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RELIABILITY DEGRADATION FORMULA BASED ON THE SUBSYSTEMS OF THE GENERALIZED ARRANGEMENT GRAPH

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Abstract. *The arrangement graph is a generalization of the star network and is considered as a good alternative to the hypercube-based topologies. Under the probability fault model, Lin et al. established an upper bound on the subgraph reliability of the arrangement graph. However, a lower bound of reliability estimation plays a more informative role in predicting the status of system availability. For this reason, this paper is aimed at deriving an analytic lower bound on the subgraph reliability of the arrangement graph in a combinatorial manner, and the numerical comparison validates the proposed formulation.*
Keywords: Availability, Reliability, Probability fault model, Arrangement graph

1. **Introduction.** In recent years, the Internet of Things (IoT) prevails in a wide range of life applications. Within the IoT, everything (including physical devices, individuals, buildings, and other items) is equipped with one or more sensors, software, and Internet access to collect and exchange numerous kinds of data in a just-in-time way. The IoT is an application-oriented network system that relies heavily upon the realization of high-performance parallel and distributed computing. To achieve this goal, it is usually the most fundamental to design a suitable underlying topology of the network system, which dominates the layout of all objects and communication links.

The star is one of the fundamental structures for interconnecting a large number of components in a network system. Based on the popularity of the star, Akers and Krishnamurthy [1] proposed the star network as a viable alternative to the hypercube-based topologies. One straightforward advantage of the star network is that it is able to connect more nodes with less connection links and less communication delay than the hypercube [2]. The promising features of the star network include low degree of nodes, small diameter, node transitivity, link symmetry, and high degree of fault tolerance, and so on [3]. The arrangement graph was proposed by Day and Tripathi [4] as a generalization of the star network, which is much more flexible in both the order and size than the star network. Many studies on the topological properties of the arrangement graph have been published [5, 6, 7, 8, 9, 10, 11].

In general, as the scale of a network system grows, the likelihood of failure occurrences increases. Reliability evaluation is extensively applied to quantifying the impact of systematic failures [12, 13, 14, 15]. The reliability of a network system is defined as the probability that the system is fully functional within a given time session [16]. A wide range of reliability models have been proposed to measure the network reliability and availability. An explicit formula of the subcube reliability of the hypercube-based network was formulated by Das and Kim [12] under the random fault model, which assumes that there are f faults distributed randomly in the hypercube. Later, Chang and Bhuyan [17] proposed the probability fault model for assessing the subcube reliability of the hypercube. The probability fault model is simple and assumes that every node has a homogeneous and independent behavior of reliability in a network. Under this model, Wu and Latifi [15] analyzed the substar reliability in star networks. They derived an upper bound on the $(n-1)$-star reliability of n-star network using the probability fault model and developed an approximate $(n-1)$-star reliability via ignoring any intersection among subgraphs. Recently, Lin et al. [18] used the same approach to calculate the reliability of $(n-1, k-1)$-subgraph in the (n, k)-arrangement graph under the probability fault model. However, only approximate and upper-bound reliability formulation is addressed in [15, 18]. It is reasonable that a lower-bound reliability may play a more informative role in the task of achieving high availability. Therefore, this paper is aimed at deriving an analytic lower bound on the subgraph reliability of the arrangement graph, as a suitable reliability degradation formula for predicting the status of system availability. Some numerical results are presented to compare the lower and upper bounds of the reliability degradation formula.

The rest of this paper is structured as follows. Section 2 introduces the topological properties of arrangement graphs and the fundamentals of the probability fault model. In Section 3, an analytic lower bound on the subgraph reliability of the arrangement graph is derived. Section 4 presents numerical comparisons between lower- and upper-bounded subgraph reliability. Finally, Section 5 concludes this paper.

2. **Background.**

A permutation over a nonempty set S of identifying codes is an order sequence containing each element of S once, and only once. For any positive integer n, let $\langle n\rangle$ denote the set of all positive integers from 1 to n; i.e., $\langle n\rangle = \{1, 2, \ldots, n\}$. Both the star network [1] and the arrangement graph [4] are based on permutations over $\langle n\rangle$.

Definition 2.1. [4] *The (n, k)-arrangement graph, denoted by $A_{n,k}$, is specified by two positive integers n and k, where $1 \le k < n$. The vertex set of $A_{n,k}$ consists of all permutations over every k-element subset of $\langle n\rangle$. Any two vertices are adjacent in $A_{n,k}$ if and only if their digits differ in exactly one position.*

According to Definition 2.1, $A_{n,k}$ is vertex-symmetric and $k(n-k)$-regular [4]. For two integers $r \in \langle k\rangle$ and $x \in \langle n\rangle$, let $V_{n,k}^{(r:x)}$ be the set of all vertices in $A_{n,k}$ whose rth digit is identical to x. Then, $\left\{V_{n,k}^{(r:x)} \mid 1 \le x \le n\right\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{(r:x)}$ denote the subgraph of $A_{n,k}$ induced by $V_{n,k}^{(r:x)}$. Then, $A_{n,k}^{(r:x)}$ is isomorphic to $A_{n-1,k-1}$. For instance, $A_{4,2}$ is partitioned into a collection of four $A_{3,1}$-subgraphs. Figure 1 illustrates $A_{4,2}$ and its partitions.

For any vertex v of $A_{n,k}$, its rth digit is denoted by $(v)_r$, and for $1 \le m \le k-1$, let $A_{n,k}^{(r_1:x_1,\ldots,r_m:x_m)}$ be the subgraph of $A_{n,k}$ induced by $\left\{v \in V(A_{n,k}) \mid (v)_{r_1} = x_1, (v)_{r_2} = x_2, \ldots, (v)_{r_m} = x_m\right\}$, where $\{r_1, r_2, \ldots, r_m\}$ and $\{x_1, x_2, \ldots, x_m\}$ are m-element subsets of $\langle k\rangle$ and $\langle n\rangle$, respectively. In this way, $A_{n,k}$ can be partitioned into a collection of $\frac{n!}{(n-m)!}$ disjoint $A_{n-m,k-m}$-subgraphs, so the total number of distinct $A_{n-m,k-m}$-subgraphs in $A_{n,k}$ is $\binom{k}{m} \times \frac{n!}{(n-m)!}$.

Under the probability fault model, every node of $A_{n,k}$ is either normal or faulty with a homogeneous node reliability p. Then, the first-order subgraph reliability of $A_{n,k}$, denoted by $R_{n,k}^{n-1,k-1}(p)$, is the probability that there exists a fault-free $A_{n-1,k-1}$-subgraph in $A_{n,k}$. Because there are nk distinct $A_{n-1,k-1}$-subgraphs in $A_{n,k}$, we denote them by

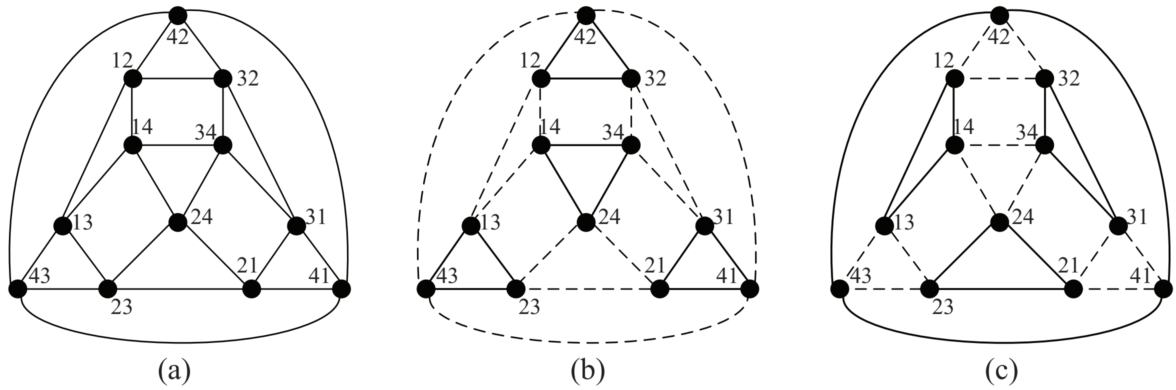


FIGURE 1. (a) $A_{4,2}$, (b) $A_{4,2}^{(2:x)}$ for $x \in \langle 4 \rangle$, and (c) $A_{4,2}^{(1:x)}$ for $x \in \langle 4 \rangle$

$A_{n-1,k-1}^{1}$, $A_{n-1,k-1}^{2}$, ..., $A_{n-1,k-1}^{nk}$ for the sake of convenience. Moreover, let $\xi_{n-1,k-1}^{i}$ denote the probabilistic event that $A_{n-1,k-1}^{i}$ is fault-free in $A_{n,k}$ for $1 \le i \le nk$. Then, $Pr\left(\xi_{n-1,k-1}^{i}\right) = p^{\frac{(n-1)!}{(n-k)!}}$ and $R_{n,k}^{n-1,k-1}(p) = Pr\left(\bigcup_{i=1}^{nk} \xi_{n-1,k-1}^{i}\right)$, where $Pr(\cdot)$ is the event probability. According to the inclusion-exclusion principle, $R_{n,k}^{n-1,k-1}(p)$ is further decomposed:

$$\begin{aligned} R_{n,k}^{n-1,k-1}(p) = &\sum_{i=1}^{nk} Pr\left(\xi_{n-1,k-1}^{i}\right) - \sum_{i<j} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j}\right) \\ &+ \sum_{i<j<l} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j} \cap \xi_{n-1,k-1}^{l}\right) \\ &- \sum_{i<j<l<h} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j} \cap \xi_{n-1,k-1}^{l} \cap \xi_{n-1}^{h}\right) \\ &+ \cdots + (-1)^{nk-1} Pr\left(\bigcap_{i=1}^{nk} \xi_{n-1,k-1}^{i}\right). \end{aligned} \tag{1}$$

Lin et al. [18] proposed two computational schemes for calculating $R_{n,k}^{n-1,k-1}(p)$. One is a binomial approximation:

$$R_{n,k}^{n-1,k-1}(p) \approx 1 - \left(1 - p^{\frac{(n-1)!}{(n-k)!}}\right)^{nk}. \tag{2}$$

The other is an upper bound on $R_{n,k}^{n-1,k-1}(p)$:

$$\begin{aligned} R_{n,k}^{n-1,k-1}(p) \le &\sum_{i=1}^{nk} Pr\left(\xi_{n-1,k-1}^{i}\right) - \sum_{i<j} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j}\right) \\ &+ \sum_{i<j<l} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j} \cap \xi_{n-1,k-1}^{l}\right) \\ = &\, nkp^{\frac{(n-1)!}{(n-k)!}} - \left[k\binom{n}{2} + n\binom{k}{2}\right] p^{\frac{2(n-1)!}{(n-k)!}} - 2\binom{n}{2}\binom{k}{2} p^{\frac{2(n-1)!-(n-2)!}{(n-k)!}} \\ &+ \left[k\binom{n}{3} + n\binom{k}{3}\right] p^{\frac{3(n-1)!}{(n-k)!}} + 6\binom{n}{3}\binom{k}{3} p^{\frac{3(n-1)!-3(n-2)!+(n-3)!}{(n-k)!}} \\ &+ 4\binom{n}{2}\binom{k}{2} p^{\frac{3(n-1)!-(n-2)!}{(n-k)!}} + (2n+2k-8)\binom{n}{2}\binom{k}{2} p^{\frac{3(n-1)!-2(n-2)!}{(n-k)!}}. \end{aligned} \tag{3}$$

3. **Lower-Bounded Approximation of $R_{n,k}^{n-1,k-1}(p)$.** A lower bound on $R_{n,k}^{n-1,k-1}(p)$ can be formed from the first four terms of Equation (1). In the rest of this section, we will derive the following combinatorial formula:

$$\begin{aligned}
&\sum_{i<j<l<h} Pr\left(\xi_{n-1,k-1}^{i} \cap \xi_{n-1,k-1}^{j} \cap \xi_{n-1,k-1}^{l} \cap \xi_{n-1,k-1}^{h}\right) \\
&= \left[k\binom{n}{4} + n\binom{k}{4}\right] p^{\frac{4(n-1)!}{(n-k)!}} + \left[6\binom{k}{2}\binom{n}{3} + 6\binom{k}{3}\binom{n}{2} + \binom{k}{2}\binom{n}{2}\right] p^{\frac{4(n-1)!-2(n-2)!}{(n-k)!}} \\
&\quad + \left[2(n-3)\binom{k}{2}\binom{n}{3} + 8\binom{k}{4}\binom{n}{2} + 6\binom{k}{2}\binom{n}{3} + 6\binom{k}{3}\binom{n}{2}\right] p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}} \\
&\quad + \left[\binom{k}{2}\binom{n}{2}\binom{n-2}{2} + 9\binom{k}{3}\binom{n}{3} + 6\binom{k}{4}\binom{n}{2}\right] p^{\frac{4(n-1)!-4(n-2)!}{(n-k)!}} \\
&\quad + \left[36\binom{k}{3}\binom{n}{4} + 36\binom{k}{4}\binom{n}{3}\right] p^{\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}} \\
&\quad + 24\binom{k}{4}\binom{n}{4} p^{\frac{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!}{(n-k)!}} + 36\binom{k}{3}\binom{n}{3} p^{\frac{4(n-1)!-4(n-2)!+(n-3)!}{(n-k)!}}. \qquad (4)
\end{aligned}$$

Below we analyze how $A_{n-1,k-1}^{i}$, $A_{n-1,k-1}^{j}$, $A_{n-1,k-1}^{l}$, and $A_{n-1,k-1}^{h}$ overlap with one another. For the sake of clarity, we associate $A_{n-1,k-1}^{i}$, $A_{n-1,k-1}^{j}$, $A_{n-1,k-1}^{l}$, $A_{n-1,k-1}^{h}$ with $A_{n,k}^{(r_1:x_1)}$, $A_{n,k}^{(r_2:x_2)}$, $A_{n,k}^{(r_3:x_3)}$, $A_{n,k}^{(r_4:x_4)}$, respectively. For simplicity, let $\mathcal{A} = A_{n,k}^{(r_1:x_1)} \cup A_{n,k}^{(r_2:x_2)} \cup A_{n,k}^{(r_3:x_3)} \cup A_{n,k}^{(r_4:x_4)}$.

Case 1: All of r_1, r_2, r_3, and r_4 are the same. Obviously, $A_{n,k}^{(r_2:x_2)}$, $A_{n,k}^{(r_3:x_3)}$, and $A_{n,k}^{(r_4:x_4)}$ are mutually disjoint. There are $\frac{4(n-1)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{1}\binom{n}{4}$ distinct groups of four $A_{n-1,k-1}$-subgraphs corresponding to this scenario of union.

Case 2: At least two of r_1, r_2, r_3, and r_4 are different. Without loss of generality, we assume $r_1 = s$ is different from $r_4 = t$.

Subcase 2.1: Both r_2 and r_3 are in $\{s,t\}$; that is, $r_2, r_3 \in \{s,t\}$.

- $|\{x_1,x_2,x_3,x_4\}| = 4$.
  - One of s and t corresponds to three identifying codes; that is, either $r_1 = r_2 = r_3 = s$ or $r_2 = r_3 = r_4 = t$. For instance, if $r_2 = r_3 = r_4$, Figure 2(a) illustrates $\mathcal{A}$. There are $\frac{4(n-1)!-3(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{2}\binom{2}{1}\binom{n}{3}\binom{n-3}{1}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.
  - Both s and t correspond to two identifying codes. Without loss of generality, we assume that $r_1 = r_2$ and $r_3 = r_4$. Figure 2(b) illustrates $\mathcal{A}$. There are $\frac{4(n-1)!-4(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{2}\binom{n}{2}\binom{n-2}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.
- $|\{x_1,x_2,x_3,x_4\}| = 3$.
  - One of s and t corresponds to three identifying codes; that is, either $r_1 = r_2 = r_3 = s$ or $r_2 = r_3 = r_4 = t$. Suppose that $r_2 = r_3 = r_4$, as illustrated in Figure 2(c). There are $\frac{4(n-1)!-2(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{2}\binom{2}{1}\binom{n}{3}\binom{3}{1} = 6\binom{k}{2}\binom{n}{3}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.
  - Both s and t correspond to two identifying codes. Without loss of generality, we assume $r_1 = r_2$, $r_3 = r_4$, and $x_2 = x_4$. See Figure 2(d). There are $\frac{4(n-1)!-3(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{2}\binom{n}{3}\binom{3}{1} \times 2! = 6\binom{k}{2}\binom{n}{3}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.

- $|\{x_1, x_2, x_3, x_4\}| = 2$. Let $\{a, b\} = \{x_1, x_2, x_3, x_4\}$. Without loss of generality, we assume $r_1 = r_2$ and $r_3 = r_4$, so we obtain $\{x_1, x_2\} = \{x_3, x_4\} = \{a, b\}$. We further assume that $x_1 = x_3 = a$ and $x_2 = x_4 = b$. See Figure 2(e). There are $\frac{4(n-1)!-2(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{2}\binom{n}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.

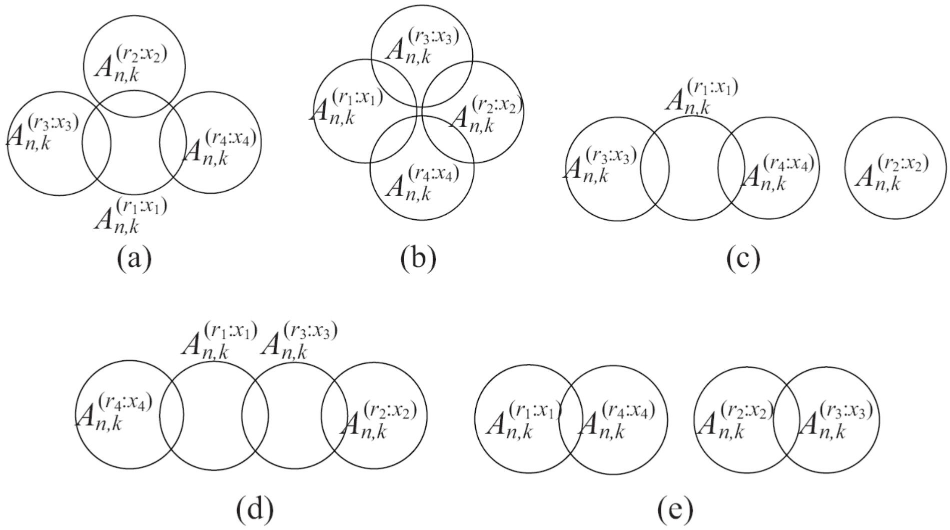


FIGURE 2. Union types of four out of nk $A_{n-1,k-1}$-subgraphs described in Subcase 2.1

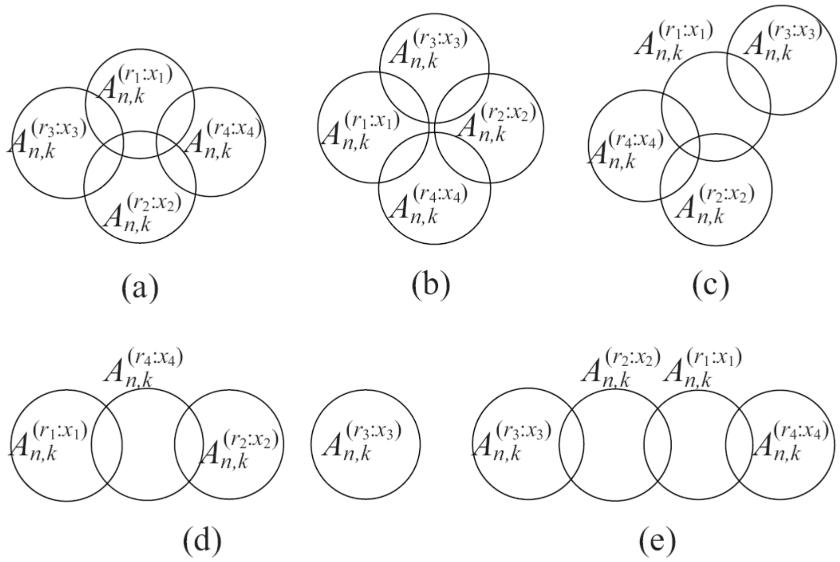


FIGURE 3. Union types of four out of nk $A_{n-1,k-1}$-subgraphs described in Subcase 2.2

Subcase 2.2: Only one of r_2 and r_3 is in $\{s, t\} = \{r_1, r_4\}$. Without loss of generality, we assume that $r_2 \notin \{s, t\}$ and $r_3 = r_4 = t$. Thus, x_3 is different from x_4, so $A_{n,k}^{(r_3:x_3)}$ and $A_{n,k}^{(r_4:x_4)}$ are disjoint.

- $|\{x_1, x_2, x_3, x_4\}| = 4$. As illustrated in Figure 3(a), there are $\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{3}\binom{3}{1}\binom{n}{4}\binom{4}{2} \times 2! = 36\binom{k}{3}\binom{n}{4}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.

- $|\{x_1, x_2, x_3, x_4\}| = 3$.
  - $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$. Then, we have $x_1 = x_2$. See Figure 3(b). Obviously, there are $\frac{4(n-1)!-4(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2} = 9\binom{k}{3}\binom{n}{3}$ distinct groups of four A_{n-1}-subgraphs with this scenario of union.
  - $\{x_1, x_2\} \cap \{x_3, x_4\} \neq \emptyset$. Without loss of generality, we assume that $x_2 = x_3$. See Figure 3(c). There are $\frac{4(n-1)!-4(n-2)!+(n-3)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2}\binom{2}{1} \times 2! = 36\binom{k}{3}\binom{n}{3}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.
- $|\{x_1, x_2, x_3, x_4\}| = 2$.
  - $x_1 = x_2$. Since $x_3 \neq x_4$, we assume that $x_1 = x_2 = x_3$. See Figure 3(d). There are $\frac{4(n-1)!-2(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{3}\binom{3}{1}\binom{n}{2}\binom{2}{1} = 6\binom{k}{3}\binom{n}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs categorized into this scenario of union.
  - $x_1 \neq x_2$. Without loss of generality, we assume that $x_1 = x_3$ and $x_2 = x_4$. Figure 3(e) illustrates $\mathcal{A}$. There are $\frac{4(n-1)!-3(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{3}\binom{3}{1}\binom{n}{2} \times 2! = 6\binom{k}{3}\binom{n}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.

Subcase 2.3: None of r_2 and r_3 is in $\{s, t\} = \{r_1, r_4\}$; that is, every two of r_1, r_2, r_3, and r_4 are different.

- $|\{x_1, x_2, x_3, x_4\}| = 4$. As illustrated in Figure 4(a), there are $\frac{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{4}\binom{n}{4} \times 4! = 24\binom{k}{4}\binom{n}{4}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.
- $|\{x_1, x_2, x_3, x_4\}| = 3$. Without loss of generality, we assume that $x_3 = x_4$. See Figure 4(b). Clearly, there are $\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{4}\binom{n}{3}\binom{3}{1}\binom{4}{2} \times 2! = 36\binom{k}{4}\binom{n}{3}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.
- $|\{x_1, x_2, x_3, x_4\}| = 2$. For convenience, let $\{x_1, x_2, x_3, x_4\} = \{a, b\}$.
  - One identifying code, a or b, is associated with three positions. Without loss of generality, we assume that $x_1 = a$ and $x_2 = x_3 = x_4 = b$. See Figure 4(c). There are $\frac{4(n-1)!-3(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{4}\binom{n}{2}\left[\binom{4}{1} + \binom{4}{3}\right] = 8\binom{k}{4}\binom{n}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.
  - Both identifying codes a and b are associated with two positions. Without loss of generality, we assume that $x_1 = x_2 = a$ and $x_3 = x_4 = b$. See Figure 4(d). There are $\frac{4(n-1)!-4(n-2)!}{(n-k)!}$ nodes in $\mathcal{A}$, and there are $\binom{k}{4}\binom{n}{2}\frac{4!}{2!2!} = 6\binom{k}{4}\binom{n}{2}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.

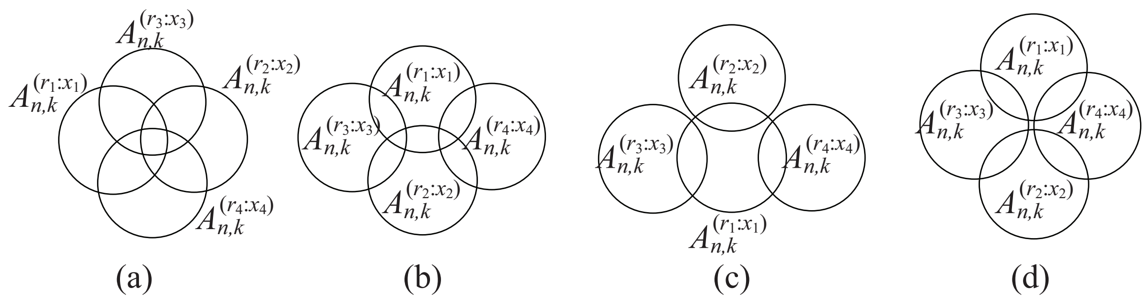


FIGURE 4. Union types of four out of nk $A_{n-1,k-1}$-subgraphs described in Subcase 2.3

- $|\{x_1, x_2, x_3, x_4\}| = 1$. Then, $A_{n,k}^{(r_1:x_1)}$, $A_{n,k}^{(r_2:x_2)}$, $A_{n,k}^{(r_3:x_3)}$, and $A_{n,k}^{(r_4:x_4)}$ are mutually disjoint, and there are $\binom{k}{4}\binom{n}{1}$ distinct groups of four $A_{n-1,k-1}$-subgraphs with this scenario of union.

Denote by $\Omega(n, k, p)$ and $\delta(n, k, p)$ the right-hand sides of Equation (3) and Equation (4), respectively. The lower bound on $R_{n,k}^{n-1,k-1}(p)$ is summarized below.

Theorem 3.1. *Given a homogeneous node reliability p of $A_{n,k}$, a lower bound of $R_{n,k}^{n-1,k-1}(p)$ is as follows:*

$$R_{n,k}^{n-1,k-1}(p) \geq \Omega(n, k, p) - \delta(n, k, p). \tag{5}$$

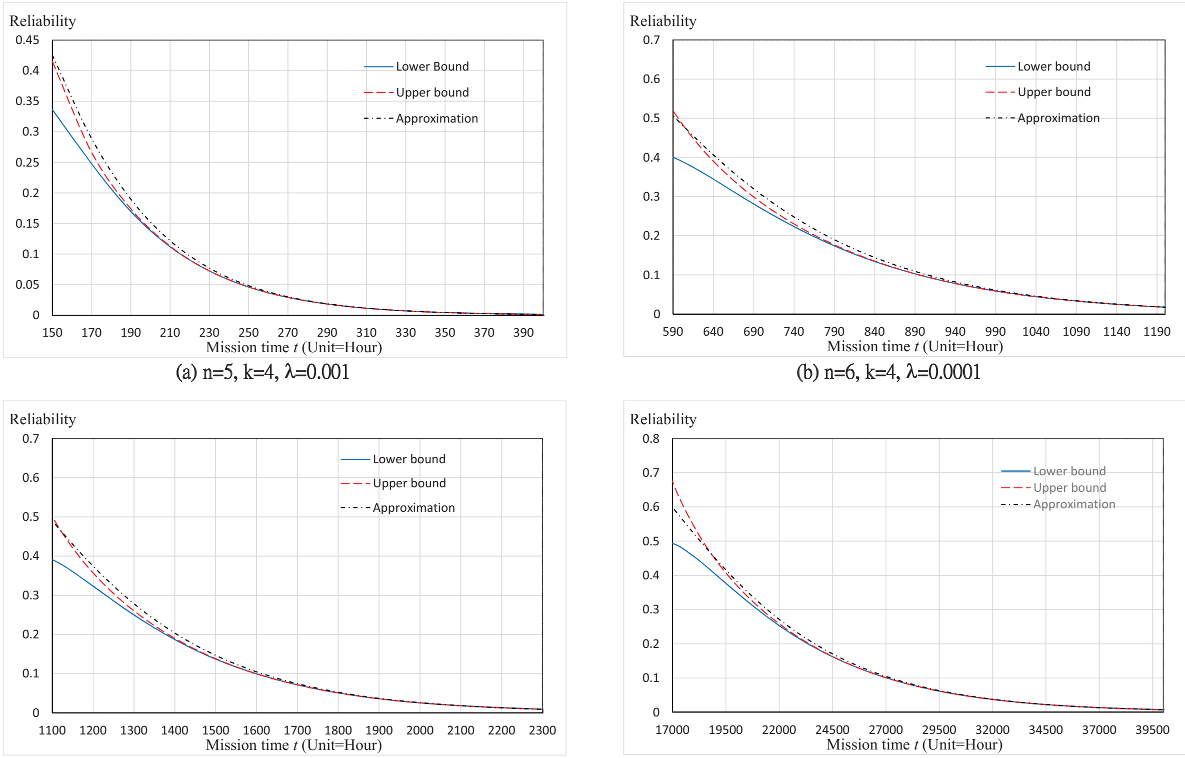
4. **Numerical Comparisons.** In [15, 18], the expected number $f(t)$ of faulty nodes in an N-node network at time t is specified by an increasing function of t: $f(t) = N \times (1 - e^{-\lambda t})$, and the node reliability function $p(t)$ is expressed by $p(t) = e^{-\lambda t}$, where λ is a constant failure rate.

In [14], a clear derivation of this exponential node reliability is given as follows. Let $T(\mathbf{v})$ be a random variable denoting the time to failure of any node $\mathbf{v}$ in $A_{n,k}$. Moreover, let us use $X_t(\mathbf{v})$ to denote the status of $\mathbf{v}$ at any moment $t > 0$: if $\mathbf{v}$ is normal, then $X_t(\mathbf{v}) = 0$; otherwise, $X_t(\mathbf{v}) = 1$. Because the node reliability is homogeneous under the probability fault model, we may simplify the random variables $T(\mathbf{v})$ and $X_t(\mathbf{v})$ as T and X_t, respectively. Thus, the node reliability function $p(t)$ is the probability that a node remains normal at moment t. Suppose that T follows an exponential distribution with a constant failure rate λ; that is, the probability density function of T is $f_T(x) = \lambda e^{-\lambda x}$. Then node reliability function $p(t)$ can be formulated:

$$p(t) = Pr(X_t = 0) = Pr(T > t) = \int_t^\infty f_T(x)dx = \int_t^\infty \lambda e^{-\lambda x} dx = e^{-\lambda t}. \tag{6}$$

The subgraph reliability is useful to evaluate the availability of a smaller-size network in a damaged system. Figure 5 plots the three estimates of $R_{n,k}^{n-1,k-1}(p(t))$ for a variety of n and k. As you can see, for example, the subgraph reliability of $A_{7,5}$ drops to about 0.2 when the $A_{7,5}$ has been operational for a period of 1400 hours, provided that the homogeneous node reliability function is $p(t) = e^{\frac{-t}{100000}}$. Similarly, the subgraph reliability of $A_{8,4}$ drops to about 0.4 after the $A_{8,4}$ has been operational for a period of 19500 hours, provided that the homogeneous node reliability function is $p(t) = e^{\frac{-t}{1000000}}$. It is noticed that the lower and upper bounds of the reliability degradation formula get close to each other rapidly as time passes by. This implies that the proposed lower bound of $R_{n,k}^{n-1,k-1}(p(t))$ is accurate and more informative than the upper bound, once at least one subgraph is available for a user's request to execute his/her programs in the current network system.

5. **Conclusions.** In this paper, a combinatorial lower bound on the subgraph reliability of the arrangement graph is derived. The lower bound of $R_{n,k}^{n-1,k-1}(p)$ plays a more informative role than the upper bound in predicting the status of system robustness, especially when a specific subgraph is available for a user to execute his/her programs in the current network. Our numerical comparison validates the proposed formulation. An accurate reliability estimation should be able to approach the real value as closely as possible. According to our numerical results, the gap between the lower and upper bounds is big for large values of p. Thus, in our future work, we plan to improve the accuracy for reducing the gap between the lower and upper bounds of reliability degradation formula.



(a) n=5, k=4, λ=0.001

(b) n=6, k=4, λ=0.0001

(c) n=7, k=5, λ=0.00001

(d) n=8, k=4, λ=0.000001

FIGURE 5. Estimations of $R_{n,k}^{n-1,k-1}(p(t))$, where $p(t) = e^{-\lambda t}$

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