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AN ALTERNATIVE HANDLING METHOD OF WINDOW INITIAL CONDITION FOR FINITE MEMORY STRUCTURE FILTER

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Abstract. *This paper proposes an alternative handling method of window initial condition for the FMS filter. Two kinds of moving windows are defined by the primary window and the auxiliary window. The primary window means the most recent filtering window and the auxiliary window means the past of the primary window. The FMS filter is obtained from the finite observations on the primary window and the window initial condition which is computed on the auxiliary window. On the other hand, the existing FMS filter is obtained from the finite observations on the primary window and the window initial condition which is also computed on the primary window, which means that the window initial condition is obtained from future observations. The gain matrix for the FMS filter incorporates knowledge about the window initial condition during its design and is shown to be time-invariant. Through extensive computer simulations, the FMS filter with the proposed method can be shown to be comparable with the Kalman filter for the nominal system and better than that for the temporarily uncertain system.*
Keywords: Estimation filtering, Filtering window, Window initial state, Finite memory structure, Infinite memory structure

1. **Introduction.** As an alternative to the recursive infinite memory structure (IMS) filter such as the well-known Kalman filter [1-3], the finite memory structure (FMS) filter has been developed for state estimation [4-11] and applied successfully for various applications such as mobile target tracking, computer network, RFID system, global positioning system, wireless sensor network, and fault diagnosis, as shown in [12-17].

However, the FMS filter has a serious issue that the window initial condition has to be handled. The window of past observations moves forward in time at each sampling time when a new observation is available. Thus, the FMS filter requires knowledge about the window initial condition as well as finite observations on the most recent window for each moving window formulation. Since the window initial state is also a state variable and thus not measurable, it is somewhat unreasonable in practical systems that knowledge about the window initial condition is assumed to be completely known. Therefore, how to handle the window initial condition might be a challenging issue in the FMS filter.

Several approaches have been researched to handle the window initial condition. The FMS filter in [5] adopted the state propagator and the Lyapunov equation to get knowledge about the window initial condition. However, the performance of this approach depends heavily on the accuracy of the state propagator, which is seriously influenced by uncertainty in the system initial state and system noises. In the FMS filter [6,7], the window initial condition was assumed to be unknown and thus have infinite covariance. However, this assumption might be somewhat heuristic and seems to have no physical meaning. The FMS filter in [8] did not consider how to handle the window initial condition, which means the window initial condition can be handled arbitrarily and heuristically. To overcome the resulting problems of existing FMS filters in [5-8], the FMS filter in [10] obtains knowledge about the window initial condition from the finite observations on the most recent filtering window, while existing FMS filters handle arbitrarily or heuristically. However, although the handling method of window initial condition in [10] has been applied successfully as shown in [16,17], it could be somewhat awkward because the window initial state is obtained from future observations.

Therefore, this paper proposes an alternative handling method of window initial condition for the FMS filter. To estimate states and handle window initial condition, a couple of moving windows are defined by the primary window and the auxiliary window. The primary window is used as the most recent filtering window to estimate states and the auxiliary window is used as the past of the primary window to handle window initial condition. On the other hand, in the existing FMS filter of [10], the finite observations on the primary window are used for both estimating states and handling window initial condition. This means that the window initial condition is obtained from future observations in the existing FMS filter of [10]. It is shown that the gain matrix for the FMS filter incorporates knowledge about the window initial condition during its design and is time-invariant for all moving windows. Finally, computer simulations show that the FMS filter with the proposed method can be comparable with the Kalman filter with IMS for the nominal system and better than that for the temporarily uncertain system.

This paper is organized as follows. In Section 2, the basic concept of FMS filtering and its window initial condition are briefly discussed. In Section 3, an alternative handling method of window initial condition is proposed for the FMS filter. In Section 4, extensive computer simulations are performed. Finally, conclusions are presented in Section 5.

2. **FMS Filtering and Window Initial Condition.** Consider the following linear discrete-time state-space model:

$$x_{i+1} = Ax_i + Gw_i, \tag{1}$$

$$z_i = Cx_i + v_i, \tag{2}$$

where $x_i \in \Re^n$ is the unknown state and $z_i \in \Re^q$ is the known observation. At the initial time i_0 of system, the state x_{i_0} is a random variable with a mean $\bar{x}_{i_0}$ and a covariance P_{i_0}. The system noise $w_i \in \Re^p$ and the observation noise $v_i \in \Re^q$ are zero-mean white Gaussian and mutually uncorrelated. The covariances of w_i and v_i are denoted by Q and R assumed to be positive definite matrices, respectively. These noises are uncorrelated with the initial condition x_{i_0}.

To overcome shortcomings of the IMS filter such as the Kalman filter, the FMS filter can be considered by combining the Kalman filter with the moving window formulation [4-10]. That is, the Kalman filter is solved over a fixed window of length M whose size does not increase with time. The window of finite observations moves forward in time at each sampling time when a new observation is available. Therefore, the FMS filter utilizes only a finite number of observations on the most recent filtering window $[i - M(\stackrel{\triangle}{=} i_M), i]$ and discards past observations outside the filtering window. As shown in [10], the FMS filter $\hat{x}_i$ can be derived from the Kalman filter on the window $[i_M, i]$ and represented by the summation form with the window initial condition $\hat{x}_{i_M}$ as follows:

$$\begin{aligned} \hat{x}_i &= \Phi_M \hat{x}_{i_M} + \sum_{j=0}^{M-1} \Phi_{M-j} \Sigma_{i_M+j} C^T R^{-1} z_{i_M+j} \\ &= \Phi_M \hat{x}_{i_M} + \big[\Phi_M \Sigma_{i_M} \ \Phi_{M-1} \Sigma_{i_M+1} \ \cdots \ \Phi_1 \Sigma_{i_M+M-1} \big] C^T R^{-1} Z_{i-1}, \end{aligned} \tag{3}$$

where the transition matrix Φ_j is given by

$$\Phi_{j+1} = \Phi_j A\big[I + \Sigma_{i_M+M-j-1} C^T R^{-1} C\big]^{-1}, \quad \Phi_0 = I, \tag{4}$$

and the error covariance Σ_{i_M+j} is given by

$$\Sigma_{i_M+j+1} = A\left(I + \Sigma_{i_M+j} C^T R^{-1} C\right)^{-1} \Sigma_{i_M+j} A^T + GQG^T, \tag{5}$$

and $0 \le j \le M-1$. The finite observation Z_{i-1} on the most recent filtering window $[i_M, i]$ is defined as follows:

$$Z_{i-1} \triangleq \begin{bmatrix} z_{i-M} \\ z_{i-M+1} \\ z_{i-M+2} \\ \vdots \\ z_{i-1} \end{bmatrix}. \tag{6}$$

As shown in (3), the FMS filter requires knowledge about the window initial condition $\hat{x}_{i_M}$ as well as finite observations Z_{i-1} on the most recent filtering window. Since the window initial state is also a state variable and thus not measurable, it is somewhat unreasonable in practical systems that knowledge about the window initial condition is assumed to be completely known. Therefore, how to handle the window initial condition might be a challenging issue in the FMS filter.

To handle the window initial condition, several approaches have been researched as shown in [5-8]. However, these approaches were shown to have some drawbacks and limitations. In order to overcome the resulting problems of these existing FMS filters which handle window initial condition arbitrarily or heuristically, the another existing work [10] obtains knowledge about the window initial condition from finite observations on the most recent window. However, the handling method in [10] could be somewhat awkward since the window initial state is obtained from future observations.

3. **Alternative Handling of Window Initial Condition for FMS Filter.** In this section, an alternative handling method of window initial condition $\hat{x}_{i_M}$ is proposed for the FMS filter in (3). Two kinds of moving windows are defined by the primary window and the auxiliary window. The primary window means the most recent filtering window, denoted by $[i_M, i]$ with the window length M, and the auxiliary window means the past of the primary window, denoted by $[i_M - N, i_M]$ with the window length N. The FMS filter $\hat{x}_i$ in (3) is obtained from the finite observations on the primary window $[i_M, i]$ and the window initial condition which is computed on the auxiliary window $[i_M - N, i_M]$. That is, the window initial condition $\hat{x}_{i_M}$ is solved over a fixed window of length N whose size does not increase with time. Note that the existing FMS filter [10] is obtained from the finite observations on the primary window $[i_M, i]$ and the window initial condition which is also computed on the primary window $[i_M, i]$, which means that the window initial condition is obtained from future observations.

A knowledge about the window initial condition $\hat{x}_{i_M}$ is denoted by $\bar{x}_{i_M}$. Then, $\bar{x}_{i_M}$ can be represented in the following matrix form:

$$\bar{x}_{i_M} = \Gamma Z_{i_M-1}, \tag{7}$$

where Γ is the gain matrix and Z_{i_M-1} is the finite observations on the auxiliary window $[i_M - N, i_M]$. The finite observations Z_{i_M-1} are expressed in terms of the window initial state x_{i_M} from (1) and (2) at the window initial time i_M as follows:

$$Z_{i_M-1} = \Lambda x_{i_M} + \Theta W_{i_M-1} + V_{i_M-1}, \tag{8}$$

where Z_{i_M-1} and matrices are defined as follows:

$$Z_{i_M-1} \triangleq \begin{bmatrix} z_{i_M-N} \\ z_{i_M-N+1} \\ z_{i_M-N+2} \\ \vdots \\ z_{i_M-1} \end{bmatrix}, \quad \Lambda \triangleq \begin{bmatrix} CA^{-N} \\ CA^{-N+1} \\ CA^{-N+2} \\ \vdots \\ CA^{-1} \end{bmatrix},$$

$$\Theta \triangleq \begin{bmatrix} CA^{-1}G & CA^{-2}G & \cdots & CA^{-N+1}G & CA^{-N}G \\ 0 & CA^{-1}G & \cdots & CA^{-N+2}G & CA^{-N+1}G \\ 0 & 0 & \cdots & CA^{-N+3}G & CA^{-N+2}G \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & CA^{-1}G \end{bmatrix}, \tag{9}$$

and W_{i_M-1}, V_{i_M-1} have the same forms with Z_{i_M-1} for w_i, v_i, respectively.

The noise term $\Theta W_{i_M-1} + V_{i_M-1}$ in (8) is zero-mean white Gaussian with covariance Π given by

$$\Pi \triangleq \Theta\Big[\mathrm{diag}(\overbrace{Q\ Q\ \cdots\ Q}^{N})\Big]\Theta^T + \Big[\mathrm{diag}(\overbrace{R\ R\ \cdots\ R}^{N})\Big]. \tag{10}$$

The following series of equations are the procedure to obtain knowledge $\bar{x}_{i_M}$ about the window initial condition $\hat{x}_{i_M}$. This procedure is based on the approach of *best linear unbiased estimation* in [18].

Taking the expectation both sides of (7), the following relation is obtained:

$$\mathbf{E}\big[\bar{x}_{i_M}\big] = \mathbf{E}\big[\Gamma Z_{i_M-1}\big] = \Gamma\Lambda\mathbf{E}\big[x_{i_M}\big].$$

Then, with the following constraint:

$$\Gamma\Lambda = I, \tag{11}$$

$\bar{x}_{i_M}$ is unbiased, i.e., $\mathbf{E}\big[\bar{x}_{i_M}\big] = \mathbf{E}\big[x_{i_M}\big]$. Thus, the constraint (11) can be called the *unbiasedness constraint* for knowledge about the window initial condition $\hat{x}_{i_M}$. The objective is now to obtain the gain matrix Γ_*, subject to the unbiasedness constraint (11), in such a way that the error of $\bar{x}_{i_M}$ has a minimum variance as follows:

$$\Gamma_* = \arg\min_{\Gamma} \mathbf{E}\left[\left(x_{i_M} - \bar{x}_{i_M}\right)^T \left(x_{i_M} - \bar{x}_{i_M}\right)\right]. \tag{12}$$

Using the approach of best linear unbiased estimation in [18], knowledge $\bar{x}_{i_M}$ about the window initial condition $\hat{x}_{i_M}$ is obtained by the solution of (12) as follows:

$$\bar{x}_{i_M} = \Gamma_* Z_{i_M-1}, \tag{13}$$

where

$$\Gamma_* = \left(\Lambda^T\Pi^{-1}\Lambda\right)^{-1}\Lambda^T\Pi^{-1}. \tag{14}$$

In addition, the error covariance of $\bar{x}_{i_M}$, denoted by $\bar{\Sigma}_{i_M}$, is obtained by

$$\begin{aligned} \bar{\Sigma}_{i_M} &= \mathbf{E}\big[(x_{i_M} - \bar{x}_{i_M})(x_{i_M} - \bar{x}_{i_M})^T\big] \\ &= \mathbf{E}\big[(x_{i_M} - \Gamma_* Z_{i_M-1})(x_{i_M} - \Gamma_* Z_{i_M-1})^T\big] \\ &= \left(\Lambda^T\Pi^{-1}\Lambda\right)^{-1}. \end{aligned} \tag{15}$$

Note that, as shown in (15), $\bar{\Sigma}_{i_M}$ is constant value. Thus, the error covariance Σ_{i_M+j} (5) defined on the primary window $[i_M, i]$ can be independent of time index i_M and thus rewritten as follows:

$$\Sigma_{j+1} = A\left(I + \Sigma_j C^T R^{-1} C\right)^{-1}\Sigma_j A^T + GQG^T \tag{16}$$

with $\Sigma_0 = \bar{\Sigma}_{i_M}$.

Therefore, knowledge $\{\bar{x}_{i_M}, \bar{\Sigma}_{i_M}\}$ obtained from the auxiliary window $[i_M - N, i_M]$ can be applied as the window initial condition $\{\hat{x}_{i_M}, \Sigma_{i_M}\}$ for the FMS filter on the primary window $[i_M, i]$ in the *unbiasedness* sense. By applying knowledge $\{\bar{x}_{i_M}, \bar{\Sigma}_{i_M}\}$ in (13) and (15), the FMS filter (3) can be represented by

$$\hat{x}_i = \Phi_M \left(\Lambda^T \Pi^{-1} \Lambda\right)^{-1} \Lambda^T \Pi^{-1} Z_{i_M - 1} + \left[\Phi_M \Sigma_0 \;\; \Phi_{M-1}\Sigma_1 \;\; \cdots \;\; \Phi_1 \Sigma_{M-1}\right] C^T R^{-1} Z_{i-1}, \tag{17}$$

where

$$\Phi_{j+1} = \Phi_j A \left[I + \Sigma_{M-j-1} C^T R^{-1} C\right]^{-1}, \quad \Phi_0 = I.$$

Finally, the FMS filter $\hat{x}_i$ with knowledge $\{\bar{x}_{i_M}, \bar{\Sigma}_{i_M}\}$ about the window initial condition $\{\hat{x}_{i_M}, \Sigma_{i_M}\}$ can be defined as the following theorem.

Theorem 3.1. *Assume that $\{A, C\}$ is observable, $M \geq n$, and $N \geq n$. Then, the FMS filter $\hat{x}_i$ on the primary window $[i_M, i]$ is defined using knowledge $\{\bar{x}_{i_M}, \bar{\Sigma}_{i_M}\}$ on the auxiliary window $[i_M - N, i_M]$ as follows:*

$$\hat{x}_i \triangleq \left[H_A \;\; H_P\right] \begin{bmatrix} Z_{i_M-1} \\ Z_{i-1} \end{bmatrix}, \tag{18}$$

where H_A is the gain matrix for the auxiliary window as follows:

$$H_A = \Phi_M \left(\Lambda^T \Pi^{-1} \Lambda\right)^{-1} \Lambda^T \Pi^{-1}, \tag{19}$$

and H_P is the gain matrix for the primary window as follows:

$$H_P = \left[\Phi_M \Sigma_0 \;\; \Phi_{M-1}\Sigma_1 \;\; \cdots \;\; \Phi_1 \Sigma_{M-1}\right] C^T R^{-1}, \tag{20}$$

and Z_{i_M-1} and Z_{i-1} are defined by (6) and (9).

The matrix Λ is of full rank since $\{A, C\}$ is observable for $N \geq n$. In addition, the matrix Π is positive definite and thus its inversion exists. Therefore, the matrix $\Lambda^T \Pi^{-1} \Lambda$ is nonsingular and thus its inversion exists. It is noted that gain matrices H_A (19) and H_P (20) require computation only on the interval $[0, N]$ and $[0, M]$, respectively, once and is time-invariant for all windows. This means the FMS filter $\hat{x}_i$ (18) is time-invariant.

4. **Computer Simulations.** To illustrate the validity of the FMS filter with the proposed handling method of the window initial condition and to compare with the Kalman filter having the recursive IMS structure, extensive computer simulations are performed for a pair of discrete-time noisy signal models with an uncertain model parameter δ_i. The Van der Pol oscillation signal model for an electronic circuit with vacuum tubes is considered as follows [19]:

$$A = \begin{bmatrix} 1 + 0.25\delta_i & T + 0.25\delta_i & T^2/2 + 0.25\delta_i \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 \\ 0.3 \\ 0.3 \end{bmatrix}, \; C = \begin{bmatrix} 1 + 0.5\delta_i & 0.5\delta_i & 0.5\delta_i \end{bmatrix}. \tag{21}$$

In the Van der Pol oscillation signal model (21), system and observation noise covariances are taken as $Q = 0.1^2$ and $R = 0.2^2$, respectively. The sampling time is taken as $T = 0.01$ as shown in [19].

The FMS filter with two kinds of windows and the Kalman filter with IMS are compared for the temporarily uncertain systems. The important issue might be how to choose an appropriate primary window length M and auxiliary window length N to make the filtering performance as good as possible. As shown in [10], it is well known that FMS structure filters have better noise suppression as the window length grows. Thus, the noise suppression of the FMS filter might be closely related to the window length. However, although the FMS filter can have greater noise suppression as the window length increases, the tracking speed of state estimate for actual state worsens as the window length grows. This illustrates the FMS filter's compromise between the noise suppression and the tracking speed of state estimate.

For the FMS filter, the primary window length is set by $M = 60$ and two cases of auxiliary window lengths are considered by $N = 20$ and $N = 30$. The uncertain model parameter is taken as $\delta_i = 0.02$ for the interval $220 \leq i \leq 250$ for the Van der Pol oscillation signal model (21), respectively. As shown in Figure 1, for both cases, the estimation error of the FMS filter is smaller than that of the Kalman filter on the interval where modeling uncertainty exists. In addition, the convergence of estimation error is much faster than that of the Kalman filter after temporary modeling uncertainty disappears. Of course, the Kalman filter can outperform the FMS filter after the effect of temporary modeling uncertainty completely disappears. Therefore, the proposed FMS filter can be more robust than the Kalman filter with the recursive IMS structure when applied to temporarily uncertain systems, although the proposed FMS filter is designed with no consideration of robustness.

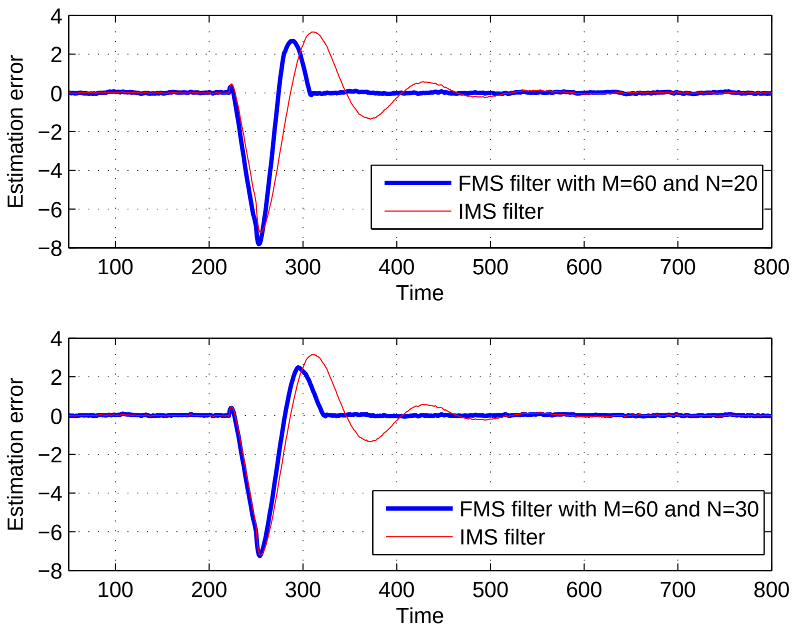

Figure 1. Estimation error for Van der Pol oscillation signal model with temporary modeling uncertainty

To show the relationship between the auxiliary window length and the tracking speed of the proposed FMS filter, four cases of auxiliary window lengths are considered by $N = 10$, $N = 20$, $N = 30$ and $N = 40$ and then compared. The primary window length is set by $M = 60$ for all cases. As shown in Figure 2, the tracking speed of state estimation for actual state can be better as the auxiliary window length N decreases. However, the magnitude of estimation error worsens in proportion with N. This observation means that there can be a tradeoff in the proposed FMS filter between the estimation error and the tracking speed of state estimate.

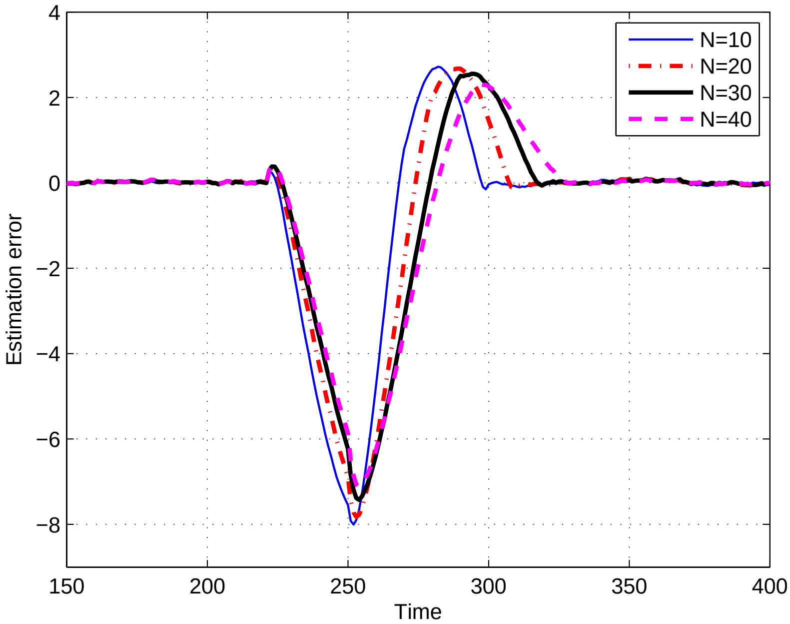

FIGURE 2. Comparison of estimation error for diverse auxiliary window lengths when $M = 60$

5. **Conclusions.** In this paper, an alternative handling method of window initial condition has been proposed for the FMS filter using two kinds of moving windows: the primary window and the auxiliary window. The FMS filter has been obtained from the finite observations on the primary window and the window initial condition which is computed on the auxiliary window. It has been shown that the gain matrix for the FMS filter incorporates knowledge about the window initial condition during its design. Via extensive computer simulations on the Van der Pol oscillation signal model, the FMS filter with the proposed method has been shown to be comparable with the Kalman filter with IMS for the nominal system and better than that for the temporarily uncertain system. Moreover, the simulation result for diverse auxiliary window lengths has shown that there can be a tradeoff between the estimation error and the tracking speed of state estimate.

In future works, the systematic procedure for choosing primary window length M and auxiliary window length N should be researched to improve filtering performance.

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