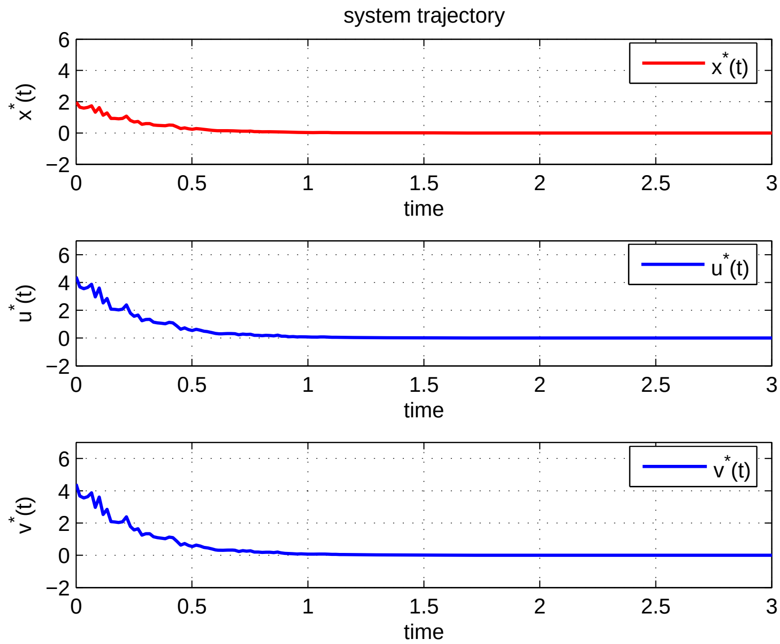

FIGURE 1. System state $x(t)$, optimal control inputs $u^*(t)$ and $v^*(t)$

case. Secondly, by means of some important definitions relevant to stochastic detectability, we have presented two important theorems on finding the Nash equilibrium strategies of LQ differential games for stochastic systems with Markov jumps and multiplicative noise. It has indicated that the Nash equilibrium strategies have close relation to the solution of four coupled GAREs, and the four coupled GAREs have been obtained on the basis of stochastic detectability. At the end of each section, the corresponding simulation examples have been presented to illustrate the main results. We believe that zero-sum LQ stochastic differential games with Markov jumps and multiplicative noise still have essential applications, and further studies on such kind of case should be continued.

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