*International Journal of Innovative Computing, Information and Control*
Volume **14**, Number **3**, June **2018**
ICIC International ©2018 ISSN 1349-4198
pp. 781–796

# EXPONENTIAL STABILITY ANALYSIS OF FUNCTIONAL OBSERVER FOR NONLINEAR SYSTEM WITH INTERVAL TIME-VARYING MIXED DELAYS

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Received September 2017; revised January 2018

Abstract. *This paper investigates exponential stability of functional observer for a class of nonlinear systems with interval time-varying mixed delays. We first address the problem of functional observer design for nonlinear systems with interval time-varying mixed delays and establish the sufficient conditions of exponential stability of functional observer for this class of systems. Second, by utilizing Lyapunov-Krasovskii approach and some well-known inequalities, we convert the stability analysis problem into the feasible problem of several linear matrix inequalities (LMIs). We proposed sufficient conditions of the exponential stability of functional observer for nonlinear systems with interval time-varying mixed delays. Furthermore, the parameters of the delay-dependent observer are also designed using the Lyapunov-Krasovskii approach. Finally, we give two numerical examples and some simulation results to illustrate the effectiveness of the obtained method.*
**Keywords:** Functional observer, Exponential stability, Nonlinear system, Time-varying delays, Linear matrix inequalities

1. **Introduction.** In the past few decades, researchers have focused on the problem of stability analysis of time-delay systems due to its theoretical and practical importance [1-6]. As we all know, time-delay happens frequently in various practical and engineering systems such as physics, mechanics, economy, population dynamics models, automatic control systems, and neural networks [1,2]. Usually, the presence of delay would deteriorate the performance of a system, sometimes causing instability [3]. Therefore, much attention has been paid to the stability analysis of time-delay systems. What is more, in many practical systems, time-delay is not constant but time-varying [4]. In other words, time-varying delay is unavoidable. The existing stability analysis criteria for time-delay systems can be classified into two types: (1) delay-dependent stability analysis; (2) delay-independent stability analysis [5]. It has been proved that delay-dependent criteria are generally less conservative than delay-independent ones [6].

A functional observer is a general form of Luenberger observer that deals with the estimation of one or more functions of the states of a system [7-9]. Observers of this type have been widely applied in systems where the observation of the whole set of system states is not necessary, including fault detection, electromechanical system and observer based control of the dynamic systems. Besides, decreasing the order of the observer can significantly cut the computational costs. Recently, the problem of observer design for time-delay systems has received a significant amount of attention (see, e.g., [3,10,11]). In

[3], the problem of nonlinear observer design for one-sided Lipschitz systems with time-varying delay and uncertainties was investigated. In [11], the problem of observer design for a class of nonlinear discrete-time systems with time-varying delay was considered.

Another popular field of research in the recent years is the functional observer of linear time-invariant systems [7-9,12-16]. In [12], a new algorithm to design minimal multi-functional observers for linear systems was presented. In [13], minimal unknown-input functional observers for multi-input multi-output linear time-invariant systems were studied. In [15], the design of linear functional state observers for systems with delays in state variables was considered. However, the above mentioned functional observers are all aimed at linear systems. Scarcely has any research focusing on the same observers of nonlinear systems.

To the best of our knowledge, none of the contributions concerning functional observer design for time-delay nonlinear systems considers multiple time-varying mixed delays in the states of the system, which will be addressed by this paper. We consider the problem of exponentially stability of functional observer for a class of nonlinear systems with interval time-varying mixed delays. We propose the design method of functional observer for this class of systems and give the novel sufficient conditions of exponential stability of functional observer. Then, we utilize Lyapunov-Krasovskii approach and some well-known inequalities to establish the criteria of delay-dependent exponential stability of the functional observer for a class of nonlinear systems with interval time-varying mixed delays. These criteria guarantee that the functional observer is exponentially stable. Furthermore, the parameters of the delay-dependent observer are designed. At last, numerical examples are given to show the effectiveness of the proposed approach.

The rest of this paper is organized as follows. Section 2 starts with problem formulation and gives preliminaries. Section 3 gives the observer structure and stability analysis. Section 4 presents the design of exponentially stable functional observer scheme and sufficient conditions of the exponential stability of functional observer for nonlinear systems with interval time-varying mixed delays. In Section 5, we give two numerical examples to illustrate the effectiveness of the proposed method which is finally followed by some conclusions in Section 6.

Notations: throughout this paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. '$*$' represents the elements below the main diagonal of a symmetric matrix. $M^T$ means the transpose of $M$. $C^\dagger$ is the pseudo-inverse or the generalized inverse of the matrix $C$; and $C^\perp$ is the right orthogonal of $C$ in a way that $CC^\perp = 0$. $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ denote the maximal and minimal eigenvalue of a matrix $P$ respectively.

## 2. **Problem Statement and Preliminaries.**

Consider a class of nonlinear systems with time-varying delays as follows:

$$\begin{aligned}
&\dot{x}(t) = A_1x(t) + A_2x(t-h(t)) + A_3x(t-\tau(t)) + Af(Cx(t)) + Bu(t),\\
&y(t) = Cx(t),\\
&z(t) = Lx(t),\\
&x(t) = \phi(t), \quad \forall\, t \in \left[\bar{M}, 0\right],
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^l$ is the functional to be estimated. $A_1$, $A_2$ and $A_3$ are system matrices with appropriate dimensions; $A$, $B$, $C$ and $L$ are known and constant matrices, and the matrices $C$ and $L$ are of full row rank. $\phi(t)$ is the initial function of the states of the system. $f(\cdot)$ is a nonlinear function with $f(0) = 0$. $h(t)$ and $\tau(t)$ are the interval time-varying delays which satisfy

$$0 \leq h_m \leq h(t) \leq h_M < \infty, \quad \dot{h}(t) \leq h < 1, \tag{2}$$

$$0 \leq \tau_m \leq \tau(t) \leq \tau_M < \infty, \; \dot{\tau}(t) \leq \tau < 1, \; \bar{M} = \max\{h_M, \tau_M\}, \tag{3}$$

where $h_m$, $h_M$, $\tau_m$, $\tau_M$, $h$ and $\tau$ are the known constants.

**Remark 2.1.** *For the sake of simplicity, in this paper, we only considered two time-varying delays. Furthermore, for the systems with more than two time-varying delays, the proposed theories can be directly extended.*

**Remark 2.2.** *As we all know, in many practical cases, time delay is not constant but time-varying. In this paper, we mainly focus on the problem of exponential stability analysis of the observer for a class of systems with interval time-varying mixed delays.*

**Definition 2.1.** [9] *A minimum-order functional observer for the system (1) is $\alpha$-exponentially stable, if there exist constants $\alpha > 0$ and $\gamma > 0$ such that the estimation error $e(t) = \hat{z}(t) - z(t)$ satisfies*

$$\|e(t)\| \leq \gamma e^{-\alpha t}\|\phi\|_c, \quad \forall t \geq 0,$$

*where* $\|\phi\|_c = \sup_{\theta\in[-\bar{M},0]}\{\|e(\theta)\|, \; \|\dot{e}(\theta)\|\}$.

**Lemma 2.1.** [1] *Suppose* $0 < h_m < h_M$, *and* $x(t) \in \mathbb{R}^n$, *for any given positive matrix* $Q \in \mathbb{R}^{n\times n}$, *then:*

$$-(h_M - h_m)\int_{t-h_M}^{t-h_m} \dot{x}^T(s)Q\dot{x}(s)ds \leq \begin{bmatrix} x(t-h_m) \\ x(t-h_M) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix} \begin{bmatrix} x(t-h_m) \\ x(t-h_M) \end{bmatrix},$$

$$-h_M \int_{t-h_M}^{t} \dot{x}^T(s)Q\dot{x}(s)ds \leq \begin{bmatrix} x(t) \\ x(t-h_M) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_M) \end{bmatrix}.$$

3. **Functional Observer and Stability Analysis.** In this section, we address the problems on exponentially stability analysis of observer for a class of nonlinear systems with interval time-varying mixed delays by means of Lyapunov-Krasovskii functional method.

A minimum-order functional observer with under structure is employed:

$$\begin{aligned} \dot{\omega}(t) &= F_1\omega(t) + F_2\omega(t-h(t)) + F_3\omega(t-\tau(t)) + F_4 f(y(t)) \\ &\quad + Gu(t) + H_1 y(t) + H_2 y(t-h(t)) + H_3 y(t-\tau(t)), \\ \hat{z}(t) &= \omega(t) + Vy(t), \\ \omega(t) &= 0, \quad \forall\, t \in [-\bar{M}, 0], \end{aligned} \tag{4}$$

where $\omega(\cdot) \in \mathbb{R}^l$ is the state of functional observer, $F_1$, $F_2$, $F_3$, $F_4$, $G$, $H_1$, $H_2$, $H_3$ and $V$ are the constant matrices of appropriate dimensions. Define the auxiliary error signal $\epsilon(\cdot) = \omega(\cdot) - Tx(\cdot)$. Then, we give the following sufficient conditions for the exponential stability of the functional observer.

**Theorem 3.1.** *The functional observer (4) is globally $\alpha$-exponentially stable if*

*(i) The error dynamics*

$$\begin{aligned} \dot{\epsilon}(t) &= F_1\epsilon(t) + F_2\epsilon(t-h(t)) + F_3\epsilon(t-\tau(t)), \\ \epsilon(\theta) &= -T\phi(\theta), \; \forall\theta \in [-\bar{M}, 0], \end{aligned} \tag{5}$$

*is $\alpha$-exponentially stable.*

*(ii) There exists a matrix $T$, such that the following matrix equations hold:*

$$T + VC - L = 0, \tag{6}$$

$$F_1T - TA_1 + H_1C = 0, \tag{7}$$
$$F_2T - TA_2 + H_2C = 0, \tag{8}$$
$$F_3T - TA_3 + H_3C = 0, \tag{9}$$
$$G = TB, \; F_4 - TA = 0. \tag{10}$$

**Proof:** Differentiating $\epsilon(t)$ along the solutions of Equations (1) and (4) gives,

$$\begin{aligned}\dot{\epsilon}(t) &= \dot{\omega}(t) - T\dot{x}(t) \\ &= F_1\epsilon(t) + F_2\epsilon(t-h(t)) + F_3\epsilon(t-\tau(t)) + (F_1T - TA_1 + H_1C)x(t) \\ &\quad + (F_2T - TA_2 + H_2C)x(t-h(t)) + (F_3T - TA_3 + H_3C)x(t-\tau(t)) \\ &\quad + (F_4 - TA)f(Cx(t)) + (G - TB)u(t).\end{aligned} \tag{11}$$

Then, if there exists a matrix $T$, such that conditions (7)-(9) and (10), as well as condition (i) are satisfied, then $\epsilon(t)$ is globally $\alpha$-exponentially stable.

Next, the calculation of the error signal $\epsilon(t)$ gives,

$$e(t) = \hat{z}(t) - z(t) = \omega(t) + Vy(t) - Lx(t) = \epsilon(t) + (T + VC - L)x(t). \tag{12}$$

For this reason, if condition (6) is achieved, then the estimated functional $\hat{z}(t)$, globally exponentially tracks its actual value with the convergence rate equal to $\alpha$. This is due to the $\alpha$-exponential stability of the error signal $\epsilon(t)$. This completes the proof of Theorem 3.1.

Now, a delay-dependent criterion is established for the exponential stability of the error dynamics (5) in the following theorem.

**Theorem 3.2.** *For given constants $\alpha > 0$, $0 \le h_m \le h_M$, $0 \le \tau_m \le \tau_M$, $h < 1$ and $\tau < 1$, the system (5) subject to (2) and (3) is globally $\tilde{\alpha}$-exponentially stable if there exist matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, \ldots, 8)$, $R_1 > 0$, $R_2 > 0$, and $N > 0$ of appropriate dimensions, such that the following matrix inequality holds:*

$$\hat{\Pi} = \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} & \hat{\Pi}_{13} & \hat{\Pi}_{14} & e^{-\alpha\tau_m}R_2 & e^{-\alpha h_M}R_1 & 0 & 0 & 0 & 0 \\ * & \hat{\Pi}_{22} & 0 & \hat{\Pi}_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Pi}_{33} & \hat{\Pi}_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Pi}_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Pi}_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\Pi}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Pi}_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Pi}_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Pi}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \hat{\Pi}_{10,10} \end{bmatrix} < 0, \tag{13}$$

*where*

$$\begin{aligned}\hat{\Pi}_{11} &= PF_1 + F_1^TP + Q_1 + Q_2 + Q_3 + Q_4 + Q_6 + Q_8 + \alpha P - e^{-\alpha h_M}R_1 - e^{-\alpha\tau_m}R_2, \\ \hat{\Pi}_{12} &= PF_2, \; \hat{\Pi}_{13} = PF_3, \; \hat{\Pi}_{14} = \tfrac{1}{2}F_1^TN^T, \; \hat{\Pi}_{22} = -(1-h)e^{-\alpha h_M}Q_3, \\ \hat{\Pi}_{24} &= \tfrac{1}{2}F_2^TN^T, \; \hat{\Pi}_{33} = -(1-\tau)e^{-\alpha\tau_M}Q_4, \; \hat{\Pi}_{34} = \tfrac{1}{2}F_3^TN^T, \\ \hat{\Pi}_{44} &= Q_5 + Q_7 + h_M^2R_1 + \tau_m^2R_2 - N, \; \hat{\Pi}_{55} = -e^{-\alpha\tau_m}Q_1 - e^{-\alpha\tau_m}R_2, \\ \hat{\Pi}_{66} &= -e^{-\alpha h_M}Q_6 - e^{-\alpha h_M}R_1, \; \hat{\Pi}_{77} = -e^{-\alpha\tau_M}Q_2, \; \hat{\Pi}_{88} = -e^{-\alpha h_m}Q_8, \\ \hat{\Pi}_{99} &= -(1-h)e^{-\alpha h_M}Q_5, \; \hat{\Pi}_{10,10} = -(1-\tau)e^{-\alpha\tau_M}Q_7, \; \tilde{\alpha} = \tfrac{\alpha}{2}.\end{aligned}$$

**Proof:** Choose a Lyapunov-Krasovskii functional candidate as:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{14}$$

where

$$\begin{aligned}
V_1(t) &= \epsilon^T(t)P\epsilon(t),\\
V_2(t) &= \int_{t-\tau_m}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_1\epsilon(s)ds + \int_{t-\tau_M}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_2\epsilon(s)ds\\
&\quad + \int_{t-h(t)}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_3\epsilon(s)ds + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_4\epsilon(s)ds\\
&\quad + \int_{t-h(t)}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)Q_5\dot\epsilon(s)ds + \int_{t-h_M}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_6\epsilon(s)ds\\
&\quad + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)Q_7\dot\epsilon(s)ds + \int_{t-h_m}^{t} e^{\alpha(s-t)}\epsilon^T(s)Q_8\epsilon(s)ds,\\
V_3(t) &= h_M\int_{-h_M}^{0}\int_{t+s}^{t} e^{\alpha(\theta-t)}\dot\epsilon^T(\theta)R_1\dot\epsilon(\theta)d\theta ds + \tau_m\int_{-\tau_m}^{0}\int_{t+s}^{t} e^{\alpha(\theta-t)}\dot\epsilon^T(\theta)R_2\dot\epsilon(\theta)d\theta ds.
\end{aligned}$$

Calculating the time derivatives of $V_i$, $i = 1, 2, 3$, along the trajectory of system (5) yields

$$\begin{aligned}
\dot V(t) + \alpha V(t) \le\ & 2\epsilon^T(t)P[F_1\epsilon(t) + F_2\epsilon(t-h(t)) + F_3\epsilon(t-\tau(t))] + \alpha\epsilon^T(t)P\epsilon(t)\\
&+ \epsilon^T(t)Q_1\epsilon(t) - e^{-\alpha\tau_m}\epsilon^T(t-\tau_m)Q_1\epsilon(t-\tau_m)\\
&+ \epsilon^T(t)Q_2\epsilon(t) - e^{-\alpha\tau_M}\epsilon^T(t-\tau_M)Q_2\epsilon(t-\tau_M)\\
&+ \epsilon^T(t)Q_3\epsilon(t) - e^{-\alpha h_M}\epsilon^T(t-h(t))Q_3\epsilon(t-h(t))(1-h)\\
&+ \epsilon^T(t)Q_4\epsilon(t) - e^{-\alpha\tau_M}\epsilon^T(t-\tau(t))Q_4\epsilon(t-\tau(t))(1-\tau)\\
&+ \dot\epsilon^T(t)Q_5\dot\epsilon(t) - e^{-\alpha h_M}\dot\epsilon^T(t-h(t))Q_5\dot\epsilon(t-h(t))(1-h)\\
&+ \epsilon^T(t)Q_6\epsilon(t) - e^{-\alpha h_M}\epsilon^T(t-h_M)Q_6\epsilon(t-h_M)\\
&+ \dot\epsilon^T(t)Q_7\dot\epsilon(t) - e^{-\alpha\tau_M}\dot\epsilon^T(t-\tau(t))Q_7\dot\epsilon(t-\tau(t))(1-\tau)\\
&+ \epsilon^T(t)Q_8\epsilon(t) - e^{-\alpha h_m}\epsilon^T(t-h_m)Q_8\epsilon(t-h_m)\\
&+ h_M^2\dot\epsilon^T(t)R_1\dot\epsilon(t) - h_M\int_{t-h_M}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)R_1\dot\epsilon(s)ds\\
&+ \tau_m^2\dot\epsilon^T(t)R_2\dot\epsilon(t) - \tau_m\int_{t-\tau_m}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)R_2\dot\epsilon(s)ds.
\end{aligned} \tag{15}$$

By using Lemma 2.1, it can be seen that:

$$\begin{aligned}
&-h_M\int_{t-h_M}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)R_1\dot\epsilon(s)ds\\
&\le e^{-\alpha h_M}\begin{bmatrix}\epsilon(t)\\ \epsilon(t-h_M)\end{bmatrix}^T\begin{bmatrix}-R_1 & R_1\\ R_1 & -R_1\end{bmatrix}\begin{bmatrix}\epsilon(t)\\ \epsilon(t-h_M)\end{bmatrix},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&-\tau_m\int_{t-\tau_m}^{t} e^{\alpha(s-t)}\dot\epsilon^T(s)R_2\dot\epsilon(s)ds\\
&\le e^{-\alpha\tau_m}\begin{bmatrix}\epsilon(t)\\ \epsilon(t-\tau_m)\end{bmatrix}^T\begin{bmatrix}-R_2 & R_2\\ R_2 & -R_2\end{bmatrix}\begin{bmatrix}\epsilon(t)\\ \epsilon(t-\tau_m)\end{bmatrix}.
\end{aligned} \tag{17}$$

The following equation holds for any matrix $N$ with appropriate dimension:

$$\dot\epsilon^T(t)N[F_1\epsilon(t) + F_2\epsilon(t-h(t)) + F_3\epsilon(t-\tau(t)) - \dot\epsilon(t)] = 0. \tag{18}$$

Combining (15)-(18), we get that

$$\dot V(t) + \alpha V(t) \le \zeta^T(t)\hat\Pi\zeta(t), \tag{19}$$

where

$$\zeta^T(t) = \begin{bmatrix}\xi^T(t) & \epsilon^T(t-h_M) & \epsilon^T(t-\tau_M) & \epsilon^T(t-h_m) & \dot\epsilon^T(t-h(t)) & \dot\epsilon^T(t-\tau(t))\end{bmatrix},$$

$$\xi^T(t) = \begin{bmatrix}\epsilon^T(t) & \epsilon^T(t-h(t)) & \epsilon^T(t-\tau(t)) & \dot\epsilon^T(t) & \epsilon^T(t-\tau_m)\end{bmatrix},$$

and $\hat\Pi$ is given by (13).

As a result, since Equation (13) holds, it is deduced from Equation (19) that

$$\dot{V}(t) + \alpha V(t) \leq 0. \tag{20}$$

Now, let us define $v(t) := e^{\alpha t} V(t)$, and differentiate it along the solution of Equation (5). It is obtained that $\dot{v}(t) < 0$. Integrating the latter inequality from 0 to $t$ and substituting from the definition of $v(t)$ result in

$$V(t) \leq e^{-\alpha t} V(0). \tag{21}$$

From (14), we have

$$V(0) \leq (\lambda_1 + \lambda_2)\|\phi\|_c^2,$$

where

$$\begin{aligned}
\lambda_1 &= \lambda_{\max}(P) + \tau_m \lambda_{\max}(Q_1) + \tau_M \lambda_M(Q_2) + h_M \lambda_{\max}(Q_3) + \tau_M \lambda_M(Q_4) \\
&\quad + h_M \lambda_{\max}(Q_5) + h_M \lambda_{\max}(Q_6) + \tau_M \lambda_{\max}(Q_7) + h_m \lambda_{\max}(Q_8) \\
\lambda_2 &= \tfrac{h_M^3}{2} \lambda_{\max}(R_1) + \tfrac{\tau_m^3}{2} \lambda_{\max}(R_2).
\end{aligned}$$

On the other hand

$$\lambda_{\min}(P)\|e(t)\|_c^2 \leq V_1(t) \leq V(t).$$

Hence, the Lyapunov-Krasovskii theorem helps us to conclude that $\epsilon(t)$ and $\dot{\epsilon}(t)$ exponentially converge to zero with the rate of $\frac{\alpha}{2}$. More specifically,

$$\|\epsilon(t)\| \leq \sqrt{\frac{\lambda_1 + \lambda_2}{\lambda_{\min}(P)}} e^{-\frac{\alpha}{2}t} \|\phi\|_c. \tag{22}$$

It completes the proof of the theorem.

4. **Functional Observer Design.** Let $\bar{C} := [C^\dagger, C^\perp]$. Firstly, we introduce the following parameters:

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = T\bar{C}, \tag{23}$$

$$\begin{bmatrix} L_1 & L_2 \end{bmatrix} = L\bar{C}, \tag{24}$$

$$\bar{C} \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = A_1 \bar{C}, \tag{25}$$

$$\bar{C} \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} = A_2 \bar{C}, \tag{26}$$

$$\bar{C} \begin{bmatrix} A_{11}^3 & A_{12}^3 \\ A_{21}^3 & A_{22}^3 \end{bmatrix} = A_3 \bar{C}, \tag{27}$$

where $T_1 \in \mathbb{R}^{l \times p}$, $T_2 \in \mathbb{R}^{l \times (n-p)}$, $L_1 \in \mathbb{R}^{l \times p}$, $L_2 \in \mathbb{R}^{l \times (n-p)}$, $A_{11}^i \in \mathbb{R}^{p \times p}$, $A_{12}^i \in \mathbb{R}^{p \times (n-p)}$, and $A_{22}^i \in \mathbb{R}^{(n-p) \times (n-p)}$, $i = \{1, 2, 3\}$. Next, post-multiplying Equations (6)-(9) by $\bar{C}$ results in the following set of equations:

$$T_2 = L_2, \tag{28}$$

$$V = L_1 - T_1, \tag{29}$$

$$H_1 = -F_1 T_1 + T_1 A_{11}^1 + T_2 A_{21}^1, \tag{30}$$

$$H_2 = -F_2 T_1 + T_1 A_{11}^2 + T_2 A_{21}^2, \tag{31}$$

$$H_3 = -F_3 T_1 + T_1 A_{11}^3 + T_2 A_{21}^3, \tag{32}$$

$$F_1 T_2 - T_1 A_{12}^1 - T_2 A_{22}^1 = 0, \tag{33}$$

$$F_2 T_2 - T_1 A_{12}^2 - T_2 A_{22}^2 = 0, \tag{34}$$

$$F_3 T_2 - T_1 A_{12}^3 - T_2 A_{22}^3 = 0. \tag{35}$$

Now, considering Equations (28) and (33)-(35), we have,

$$\begin{bmatrix} F_1 & F_2 & F_3 & -T_1 \end{bmatrix} \Omega = \Phi, \tag{36}$$

where

$$\Omega := \begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \\ A_{12}^1 & A_{12}^2 & A_{12}^3 \end{bmatrix},$$

and $\Phi = \begin{bmatrix} L_2 A_{22}^1 & L_2 A_{22}^2 & L_2 A_{22}^3 \end{bmatrix}$. It can be shown that Equation (36) has a solution if and only if the below condition is fulfilled:

Condition I:

$$rank\left(\begin{bmatrix} L_2A_{22}^1 & L_2A_{22}^2 & L_2A_{22}^3 \\ A_{12}^1 & A_{12}^2 & A_{12}^3 \\ L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \end{bmatrix}\right) = rank\left(\begin{bmatrix} A_{12}^1 & A_{12}^2 & A_{12}^3 \\ L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \end{bmatrix}\right). \tag{37}$$

**Remark 4.1.** *In fact, (36) is equivalent to the following equation:*

$$\Omega^T \begin{bmatrix} F_1 & F_2 & F_3 & -T_1 \end{bmatrix}^T = \Phi^T, \tag{37a}$$

*and (37a) has a solution if and only if the following rank condition is satisfied:*

$$rank\left(\Omega^T\right) = rank\left(\left[\Omega^T, \Phi^T\right]\right). \tag{37b}$$

*(37b) holds if and only if*

$$rank(\Omega) = rank\begin{pmatrix} \Omega \\ \Phi \end{pmatrix}$$

*holds, which is equivalent to (37) holds.*

If Condition I is achieved, then it is concluded from Equation (36) that,

$$\begin{bmatrix} F_1 & F_2 & F_3 & -T_1 \end{bmatrix} = U_1 + \bar{Z}U_2, \tag{38}$$

where $U_1 := \Phi\Omega^\dagger$, $U_2 := I_{3l+p} - \Omega\Omega^\dagger$, and $\bar{Z} \in R^{l\times(3l+p)}$ is an arbitrary parameter. Hence, the below can be written from Equation (38):

$$F_1 = U_{11} + \bar{Z}U_{21}, \tag{38a}$$

$$F_2 = U_{12} + \bar{Z}U_{22}, \tag{38b}$$

$$F_3 = U_{13} + \bar{Z}U_{23}, \tag{38c}$$

$$-T_1 = U_{14} + \bar{Z}U_{24}, \tag{38d}$$

where $U_{11}$, $U_{12}$, $U_{13}$, $U_{14}$, $U_{21}$, $U_{22}$, $U_{23}$ and $U_{24}$ are the partitions of $U_1$ and $U_2$ with appropriate dimensions, respectively.

**Theorem 4.1.** *Assume that Condition I is satisfied. For given constants $\alpha > 0$, $0 \le h_m \le h_M$, $0 \le \tau_m \le \tau_M$, $h < 1$ and $\tau < 1$, the functional observer (4) is globally $\tilde{\alpha}$-exponentially stable if*

*(a) there exist matrices $M > 0$, $\bar{Q} > 0$, $\bar{R} > 0$, $K_1$, $K_2$, and $K_3$ of appropriate dimensions, and positive scalars $\alpha_i$ $(i = 2, 3, \ldots, 8)$ and $\beta$ such that the following linear matrix inequality holds:*

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} & \tilde{\Pi}_{14} & e^{-\alpha\tau_m}\beta\bar{R} & e^{-\alpha h_M}\bar{R} & 0 & 0 & 0 & 0 \\ * & \tilde{\Pi}_{22} & 0 & \tilde{\Pi}_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Pi}_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \tilde{\Pi}_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Pi}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Pi}_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \tilde{\Pi}_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \tilde{\Pi}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \tilde{\Pi}_{10,10} \end{bmatrix} < 0, \quad (39)$$

*where*

$$\begin{aligned}
\tilde{\Pi}_{11} &= U_{11}M^T + MU_{11}^T + K_1 + K_1^T + \bar{\alpha}\bar{Q} + \alpha M^T - \left(e^{-\alpha h_M} + e^{-\alpha\tau_m}\beta\right)\bar{R},\\
\tilde{\Pi}_{12} &= U_{12}M^T + K_2,\ \tilde{\Pi}_{13} = U_{13}M^T + K_3,\ \tilde{\Pi}_{14} = \tfrac{1}{2}MU_{11}^T + \tfrac{1}{2}K_1^T,\\
\tilde{\Pi}_{22} &= -(1-h)e^{-\alpha h_M}\alpha_3\bar{Q},\ \tilde{\Pi}_{24} = \tfrac{1}{2}MU_{12}^T + \tfrac{1}{2}K_2^T,\\
\tilde{\Pi}_{33} &= -(1-\tau)e^{-\alpha\tau_M}\alpha_4\bar{Q},\ \tilde{\Pi}_{34} = \tfrac{1}{2}MU_{13}^T + \tfrac{1}{2}K_3^T,\\
\tilde{\Pi}_{44} &= \alpha_5\bar{Q} + \alpha_7\bar{Q} + h_M^2\bar{R} + \tau_m^2\beta\bar{R} - M^T,\ \tilde{\Pi}_{55} = -e^{-\alpha\tau_m}\bar{Q} - e^{-\alpha\tau_m}\beta\bar{R},\\
\tilde{\Pi}_{66} &= -e^{-\alpha h_M}\alpha_6\bar{Q} - e^{-\alpha h_M}\bar{R},\ \tilde{\Pi}_{77} = -e^{-\alpha\tau_M}\alpha_2\bar{Q},\ \tilde{\Pi}_{88} = -e^{-\alpha h_m}\alpha_8\bar{Q},\\
\tilde{\Pi}_{99} &= -(1-h)e^{-\alpha h_M}\alpha_5\bar{Q},\ \tilde{\Pi}_{10,10} = -(1-\tau)e^{-\alpha\tau_M}\alpha_7\bar{Q},\\
\bar{\alpha} &= 1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8,\ \tilde{\alpha} = \tfrac{\alpha}{2}.
\end{aligned}$$

*(b) the following rank condition is fulfilled:*

$$rank\left(\begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \\ K_1 & K_2 & K_3 \end{bmatrix}\right) = rank\left(\begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \end{bmatrix}\right). \quad (40)$$

*Furthermore, the observer design parameter $\bar{Z}$ can be computed from the following equation:*

$$\bar{Z} = \bar{K}\Psi^{\dagger}, \quad (41)$$

*where $\bar{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$, and $\Psi = \begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \end{bmatrix}$.*

**Proof:** Considering $\hat{\Pi}$ is not a linear matrix inequality. It is pre- and post-multiplied by a matrix $\Delta_1 := diag\left(M^T, M^T, M^T, N^{-1}, M^T, M^T, M^T, M^T, M^T, M^T\right)$, where $M = P^{-1}$. Denoting $\bar{\Pi} := \Delta_1^T\hat{\Pi}\Delta_1$, (13) is equivalent to $\bar{\Pi} < 0$, where

$$\bar{\Pi} = \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} & \bar{\Pi}_{14} & e^{-\alpha\tau_m}MR_2M^T & e^{-\alpha h_M}MR_1M^T & 0 & 0 & 0 & 0 \\ * & \bar{\Pi}_{22} & 0 & \bar{\Pi}_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{\Pi}_{33} & \bar{\Pi}_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \bar{\Pi}_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \bar{\Pi}_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \bar{\Pi}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \bar{\Pi}_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \bar{\Pi}_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \bar{\Pi}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \bar{\Pi}_{10,10} \end{bmatrix}$$

$$\begin{aligned}
\bar{\Pi}_{11} &= F_1M^T + MF_1^T + MQ_1M^T + MQ_2M^T + MQ_3M^T + MQ_4M^T \\
&\quad + MQ_6M^T + MQ_8M^T + \alpha M^T - e^{-\alpha h_M}MR_1M^T - e^{-\alpha\tau_m}MR_2M^T, \\
\bar{\Pi}_{12} &= F_2M^T,\ \bar{\Pi}_{13} = F_3M^T,\ \bar{\Pi}_{14} = \tfrac{1}{2}MF_1^T,\ \bar{\Pi}_{22} = -(1-h)e^{-\alpha h_M}MQ_3M^T, \\
\bar{\Pi}_{24} &= \tfrac{1}{2}MF_2^T,\ \bar{\Pi}_{33} = -(1-\tau)e^{-\alpha\tau_M}MQ_4M^T,\ \bar{\Pi}_{34} = \tfrac{1}{2}MF_3^T, \\
\bar{\Pi}_{44} &= N^{-1}Q_5N^{-T} + N^{-1}Q_7N^{-T} + h_M^2N^{-1}R_1N^{-T} + \tau_m^2N^{-1}R_2N^{-T} - N^{-T}, \\
\bar{\Pi}_{55} &= -e^{-\alpha\tau_m}MQ_1M^T - e^{-\alpha\tau_m}MR_2M^T,\ \bar{\Pi}_{66} = -e^{-\alpha h_M}MQ_6M^T - e^{-\alpha h_M}MR_1M^T, \\
\bar{\Pi}_{77} &= -e^{-\alpha\tau_M}MQ_2M^T,\ \bar{\Pi}_{88} = -e^{-\alpha h_m}MQ_8M^T, \\
\bar{\Pi}_{99} &= -(1-h)e^{-\alpha h_M}MQ_5M^T,\ \bar{\Pi}_{10,10} = -(1-\tau)e^{-\alpha\tau_M}MQ_7M^T.
\end{aligned}$$

In this line, it is assumed that

$$\begin{aligned}
&Q_1 = Q,\ Q_i = \alpha_iQ\ (i = 2, 3, \dots, 8),\ R_1 = R,\ R_2 = \beta R, \\
&\bar{\alpha} = 1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8.
\end{aligned}$$

Let $\bar{Q} = MQM^T$, $\bar{R} = MRM^T$, $K_1 = \bar{Z}U_{21}M^T$, $K_2 = \bar{Z}U_{22}M^T$, $K_3 = \bar{Z}U_{23}M^T$.

Furthermore, from (38a)-(38c), we know that $\bar{\Pi} < 0$ is equivalent to

$$\Pi = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & e^{-\alpha\tau_m}\beta\bar{R} & e^{-\alpha h_M}\bar{R} & 0 & 0 & 0 & 0 \\
* & \Pi_{22} & 0 & \Pi_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Pi_{33} & \Pi_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{66} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{77} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Pi_{88} & 0 & 0 \\
* & * & * & * & * & * & * & * & \Pi_{99} & 0 \\
* & * & * & * & * & * & * & * & * & \Pi_{10,10}
\end{bmatrix} < 0,$$

where

$$\begin{aligned}
\Pi_{11} &= U_{11}M^T + MU_{11}^T + K_1 + K_1^T + \bar{\alpha}\bar{Q} + \alpha M^T - \left(e^{-\alpha h_M} + e^{-\alpha\tau_m}\beta\right)\bar{R}, \\
\Pi_{12} &= U_{12}M^T + K_2,\ \Pi_{13} = U_{13}M^T + K_3,\ \Pi_{14} = \tfrac{1}{2}MU_{11}^T + \tfrac{1}{2}K_1^T, \\
\Pi_{22} &= -(1-h)e^{-\alpha h_M}\alpha_3\bar{Q},\ \Pi_{24} = \tfrac{1}{2}MU_{12}^T + \tfrac{1}{2}K_2^T, \Pi_{33} = -(1-\tau)e^{-\alpha\tau_M}\alpha_4\bar{Q}, \\
\Pi_{34} &= \tfrac{1}{2}MU_{13}^T + \tfrac{1}{2}K_3^T,\ \Pi_{55} = -e^{-\alpha\tau_m}\bar{Q} - e^{-\alpha\tau_m}\beta\bar{R},\ \Pi_{66} = -e^{-\alpha h_M}\alpha_6\bar{Q} - e^{-\alpha h_M}\bar{R}, \\
\Pi_{44} &= \alpha_5N^{-1}QN^{-T} + \alpha_7N^{-1}QN^{-T} + h_M^2N^{-1}RN^{-T} + \tau_m^2\beta N^{-1}RN^{-T} - N^{-T}, \\
\Pi_{77} &= -e^{-\alpha\tau_M}\alpha_2\bar{Q},\ \Pi_{88} = -e^{-\alpha h_m}\alpha_8\bar{Q},\quad \Pi_{99} = -(1-h)e^{-\alpha h_M}\alpha_5\bar{Q}, \\
\Pi_{10,10} &= -(1-\tau)e^{-\alpha\tau_M}\alpha_7\bar{Q},\quad \bar{\alpha} = 1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8.
\end{aligned}$$

In order to linearize the quadratic terms in $\Pi_{44}$, some simplifications are considered. From the definitions of $\bar{Q}$ and $\bar{R}$, we can obtain that $Q = M^{-1}\bar{Q}(M^T)^{-1}, R = M^{-1}\bar{R}$ $(M^T)^{-1}$, then

$$\begin{aligned}
\Pi_{44} &= \alpha_5N^{-1}QN^{-T} + \alpha_7N^{-1}QN^{-T} + h_M^2N^{-1}RN^{-T} + \tau_m^2\beta N^{-1}RN^{-T} - N^{-T} \\
&= \alpha_5N^{-1}M^{-1}\bar{Q}(M^T)^{-1}N^{-T} + \alpha_7N^{-1}M^{-1}\bar{Q}(M^T)^{-1}N^{-T} \\
&\quad + h_M^2N^{-1}M^{-1}\bar{R}(M^T)^{-1}N^{-T} + \tau_m^2\beta N^{-1}M^{-1}\bar{R}(M^T)^{-1}N^{-T} - N^{-T}.
\end{aligned}$$

Let $N^{-1} = M$, then $\Pi_{44} = \alpha_5\bar{Q} + \alpha_7\bar{Q} + h_M^2\bar{R} + \tau_m^2\beta\bar{R} - M^T$. Moreover, employing the above similarity transformation, it can be seen that $\hat{\Pi} < 0$ is equivalent to

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} & \tilde{\Pi}_{14} & e^{-\alpha\tau_m}\beta\bar{R} & e^{-\alpha h_M}\bar{R} & 0 & 0 & 0 & 0 \\ * & \tilde{\Pi}_{22} & 0 & \tilde{\Pi}_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Pi}_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \tilde{\Pi}_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Pi}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Pi}_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \tilde{\Pi}_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \tilde{\Pi}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \tilde{\Pi}_{10,10} \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\tilde{\Pi}_{11} &= U_{11}M^T + MU_{11}^T + K_1 + K_1^T + \bar{\alpha}\bar{Q} + \alpha M^T - \left(e^{-\alpha h_M} + e^{-\alpha\tau_m}\beta\right)\bar{R}, \\
\tilde{\Pi}_{12} &= U_{12}M^T + K_2,\ \tilde{\Pi}_{13} = U_{13}M^T + K_3,\ \tilde{\Pi}_{14} = \tfrac{1}{2}MU_{11}^T + \tfrac{1}{2}K_1^T, \\
\tilde{\Pi}_{22} &= -(1-h)e^{-\alpha h_M}\alpha_3\bar{Q},\ \tilde{\Pi}_{24} = \tfrac{1}{2}MU_{12}^T + \tfrac{1}{2}K_2^T, \\
\tilde{\Pi}_{33} &= -(1-\tau)e^{-\alpha\tau_M}\alpha_4\bar{Q},\ \tilde{\Pi}_{34} = \tfrac{1}{2}MU_{13}^T + \tfrac{1}{2}K_3^T, \\
\tilde{\Pi}_{44} &= \alpha_5\bar{Q} + \alpha_7\bar{Q} + h_M^2\bar{R} + \tau_m^2\beta\bar{R} - M^T,\ \tilde{\Pi}_{55} = -e^{-\alpha\tau_m}\bar{Q} - e^{-\alpha\tau_m}\beta\bar{R}, \\
\tilde{\Pi}_{66} &= -e^{-\alpha h_M}\alpha_6\bar{Q} - e^{-\alpha h_M}\bar{R},\ \tilde{\Pi}_{77} = -e^{-\alpha\tau_M}\alpha_2\bar{Q},\ \tilde{\Pi}_{88} = -e^{-\alpha h_m}\alpha_8\bar{Q}, \\
\tilde{\Pi}_{99} &= -(1-h)e^{-\alpha h_M}\alpha_5\bar{Q},\ \tilde{\Pi}_{10,10} = -(1-\tau)e^{-\alpha\tau_M}\alpha_7\bar{Q}, \\
\bar{\alpha} &= 1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8.
\end{aligned}$$

Finally, according to the definitions of $K_1$, $K_2$ and $K_3$, it can be seen that $\bar{Z}\Psi = \bar{K}$.

The parameter $\bar{Z}$ has a unique solution as Equation (41), if and only if Condition (b) is satisfied. As a result, the observer parameters $F_i$, $T_1$, $H_i$, and $V$, $(i = 1, 2, 3)$ can be respectively computed from Equations (38a)-(38c), (38d), and (30)-(32), which implies that Condition (b) of Theorem 4.1 is also satisfied. This completes the proof.

**Remark 4.2.** *In the case that $h(t)$ and $\tau(t)$ are the time-varying delays, and they are assumed to satisfy*

$$0 \le h(t) \le h_M < \infty,\ \dot{h}(t) \le h < 1, \tag{42}$$

$$0 \le \tau(t) \le \tau_M < \infty,\ \dot{\tau}(t) \le \tau < 1, \tag{43}$$

*then we have the following corollary.*

**Corollary 4.1.** *Assume that Condition I is satisfied. For given constants $\alpha > 0$, $h_M \ge 0$, $\tau_M \ge 0$, $h < 1$ and $\tau < 1$, the functional observer (4) is globally $\tilde{\alpha}$-exponentially stable if*

*(a) there exist matrices $M > 0$, $\bar{Q} > 0$, $\bar{R} > 0$, $K_1$, $K_2$ and $K_3$ of appropriate dimensions and positive scalars $\alpha_i > 0$ $(i = 2, 3, \ldots, 7)$ such that the following linear matrix inequality holds:*

$$\check{\Pi} = \begin{bmatrix} \check{\Pi}_{11} & \check{\Pi}_{12} & \check{\Pi}_{13} & \check{\Pi}_{14} & e^{-\alpha h_M}\bar{R} & 0 & 0 & 0 \\ * & \check{\Pi}_{22} & 0 & \check{\Pi}_{24} & 0 & 0 & 0 & 0 \\ * & * & \check{\Pi}_{33} & \check{\Pi}_{34} & 0 & 0 & 0 & 0 \\ * & * & * & \check{\Pi}_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \check{\Pi}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Pi}_{66} & 0 & 0 \\ * & * & * & * & * & * & \check{\Pi}_{77} & 0 \\ * & * & * & * & * & * & * & \check{\Pi}_{88} \end{bmatrix} < 0, \tag{44}$$

*where*

$$\begin{aligned} \check{\Pi}_{11} &= U_{11}M^T + MU_{11}^T + K_1 + K_1^T + \check{\alpha}\bar{Q} + \alpha M^T - e^{-\alpha h_M}\bar{R}, \\ \check{\Pi}_{12} &= U_{12}M^T + K_2,\ \check{\Pi}_{13} = U_{13}M^T + K_3,\ \check{\Pi}_{14} = \tfrac{1}{2}MU_{11}^T + \tfrac{1}{2}K_1^T, \\ \check{\Pi}_{22} &= -(1-h)e^{-\alpha h_M}\alpha_3\bar{Q},\ \check{\Pi}_{24} = \tfrac{1}{2}MU_{12}^T + \tfrac{1}{2}K_2^T,\ \check{\Pi}_{33} = -(1-\tau)e^{-\alpha\tau_M}\alpha_4\bar{Q}, \\ \check{\Pi}_{34} &= \tfrac{1}{2}MU_{13}^T + \tfrac{1}{2}K_3^T,\ \check{\Pi}_{44} = \alpha_5\bar{Q} + \alpha_7\bar{Q} + h_M^2\bar{R} - M^T, \\ \check{\Pi}_{55} &= -e^{-\alpha h_M}\alpha_6\bar{Q} - e^{-\alpha h_M}\bar{R},\ \check{\Pi}_{66} = -e^{-\alpha\tau_M}\alpha_2\bar{Q},\ \check{\Pi}_{77} = -(1-h)e^{-\alpha h_M}\alpha_5\bar{Q}, \\ \check{\Pi}_{88} &= -(1-\tau)e^{-\alpha\tau_M}\alpha_7\bar{Q},\ \check{\alpha} = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6,\ \tilde{\alpha} = \tfrac{\alpha}{2}. \end{aligned}$$

*(b) the following rank condition is fulfilled:*

$$rank\left(\begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \\ K_1 & K_2 & K_3 \end{bmatrix}\right) = rank\left(\begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \end{bmatrix}\right). \tag{45}$$

*Furthermore, the observer design parameter $\bar{Z}$ can be computed from the following equation:*

$$\bar{Z} = \bar{K}\Psi^\dagger, \tag{46}$$

*where $\bar{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$, and $\Psi = \begin{bmatrix} U_{21}M^T & U_{22}M^T & U_{23}M^T \end{bmatrix}$.*

**Remark 4.3.** *If $F_1 = 0$, $F_2 = 0$, then we have the following observer structure:*

$$\begin{aligned} \dot{\omega}(t) &= F_1\omega(t) + Gu(t) + F_4 f(y(t)) + H_1 y(t) + H_2 y(t-h(t)) + H_3 y(t-\tau(t)), \\ \hat{z}(t) &= \omega(t) + Vy(t), \\ \omega(t) &= 0,\ \forall t \in [-\bar{M}, 0], \end{aligned} \tag{47}$$

*where $F_1$, $F_4$, $H_1$, $H_2$, $H_3$, $G$ and $V$ are as defined for the observer structure (4).*
*Furthermore, Condition I can be changed into the following condition:*
*Condition II:*

$$rank\left(\begin{bmatrix} L_2A_{22}^1 & L_2A_{22}^2 & L_2A_{22}^3 \\ A_{12}^1 & A_{12}^2 & A_{12}^3 \\ L_2 & 0 & 0 \end{bmatrix}\right) = rank\left(\begin{bmatrix} A_{12}^1 & A_{12}^2 & A_{12}^3 \\ L_2 & 0 & 0 \end{bmatrix}\right). \tag{48}$$

**Corollary 4.2.** *Assume that Condition II is satisfied. For given constants $\alpha > 0$, $0 \le h_m \le h_M$, $0 \le \tau_m \le \tau_M$, the functional observer (47) is globally $\tilde{\alpha}$-exponentially stable if*
*(a) there exist matrices $M > 0$, $\bar{Q} > 0$, $\bar{R} > 0$, $K_1$ of appropriate dimensions, and positive scalars $\alpha_i > 0$ $(i = 2, 6, 8)$ and $\beta$ such that the following linear matrix inequality*

*holds:*

$$\acute{\Pi} = \begin{bmatrix} \acute{\Pi}_{11} & \acute{\Pi}_{12} & e^{-\alpha\tau_m}\beta\bar{R} & e^{-\alpha h_M}\bar{R} & 0 & 0 \\ * & \acute{\Pi}_{22} & 0 & 0 & 0 & 0 \\ * & * & -e^{-\alpha\tau_m}\bar{Q} - e^{-\alpha\tau_m}\beta\bar{R} & 0 & 0 & 0 \\ * & * & * & -e^{-\alpha h_M}\alpha_6\bar{Q} - e^{-\alpha h_M}\bar{R} & 0 & 0 \\ * & * & * & * & \acute{\Pi}_{55} & 0 \\ * & * & * & * & * & \acute{\Pi}_{66} \end{bmatrix} < 0, \quad (49)$$

*where*

$$\acute{\Pi}_{11} = U_{11}M^T + MU_{11}^T + K_1 + K_1^T + \acute{\alpha}\bar{Q} + \alpha M^T - (e^{-\alpha h_M} + e^{-\alpha\tau_m}\beta)\bar{R},$$
$$\acute{\Pi}_{12} = \tfrac{1}{2}MU_{11}^T + \tfrac{1}{2}K_1^T,\ \acute{\Pi}_{22} = h_M^2\bar{R} + \tau_m^2\beta\bar{R} - M^T,\ \tilde{\alpha} = \tfrac{\alpha}{2},$$
$$\acute{\Pi}_{55} = -e^{-\alpha\tau_M}\alpha_2\bar{Q},\ \acute{\Pi}_{66} = -e^{-\alpha h_m}\alpha_8\bar{Q},\ \acute{\alpha} = 1 + \alpha_2 + \alpha_6 + \alpha_8.$$

*(b) the following rank condition is fulfilled:*

$$rank\left(\begin{bmatrix} U_{21}M^T \\ K_1 \end{bmatrix}\right) = rank\left(\begin{bmatrix} U_{21}M^T \end{bmatrix}\right). \quad (50)$$

*Furthermore, the observer design parameter $\bar{Z}$ can be computed from the following equation:*

$$\bar{Z} = \bar{K}\Psi^{\dagger}, \quad (51)$$

*where $\bar{K} = \begin{bmatrix} K_1 \end{bmatrix}$, and $\Psi = \begin{bmatrix} U_{21}M^T \end{bmatrix}$.*

**Proof:** The proof is similar to the proof of Theorem 4.1. Thus, it is omitted.

5. **Numerical Examples.** In this section, we will provide two examples to illustrate the effectiveness of the obtained results.

**Example 5.1.** *Consider system (1) with the following parameters:*

$$A_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0.1 \\ 2 & 3 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.4 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0.2 & 3 & 0.5 \\ 0.3 & 2 & 7 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.2 & 1 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0.4 & 0.2 & 0 \end{bmatrix},\ f(Cx(t)) = [\sin(x_1(t))\ \ 0\ \ \sin(x_2(t))\ \ 0]^T,\ \alpha = 2,$$
$$\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 1,\ \bar{\alpha} = 6,\ \beta = 1,\ \tau_m = 1,\ \tau_M = 3,$$
$$\tau = 0.8,\ h_m = 1,\ h_M = 3,\ h = 0.8,\ u(t) = 0.$$

*It can be observed that Condition I is satisfied. Applying Theorem 4.1 and using the Matlab LMI control toolbox, we solve (39) and obtain a set of feasible solutions as follows:*

$$M = 917290,\ K_1 = -1756200,\ K_2 = 0,\ K_3 = 0,\ \bar{Q} = 65915,\ \bar{R} = 9179.5.$$

*In addition, the observer parameters were obtained as*

$$F_1 = -1.9530,\ F_2 = 0,\ F_3 = 0,\ F_4 = [0.2106\ \ 0.0781\ \ 0.5918\ \ 0.2553],$$
$$G = 0,\ H_1 = [0.7722\ \ 0.6],\ H_2 = [0\ \ 0],\ H_3 = [0\ \ 0],\ V = [-0.1906\ \ 0.4000].$$

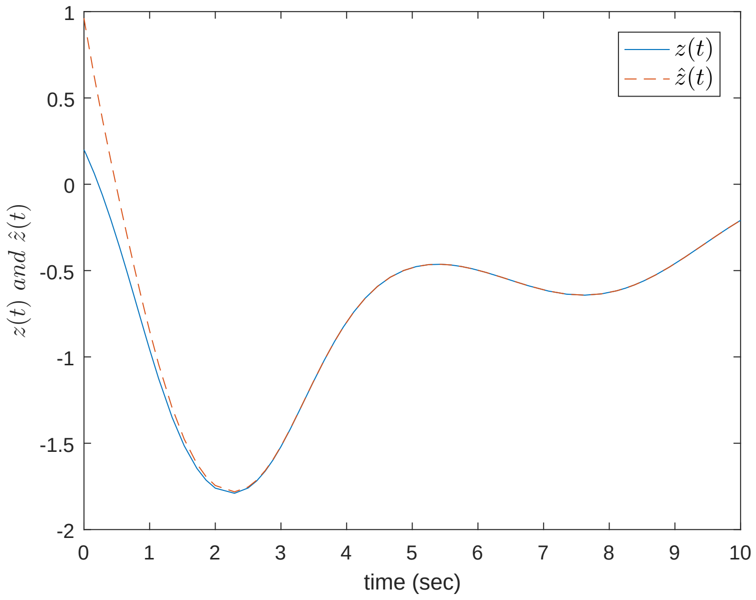


FIGURE 1. Responses of state $z(t)$ and the estimate of $z(t)$ in Example 5.1

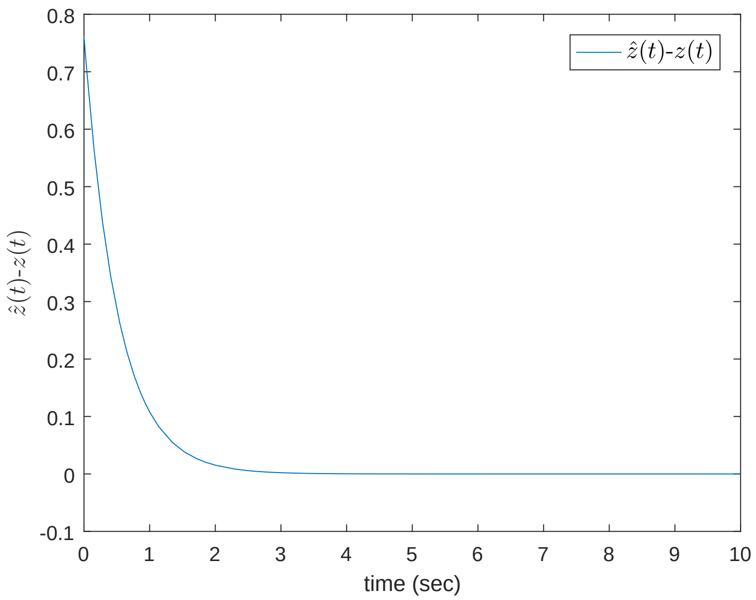


FIGURE 2. Responses of state $\hat{z}(t) - z(t)$ in Example 5.1

*The simulation results are given in Figures 1 and 2. From Figure 2, it can be seen that* $\hat{z}(t) - z(t)$ *converges to* 0 *rapidly.*

**Example 5.2.** *Consider system (1) with the following parameters:*

$$A_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0.1 \\ 2 & 1 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.2 & 0 & 0.4 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0.5 & -2 \\ 0.3 & 0 & 3 & -0.2 \\ 1 & 0.8 & 1 & -2 \\ -0.3 & 0 & -1 & 3 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0.5 & 0.2 & 0 \end{bmatrix}, \ f(Cx(t)) = [0.1\sin(x_1(t)) \ 0 \ 0.2\sin(x_2(t)) \ 0]^T, \ \alpha = 3.9,$$
$$\alpha_i = 1, \ i = 2, 3, \ldots, 8, \ \bar{\alpha} = 6, \ \beta = 1, \ \tau_m = 1, \ \tau_M = 3, \ \tau = 0.8, \ h_m = 1, \ h_M = 3,$$
$$h = 0.8, \ u(t) = e^{-0.4t}\cos(2t).$$

*It can be observed that Condition I is satisfied. Applying Theorem 4.1 and using the Matlab LMI control toolbox, we solve (39) and obtain a set of feasible solutions as follows:*

$$M = 2.4647 \times 10^8, \ K_1 = -9.4289 \times 10^8, \ K_2 = 0, \ K_3 = 0, \ \bar{Q} = 1151800, \ \bar{R} = 91717.$$

*In addition, the observer parameters were obtained as*

$$F_1 = -3.8641, \ F_2 = 0, \ F_3 = 0, \ F_4 = [0.7728 \ \ 0.1600 \ \ 0.4864 \ \ -0.5456], \ G = 0.5728,$$
$$H_1 = \begin{bmatrix} 2.6134 & 0.6 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ H_3 = \begin{bmatrix} 0.5728 & 0.2 \end{bmatrix}, \ V = \begin{bmatrix} 0.4272 & 0.5 \end{bmatrix}.$$

*The simulation results are given in Figures 3 and 4.*

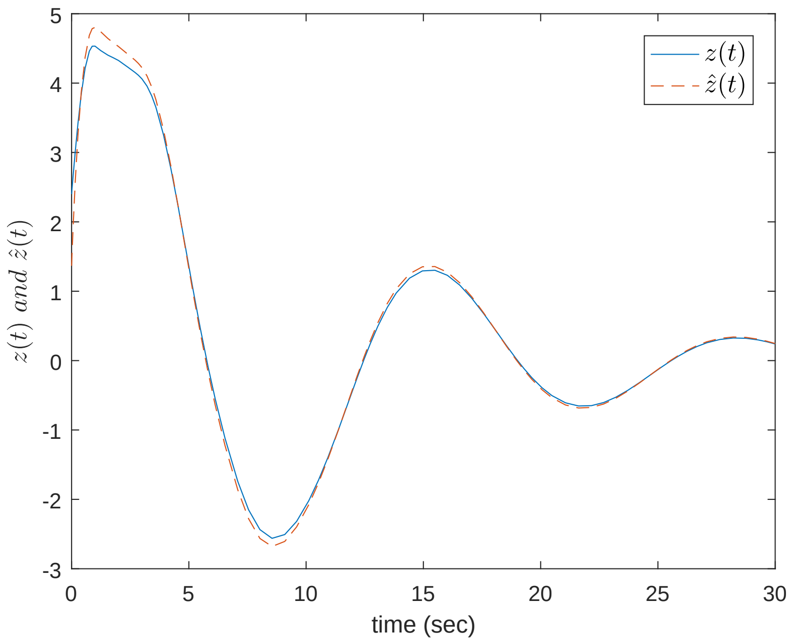


FIGURE 3. Responses of state $z(t)$ and the estimate of $z(t)$ in Example 5.2

*From Figure 4, it can be seen that $\hat{z}(t) - z(t)$ converges to 0 rapidly.*

**Remark 5.1.** *In Example 5.1 and Example 5.2, we deal with the functional observer for the system with $x(\cdot) \in R^4$, $f(\cdot) \in R^4$, $A_i \in R^{4\times4}$, $i = 1, 2, 3$, $A \in R^{4\times4}$, $C \in R^{2\times4}$, $L \in R^{1\times4}$, $B \in R^{4\times1}$. For the general system, the above method can be used to construct the functional observer.*

6. **Conclusions.** This paper addresses the problem of exponential stability analysis of functional observer for a class of nonlinear systems with interval time-varying mixed delays by designing a delay-dependent minimum-order functional observer. The sufficient conditions of exponential stability of functional observer have been given. Then, by utilizing Lyapunov-Krasovskii approach and some well-known inequalities, we have proposed the

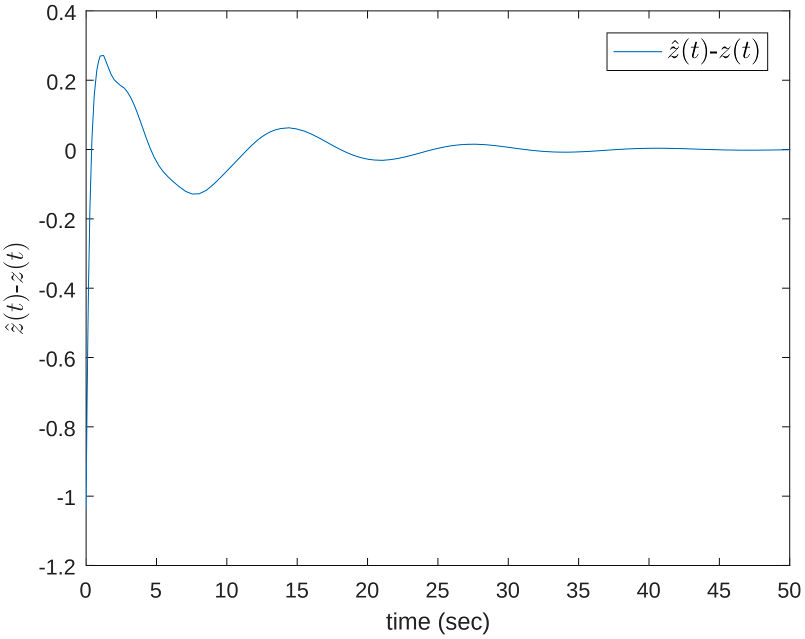


FIGURE 4. Responses of state $\hat{z}(t) - z(t)$ in Example 5.2

sufficient conditions of the exponential stability of functional observer in terms of linear matrix inequality for nonlinear systems with interval time-varying mixed delays. We also propose the computational method of the parameters of the delay-dependent functional observer that we have designed. The effectiveness of the proposed method is illustrated by two numerical examples and simulation results. Our future research may expand to cover the design of functional observer for uncertain nonlinear systems with interval time-varying mixed delays and switched nonlinear systems with interval time-varying mixed delays.

**Acknowledgment.** This work is supported by the National Natural Science Foundation of China under Grant 61603272.

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