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FIXED-END MOMENTS FOR BEAMS SUBJECTED TO A CONCENTRATED FORCE LOCALIZED ANYWHERE TAKING INTO ACCOUNT THE SHEAR DEFORMATIONS

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Abstract. *This paper presents a mathematical model for beams subjected to a concentrated force localized anywhere of member taking into account the shear deformations to obtain the fixed-end moments. The consistent deformation method is used to solve such problems; a method based on the superposition of its effects and by the Bernoulli-Euler theory are obtained the deformations anywhere of the beam. Traditional methods used for beams subjected to a concentrated force are not considered the shear deformations. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the displacements, fixed-end moments are calculated for any cross section of the beam using the respective integral representations as mathematical formulas.*

Keywords: Shear deformations, Poisson's ratio, Elasticity modulus, Shear modulus and shear area

1. **Introduction.** Structural analysis is the study of structures such as discrete systems. The theory of the structures is essentially based on the fundamentals of mechanics with which are formulated the different structural members. The laws or rules that define the balance and continuity of a structure can be expressed in different ways, including partial differential equations of continuous medium three-dimensional, ordinary differential equations that define a member or the theories several of beams, or simply, algebraic equations for a discrete structure [1].

Structural analysis can be addressed using three main approaches [2]: a) tensorial formulation (Newtonian mechanics and vectorial), b) formulation based on the principles of virtual work, c) formulation based on classical mechanics [3,4].

As regards the conventional techniques of structural analysis of beams and rigid frames to obtain the fixed-end moments, the common practice considers only the bending deformations [5,6].

Recently, a method of structural analysis for statically indeterminate beams and rigid frames was developed, the method takes into account the bending deformations and shear to generate a system of equations in function of rotations and displacements [7-9]. Also a moments-distribution method considering the bending deformations and shear was presented [10]. These methods do not consider shear deformations in the fixed-end moments.

After, a mathematical model is presented to obtain the fixed-end moments of a beam subjected to a uniformly distributed load and also to a triangularly distributed load taking into account the bending deformations and shear [11,12].

This paper presents a mathematical model for beams subjected to a concentrated force localized anywhere of member considering the bending deformations and shear to obtain fixed-end moments. Also, a comparison is realized between traditional model and proposed model to observe differences.

2. **Mathematical Development of the Proposed Model.** The scheme of deformations of a structural member is illustrated in Figure 1, which shows the difference between the Timoshenko theory and Euler-Bernoulli theory: the first "θ_z" and "dy/dx" do not coincide necessarily, while in the second are equal [7-16].

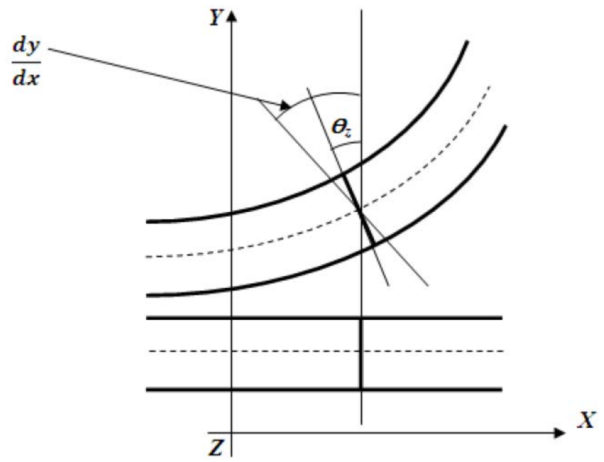

FIGURE 1. Deformation of a structure member

The fundamental difference between Euler-Bernoulli theory and Timoshenko theory is that in the first the relative rotation of the section is approximated by the derivative of vertical displacement; this is an approximation valid only for long members in relation to the dimensions of cross section, and then it happens due to the fact that shear deformations are negligible in comparison with the deformations caused by moment. On the Timoshenko theory, which considers the deformation due to shear, i.e., and is valid therefore for short members and long, the equation of the elastic curve is given by the complex system of equations:

$$G\left(\frac{dy}{dx} - \theta_z\right) = \frac{V_y}{A_s} \tag{1}$$

$$E\left(\frac{d\theta_z}{dx}\right) = \frac{M_z}{I_z} \tag{2}$$

where G is shear modulus, dy/dx is the total rotation around axis "Z", θ_z is rotation around axis "Z" due to the bending, V_y is shear force in direction "Y", A_s is shear area, $d\theta_z/dx$ is d^2y/dx^2, E is elasticity modulus, M_z is moment around axis "Z", and I_z is moment of inertia around axis "Z".

Deriving Equation (1) and substituting into Equation (2), it is arrived at the equation of the elastic curve including the effect of shear stress:

$$\frac{d^2y}{dx^2} = \frac{1}{GA_s}\frac{dV_y}{dx} + \frac{M_z}{EI_z} \tag{3}$$

Equation (3) is integrated to obtain the rotation anywhere:

$$\frac{dy}{dx} = \frac{V_y}{GA_s} + \int \frac{M_z}{EI_z}dx \tag{4}$$

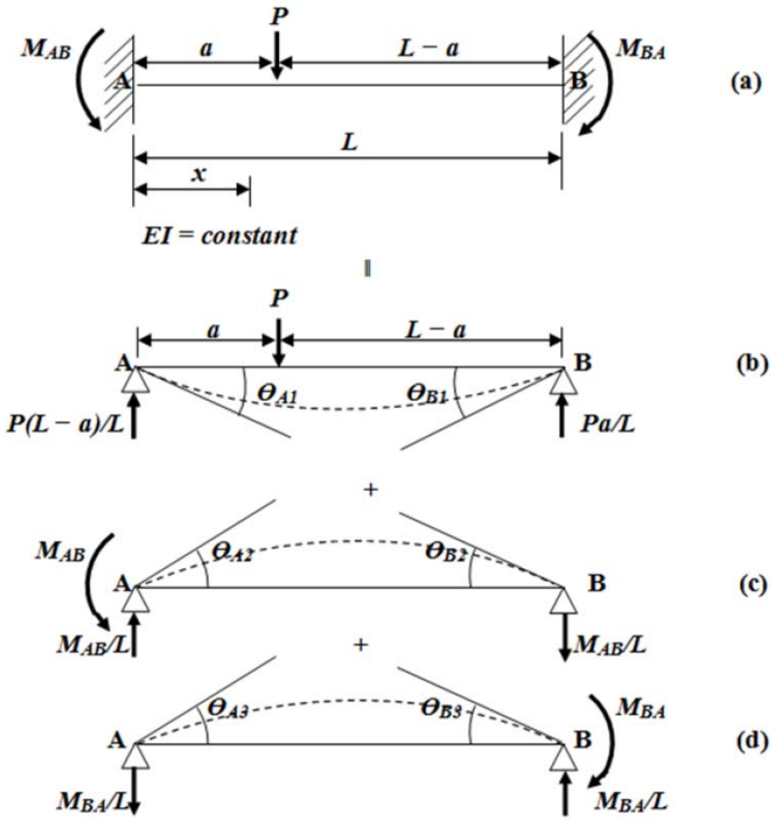

FIGURE 2. Derivation of equations for moments

Figure 2(a) shows the beam "AB" subjected to a concentrated force "P" and fixed-ends. The fixed-end moments are found by the sum of the effects. The moments are considered positive in counterclockwise and the moments are considered negative in clockwise. Figure 2(b) presents the same beam simply supported at their ends with the force applied to find the rotations "θ_{A1}" and "θ_{B1}". Now, the rotations "θ_{A2}" and "θ_{B2}" are caused by the moment "M_{AB}" applied in the support "A", according to Figure 2(c), and in terms of "θ_{A3}" and "θ_{B3}" are caused by the moment "M_{BA}" applied in the support "B", see Figure 2(d) [7-16].

The conditions of geometry are [7-12]:

$$\theta_{A1} + \theta_{A2} + \theta_{A3} = 0 \tag{5}$$

$$\theta_{B1} + \theta_{B2} + \theta_{B3} = 0 \tag{6}$$

The beam of Figure 2(b) is analyzed to find "θ_{A1}" and "θ_{B1}" by Euler-Bernoulli theory to obtain the deflections [7-16].

Shear force and moment anywhere of the beam on axis "x" is:

To $0 \leq x \leq a$:

$$V_x = -\frac{P(L-a)}{L} \tag{7}$$

$$M_x = -\frac{P(L-a)x}{L} \tag{8}$$

To $a \leq x \leq L$:

$$V_x = \frac{Pa}{L} \tag{9}$$

$$M_x = -\frac{Pa(L-x)}{L} \tag{10}$$

where: $V_x = V_y$, and $M_x = M_y$.

We analyze for $0 \leq x \leq a$.

Equations (7) and (8) are substituted into Equation (4):

$$\frac{dy}{dx} = -\frac{P(L-a)}{GA_sL} - \frac{P(L-a)}{EI_zL}\int (x)dx \tag{11}$$

The integral of Equation (11) is developed:

$$\frac{dy}{dx} = -\frac{P(L-a)}{GA_sL} - \frac{P(L-a)}{EI_zL}\left(\frac{x^2}{2} + C_1\right) \tag{12}$$

Substituting $x = a$, into Equation (12) to find the rotation $dy/dx = \theta_{a1}$, where the concentrated force is localized:

$$\theta_{a1} = -\frac{P(L-a)}{GA_sL} - \frac{P(L-a)}{EI_zL}\left(\frac{a^2}{2} + C_1\right) \tag{13}$$

Equation (13) is integrated to obtain the displacements, because there are unknown conditions for rotations, this is as follows:

$$y = -\frac{P(L-a)}{GA_sL}x + C_3 - \frac{P(L-a)}{EI_zL}\left(\frac{x^3}{6} + C_1x + C_2\right) \tag{14}$$

Then, shear deformations and bending must be separated to obtain the integration constants, this is as follows.

Shear deformation is:

$$y_s = -\frac{P(L-a)}{GA_sL}x + C_3 \tag{15}$$

The boundary conditions are considered into Equation (15), when $x = 0$; $y = 0$ to find $C_3 = 0$.

Now, value of "C_3" is substituted into Equation (15):

$$y_s = -\frac{P(L-a)}{GA_sL}x \tag{16}$$

Substituting $x = a$ into Equation (16) to find the displacement $y = y_{as1}$, where the concentrated force is localized:

$$y_{as1} = -\frac{Pa(L-a)}{GA_sL} \tag{17}$$

Bending deformation is:

$$y_f = -\frac{P(L-a)}{EI_zL}\left(\frac{x^3}{6} + C_1x + C_2\right) \tag{18}$$

The boundary conditions are substituted into Equation (18), when $x = 0$; $y = 0$ to find $C_2 = 0$.

Now, value of "C_2" is substituted into Equation (18):

$$y_f = -\frac{P(L-a)}{EI_zL}\left(\frac{x^3}{6} + C_1x\right) \tag{19}$$

Substituting $x = a$, into Equation (19) to find the displacement $y = y_{af1}$, where the concentrated force is localized:

$$y_{af1} = -\frac{P(L-a)}{EI_zL}\left(\frac{a^3}{6} + C_1a\right) \tag{20}$$

Then, y_{af1} and y_{as1} are summed to find the total displacement y_{at1}:

$$y_{at1} = -\frac{Pa(L-a)}{GA_sL} - \frac{P(L-a)}{EI_zL}\left(\frac{a^3}{6} + C_1a\right) \tag{21}$$

We analyze for $a \leq x \leq L$.

Equations (9) and (10) are substituted into Equation (4):

$$\frac{dy}{dx} = \frac{Pa}{GA_sL} - \frac{Pa}{EI_zL}\int (L-x)dx \tag{22}$$

The integral of Equation (11) is developed:

$$\frac{dy}{dx} = \frac{Pa}{GA_sL} - \frac{Pa}{EI_zL}\left(Lx - \frac{x^2}{2} + C_4\right) \tag{23}$$

Substituting $x = a$ into Equation (23) to find the rotation $dy/dx = \theta_{a2}$, where the concentrated load is localized:

$$\theta_{a2} = \frac{Pa}{GA_sL} - \frac{Pa}{EI_zL}\left(La - \frac{a^2}{2} + C_4\right) \tag{24}$$

Equation (23) is integrated to obtain the displacements, because there are unknown conditions for rotations, this is as follows:

$$y = \frac{Pa}{GA_sL}x + C_6 - \frac{Pa}{EI_zL}\left(\frac{Lx}{2} - \frac{x^3}{6} + C_4x + C_5\right) \tag{25}$$

Then, shear deformations and bending are separated to obtain the integration constants, this is as follows.

Shear deformation is:

$$y_s = \frac{Pa}{GA_sL}x + C_6 \tag{26}$$

The boundary conditions are substituted into Equation (26), when $x = L$; $y = 0$ to obtain $C_6 = -Pa/GA_s$.

Now, value of "C_6" is substituted into Equation (26):

$$y_s = \frac{Pa}{GA_sL}x - \frac{Pa}{GA_s} \tag{27}$$

Substituting $x = a$, into Equation (27) to find the displacement $y = y_{as2}$, where the concentrated load is localized:

$$y_{as2} = \frac{P(a^2 - La)}{GA_sL} \tag{28}$$

Bending deformation is:

$$y_f = -\frac{Pa}{EI_zL}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_4x + C_5\right) \tag{29}$$

The boundary conditions are substituted into Equation (29), when $x = L$; $y = 0$ to find "C_5" in function of "C_4".

$$C_5 = -\frac{L^3}{3} - C_4L \tag{30}$$

Now, value of "C_5" is substituted into Equation (29):

$$y_f = -\frac{Pa}{EI_zL}\left[\frac{Lx^2}{2} - \frac{x^3}{6} + C_4(x-L) - \frac{L^3}{3}\right] \tag{31}$$

Substituting $x = a$, into Equation (31) to find the displacement $y = y_{af2}$, where the concentrated load is localized:

$$y_{af2} = -\frac{Pa}{EI_zL}\left[\frac{La^2}{2} - \frac{a^3}{6} + C_4(a-L) - \frac{L^3}{3}\right] \tag{32}$$

Then, y_{af2} and y_{as2} are summed to find the total displacement y_{at2}:

$$y_{at2} = \frac{P(a^2-La)}{GA_sL} - \frac{Pa}{EI_zL}\left[\frac{La}{2} - \frac{a^3}{6} + C_4(a-L) - \frac{L^3}{3}\right] \tag{33}$$

Then, Equations (13) and (24) are equalized, because rotations must be equal at the point $x = a$, where the force is applied to find the constant "C_4" in function of "C_1", this is:

$$C_4 = \frac{EI_zL}{aGA_s} - \frac{La}{2} + \frac{C_1(L-a)}{a} \tag{34}$$

Also, Equations (21) and (33) are equalized, because the displacements must be equal at the point $x = a$, where the force is applied and subsequently Equation (34) is substituted to find the constant "C_1", this value is:

$$C_1 = -\frac{EI_z}{GA_s} - \frac{a(a^2-3aL+2L^2)}{6(L-a)} \tag{35}$$

Equation (35) is substituted into Equation (34) to find the constant "C_4":

$$C_4 = \frac{EI_z}{GA_s} - \frac{(a^2+2L^2)}{6} \tag{36}$$

Equation (35) is substituted into Equation (12) to obtain the rotations anywhere of the segment $0 \le x \le a$:

$$\frac{dy}{dx} = -\frac{P(L-a)}{GA_sL} - \frac{P(L-a)}{EI_zL}\left[\frac{x^2}{2} - \frac{EI_z}{GA_s} - \frac{a(a^2-3aL+2L^2)}{6(L-a)}\right] \tag{37}$$

Substituting $x = 0$ into Equation (37) to find the rotation in support "A", this is:

$$\theta_{A1} = \frac{Pa(a^2-3aL+2L^2)}{6EI_zL} \tag{38}$$

Equation (36) is substituted into Equation (23) to obtain the rotations anywhere of the segment $a \le x \le L$:

$$\frac{dy}{dx} = \frac{Pa}{GA_sL} - \frac{Pa}{EI_zL}\left(Lx - \frac{x^2}{2} + \frac{EI_z}{GA_s} - \frac{a^2+2L^2}{6}\right) \tag{39}$$

Substituting $x = L$ into Equation (39) to find the rotation in support "B", this is:

$$\theta_{B1} = \frac{Pa(a^2-L^2)}{6EI_zL} \tag{40}$$

The beam of Figure 2(c) is analyzed to find "θ_{A2}" and "θ_{B2}" in function of "M_{AB}" [7-12].

Shear force and moment anywhere of the beam on axis "x" is:

$$V_x = -\frac{M_{AB}}{L} \tag{41}$$

$$M_x = \frac{M_{AB}}{L}(L-x) \tag{42}$$

Equations (41) and (42) are substituted into Equation (4):

$$\frac{dy}{dx} = -\frac{M_{AB}}{GA_sL} + \frac{M_{AB}}{EI_zL}\int(L-x)dx \tag{43}$$

The integral of Equation (43) is developed:

$$\frac{dy}{dx} = -\frac{M_{AB}}{GA_sL} + \frac{M_{AB}}{EI_zL}\left(Lx - \frac{x^2}{2} + C_1\right) \tag{44}$$

Equation (43) is integrated to obtain the displacements, because there are unknown conditions for rotations, this is as follows:

$$y = -\frac{M_{AB}}{GA_sL}x + C_3 + \frac{M_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} + C_1x + C_2\right) \tag{45}$$

Then, shear deformations and bending are separated to obtain the integration constants, this is as follows.

Shear deformation is:

$$y = -\frac{M_{AB}}{GA_sL}x + C_3 \tag{46}$$

The boundary conditions are substituted into Equation (46), when $x = 0$; $y = 0$ to find $C_3 = 0$.

Now, value of "C_3" is substituted into Equation (46):

$$y = -\frac{M_{AB}}{GA_sL}x \tag{47}$$

Bending deformation is:

$$y = \frac{M_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} + C_1x + C_2\right) \tag{48}$$

The boundary conditions are substituted into Equation (48), when $x = 0$; $y = 0$ to find $C_2 = 0$.

Now, value of "C_2" is substituted into Equation (48):

$$y = \frac{M_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} + C_1x\right) \tag{49}$$

Now, the boundary conditions are substituted into Equation (49), when $x = L$; $y = 0$ to find $C_1 = -L^2/3$.

Then, the value of "C_1" is substituted into Equation (44):

$$\frac{dy}{dx} = -\frac{M_{AB}}{GA_sL} + \frac{M_{AB}}{EI_zL}\left(Lx - \frac{x^2}{2} - \frac{L^2}{3}\right) \tag{50}$$

Substituting $x = 0$ into Equation (50) to find the rotation in support "A", this is:

$$\theta_{A2} = -\frac{M_{AB}L}{3EI_z} - \frac{M_{AB}}{GA_sL} = -\frac{M_{AB}L}{12EI_z}\left(4 + \frac{12EI_z}{GA_sL^2}\right) \tag{51}$$

Being [7-12]:

$$\varnothing = \frac{12EI_z}{GA_sL^2} \tag{52}$$

where $\varnothing$ is shape factor.

Then, Equation (52) is substituted into Equation (51), this is as follows:

$$\theta_{A2} = -\frac{M_{AB}L}{12EI_z}(4 + \varnothing) \tag{53}$$

Substituting $x = L$ into Equation (50) to find the rotation in support "B", this is:

$$\theta_{B2} = \frac{M_{AB}L}{6EI_z} - \frac{M_{AB}}{GA_sL} = \frac{M_{AB}L}{12EI_z}\left(2 - \frac{12EI_z}{GA_sL^2}\right) \tag{54}$$

Then, Equation (52) is substituted into Equation (54), and this is as follows:

$$\theta_{B2} = \frac{M_{AB}L}{12EI_z}(2 - \varnothing) \tag{55}$$

Now, the beam of Figure 2(d) is analyzed to find "θ_{A3}" and "θ_{B3}" in function of "M_{BA}" of the same way as was done in Figure 2(c), it is obtained:

$$\theta_{A3} = -\frac{M_{BA}L}{12EI_z}(2 - \varnothing) \tag{56}$$

$$\theta_{B3} = \frac{M_{BA}L}{12EI_z}(4 + \varnothing) \tag{57}$$

Now, Equations (38), (53) and (56) are substituted into Equation (5) and Equations (40), (55) and (57) into Equation (6) are presented:

$$\frac{Pa(a^2 - 3aL + 2L^2)}{6EI_zL} - \frac{M_{AB}L}{12EI_z}(4 + \varnothing) - \frac{M_{BA}L}{12EI_z}(2 - \varnothing) = 0 \tag{58}$$

$$\frac{Pa(a^2 - L^2)}{6EI_zL} + \frac{M_{AB}L}{12EI_z}(2 - \varnothing) + \frac{M_{BA}L}{12EI_z}(4 + \varnothing) = 0 \tag{59}$$

We develop Equations (58) and (59) to find "M_{AB}" and "M_{BA}" shown:

$$M_{AB} = \frac{Pa(L - a)[2(L - a) + L\varnothing]}{2L^2(1 + \varnothing)} \tag{60}$$

$$M_{BA} = \frac{Pa(L - a)(2a + L\varnothing)}{2L^2(1 + \varnothing)} \tag{61}$$

Therefore, if shear deformations are neglected ($\varnothing = 0$), we arrive at:

$$M_{AB} = \frac{Pa(L - a)^2}{L^2} \tag{62}$$

$$M_{BA} = \frac{Pa^2(L - a)}{L^2} \tag{63}$$

3. **Application.** Then, a steel beam is presented to obtain the fixed-end moments by traditional Equation (shear deformations are neglected) and the proposed Equation (shear deformations are considered), the beam used is of profile W24X94 and the beam length varies from 1 to 10 m, the profile properties are:

$E = 20019.6$ kN/cm^2
$A = 173.12$ cm^2
$A_s = 78.25$ cm^2
$I = 105469$ cm^4
$v = 0.32$

The shear modulus is obtained as follows:

$$G = \frac{E}{2(1 + v)}$$

$$G = \frac{20019.6}{2(1 + 0.32)} = 7583.18 \text{ kN/cm}^2$$

Equation (52) is used to find the shape factor. By means of Equations (62) and (63) are obtained the fixed-end moments, when shear deformations are neglected. By means of Equations (60) and (61) are found the fixed-end moments, when shear deformations are considered. Table 1 presents the fixed-end moments for $a = 0.1$L, and Figure 3 shows the behavior of the fixed-end moments with respect to length of the beam. Table 2 presents the fixed-end moments for $a = 0.3$L, and Figure 4 shows the behavior of the fixed-end

TABLE 1. Fixed-end moments to $a = 0.1$L

Beam length (m)	Shape factor	Fixed-end moments neglecting the shear deformations		Fixed-end moments considering the shear deformations		Comparison of results	
		M_{ABN}	M_{BAN}	M_{ABC}	M_{BAC}	M_{ABN}/M_{ABC}	M_{BAN}/M_{BAC}
1.00	4.2700	0.0810PL	0.0090PL	0.0518PL	0.0382PL	1.5637	0.2356
2.00	1.0675	0.0810PL	0.0090PL	0.0624PL	0.0276PL	1.2981	0.3261
3.00	0.4744	0.0810PL	0.0090PL	0.0694PL	0.0206PL	1.1671	0.4369
4.00	0.2669	0.0810PL	0.0090PL	0.0734PL	0.0166PL	1.1035	0.5422
5.00	0.1708	0.0810PL	0.0090PL	0.0757PL	0.0143PL	1.0700	0.6294
6.00	0.1186	0.0810PL	0.0090PL	0.0772PL	0.0128PL	1.0492	0.7031
7.00	0.0871	0.0810PL	0.0090PL	0.0781PL	0.0119PL	1.0371	0.7563
8.00	0.0667	0.0810PL	0.0090PL	0.0787PL	0.0113PL	1.0292	0.7965
9.00	0.0527	0.0810PL	0.0090PL	0.0792PL	0.0108PL	1.0227	0.8333
10.00	0.0427	0.0810PL	0.0090PL	0.0795PL	0.0105PL	1.0189	0.8571

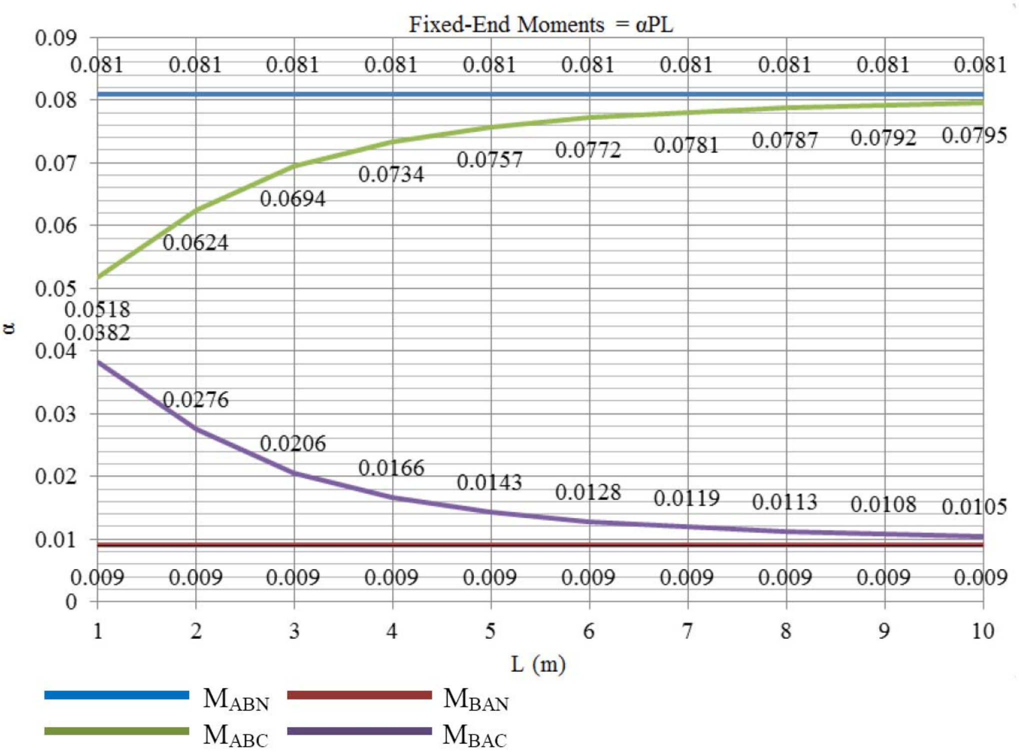

FIGURE 3. Fixed-end moments to $a = 0.1$L

moments with respect to length of the beam. Table 3 presents the fixed-end moments for $a = 0.5$L, but in this relationship it was not necessary to show a graph, because the values are the same in both models.

4. **Results.** Table 1 presents the fixed-end moments at the supports, when the force is applied to a distance $a = 0.1$L of beam. The results showed that the differences are significant for short members. The fixed-end moments neglecting the shear deformations have an increase of *56.37%* with regard to the fixed-end moments considering the shear deformations for the support "*A*". However, to the support "*B*" the fixed-end moments

Table 2. Fixed-end moments to $a = 0.3$L

Beam length (m)	Shape factor	Fixed-end moments neglecting the shear deformations		Fixed-end moments considering the shear deformations		Comparison of results	
		M_{ABN}	M_{BAN}	M_{ABC}	M_{BAC}	M_{ABN}/M_{ABC}	M_{BAN}/M_{BAC}
1.00	4.2700	0.1470PL	0.0630PL	0.1130PL	0.0970PL	1.3009	0.6495
2.00	1.0675	0.1470PL	0.0630PL	0.1253PL	0.0847PL	1.1732	0.7438
3.00	0.4744	0.1470PL	0.0630PL	0.1335PL	0.0765PL	1.1011	0.8235
4.00	0.2669	0.1470PL	0.0630PL	0.1382PL	0.0718PL	1.0637	0.8774
5.00	0.1708	0.1470PL	0.0630PL	0.1409PL	0.0691PL	1.0433	0.9117
6.00	0.1186	0.1470PL	0.0630PL	0.1425PL	0.0675PL	1.0316	0.9333
7.00	0.0871	0.1470PL	0.0630PL	0.1436PL	0.0664PL	1.0237	0.9488
8.00	0.0667	0.1470PL	0.0630PL	0.1444PL	0.0656PL	1.0180	0.9604
9.00	0.0527	0.1470PL	0.0630PL	0.1449PL	0.0651PL	1.0145	0.9677
10.00	0.0427	0.1470PL	0.0630PL	0.1453PL	0.0647PL	1.0117	0.9737

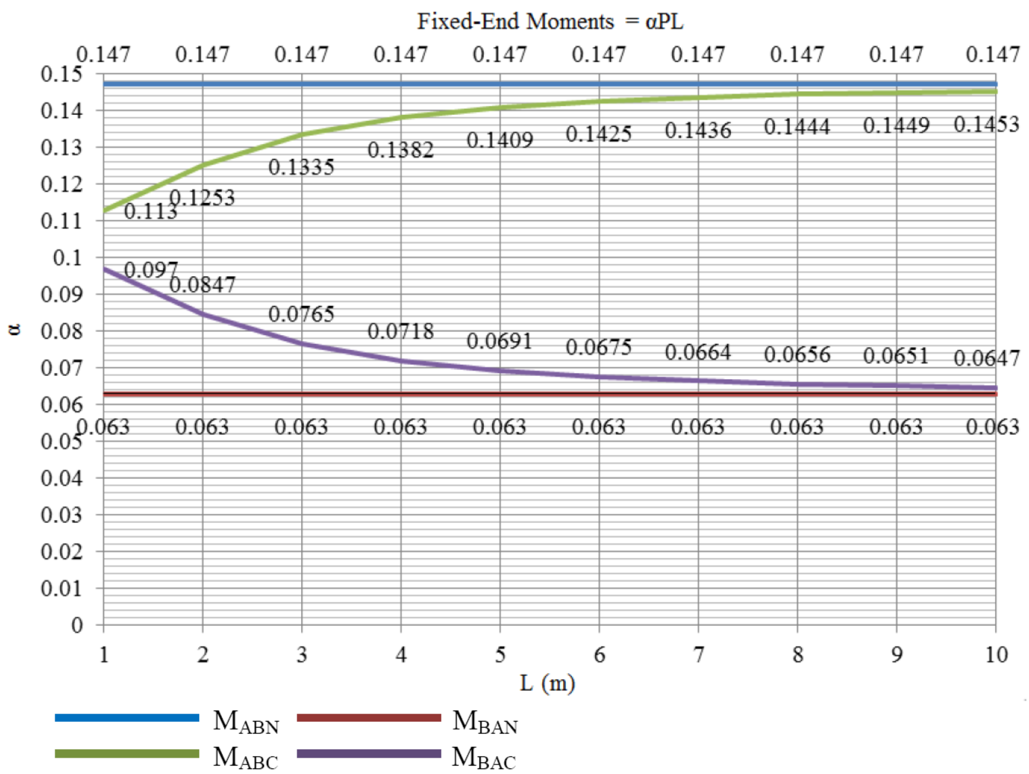

Figure 4. Fixed-end moments to $a = 0.3$L

neglecting the shear deformations have a decrease of *76.44%* with regard to the fixed-end moments considering the shear deformations, these are presented in length of 1.00m.

Table 2 shows the fixed-end moments at the supports, when the force is applied to a distance $a = 0.3$L of beam. The results showed that the differences are significant for short members. The fixed-end moments neglecting the shear deformations have an increase of *30.09%* with regard to the fixed-end moments considering the shear deformations for the support "*A*". However, to the support "*B*" the fixed-end moments neglecting the shear deformations have a decrease of *35.05%* with regard to the fixed-end moments considering the shear deformations, these are presented in length of 1.00m.

Table 3. Fixed-end moments to $a = 0.5$L

Beam length (m)	Shape factor	Fixed-end moments neglecting the shear deformations		Fixed-end moments considering the shear deformations		Comparison of results	
		M_{ABN}	M_{BAN}	M_{ABC}	M_{BAC}	M_{ABN}/M_{ABC}	M_{BAN}/M_{BAC}
1.00	4.2700	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
2.00	1.0675	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
3.00	0.4744	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
4.00	0.2669	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
5.00	0.1708	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
6.00	0.1186	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
7.00	0.0871	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
8.00	0.0667	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
9.00	0.0527	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000
10.00	0.0427	0.1250PL	0.1250PL	0.1250PL	0.1250PL	1.0000	1.0000

Table 3 presents the fixed-end moments at the supports, when the force is applied to a distance $a = 0.5$L of beam. The results showed that there are not differences, because these are equal.

5. **Conclusions.** This paper presented a mathematical model to obtain the fixed-end moments of beams subjected to a concentrated force localized anywhere taking into account the bending deformations and shear. The mathematical technique presented in this research is very adequate to obtain the fixed-end moments and rotations for beams, because it presents the mathematical expression.

The significant application of fixed-end moments, rotations and displacements is in the matrix methods of structural analysis for the moments acting on member and stiffness of the beam.

Besides the efficiency and accuracy of the method developed in this investigation, a significant advantage is that the rotations and displacements and moments acting are obtained in any cross section of the beam using the respective integral representations as mathematical formulas.

The mathematical model presented in this paper is applied only for fixed-end moments subjected to a concentrated force localized anywhere of the member for constant cross section. The suggestions for future research: when the member presented a variable cross section, by example of rectangular type, drawer type, “T” and “I”.

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