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HESITANT FUZZY MUIRHEAD MEAN OPERATORS AND ITS APPLICATION TO MULTIPLE ATTRIBUTE DECISION MAKING

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Abstract. *In this paper, we investigate the multiple attribute decision making (MADM) problems with the hesitant fuzzy information based on a new aggregation operator. To begin with, we present the new hesitant fuzzy Muirhead mean operator to deal with MADM problems with hesitant fuzzy information, including the hesitant fuzzy Muirhead mean (HFMM) operator, the hesitant fuzzy weighted Muirhead mean (HFWMM) operator, the main advantages of these aggregation operators are that they can capture interrelationships of multiple attributes among any number of attributes by a parameter vector P and make information aggregation process more flexible by the parameter vector P, whilst, HFMM and HFWMM are also a generalization of hesitant fuzzy Maclaurin symmetric mean (HFMSM) operator. In addition, some properties of these new aggregation operators are obtained and some special cases are discussed where the parameter vector takes some different values. Moreover, we present a new method to solve the MADM problems with hesitant fuzzy information. Finally, an illustrative example is provided to show the feasibility and validity of the new method, the influences of parameter vector P on the decision making results are investigated and the advantages of the proposed methods by comparing with the other existing methods are also analyzed by the example.*

Keywords: Hesitant fuzzy set, Murihead mean, Aggregation operator, Multiple attribute decision making

1. **Introduction.** Multiple attribute decision making (MADM), as an effective framework for comparison, has always been used to find the most desirable one from a finite set of alternatives on the predefined attributes. An important problem of decision process is to express the attribute value. However, due to the intrinsic complexity of natural objects, there exists much uncertain information in many real-world problems. So, it is difficult for experts or decision makers (DMs) to give their assessments on attributes by crisp numbers. In 2010, Torra [1] introduced an important extension of fuzzy sets named hesitant fuzzy sets (HFSs) which permit the membership degree of an element to a set to

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be represented as several possible values between 0 and 1, human beings hesitate among a set of membership degrees and they need to represent such a hesitation. Rodriguez et al. [2, 3, 4] recently provided a position and perspective analysis of HFSs in decision making, which gave a detailed review on HFS and its application in decision problems, especially pointed out some important challenges. Since HFS was proposed, a lot of research achievements about theory and methods have been made, and it has the following aspects: (A) the basic theory, such as distance and similarity degree [5, 6, 7, 8], entropy and cross entropy [9]; (B) the decision methods [10, 11, 12, 13, 14, 15, 16, 17, 20] based on some hesitant fuzzy aggregation operators.

In the field of information fusion, information aggregation is an important research topic as it is a critical process of gathering relevant information from multiple sources. However, aggregation operator as a tool to aggregate relevant information has been focused on and also used in many decision making problems. In real decision making, there exist the interrelationships among the attributes in MADM or MAGDM problems. For example, a company wants to choose a supplier. Suppose that some suppliers which are regarded as the alternatives and $C = \{c_1, c_2, c_3, c_4, c_5\}$ is a group of attributes, $(c_1, c_2, c_3, c_4, c_5)$ stand for 'production cost', 'production quality', 'supplier's service performance', 'the profile of supplier' and 'risk factor', respectively. In the process of decision, the interrelationships of the five attribute should be considered, usually, we use the parameter vector $P = (p_1, \ldots, p_5)$ to control this interrelationship, for example, $P = (p_1, \ldots, p_5)$ (where $p_i \neq 0$, $i = 1, 2, \ldots, 5$) means that the interrelationship of five attributes is considered, $P = (1, 1, 0, 0, 0)$ means that the interrelationship between only two attributes can be considered, of course, $P = (1, 0, 0, 0, 0)$ means that the interrelationship of attributes is not considered. Actually, the parameter vector P can be regarded as a utility measure which helps the DM to obtain the compromise solution by assigning appropriate values of the parameters, the quality, and flexibility of decision making can be improved by this investigation. Muirhead mean (MM) [18] is a well-known aggregation operator for it can consider the interrelationships among any number of aggregation arguments and the main advantage of the MM is exactly that it can capture interrelationships among many arguments. Whilst, MM is also a universal operator since it contains other general operators by assessing different parameters and MM is also a generalization of Maclaurin symmetric mean (MSM) [19]. When the parameter vector is assessed of different values, MM will reduce to some existing operators, such as arithmetic and geometric operators which do not consider the interrelationships of aggregation arguments, intuitionistic fuzzy and hesitant fuzzy Maclaurin symmetric mean [21, 22, 23, 24], were the special cases of MM operators and applied to solving the some decision making problems. In current hesitant fuzzy aggregations operators, it can be divided into two categories from the interrelationships of the attributes: (1) some hesitant fuzzy aggregation operators in which the interrelationships of the attributes are not considered, such as hesitant weighted averaging operator (HFWA), hesitant weighted geometric operator (HFWGA), hesitant fuzzy Hamacher weighted aggregation operators (HFHWA); (2) some hesitant fuzzy aggregation operators in which the interrelationships of attributes are considered, such as, hesitant fuzzy geometric Bonferroni mean (HFGBM) [15], hesitant fuzzy Bonferroni mean (HFBM) operator [14] and hesitant fuzzy Heronian mean (HFHM) operator [27], hesitant fuzzy Maclaurin systems mean (HFMSM) operator [23, 24]. However, although HFGBM, HFBM and HFHM operators can capture the interrelationship of aggregation arguments, they can only consider the interrelationship between any two arguments. As far as HFMSM is concerned, since MM is a generalization of MSM, it is meaningful to extend HFMSM to hesitant fuzzy Murihead mean (HFMM) despite the fact that HFMSM is capable of capturing the interrelationships of multiple attributes. Therefore, it is necessary and significant to develop some new aggregation operators based on MM that not only accommodate hesitant fuzzy information but also can capture the interrelationships among multi-input arguments.

The goal of this paper is to develop a method for MADM problems with hesitant fuzzy information based on new hesitant fuzzy MM (HFMM) operators by combining MM and hesitant fuzzy information. To begin with, we present a new hesitant fuzzy Muirhead mean operator to deal with MADM problems with hesitant fuzzy information, including the hesitant fuzzy Muirhead mean (HFMM) operator, hesitant fuzzy weighted Muirhead mean (HFWMM) operator. In addition, some properties of these new aggregation operators are obtained and some special cases are discussed. Finally, a new method is presented to solve an MADM problem with hesitant fuzzy information. To do so, the rest of the paper is organized as follows. In Section 2, we review some definitions on HFSs, HFEs and Muirhead mean, which are used in the analysis throughout this paper. Section 3 is devoted to the main results concerning HFMM operator along with their properties. Section 4 focuses on HFWMM operator along with their properties. In Section 5, we construct an MADM approach based on HFWMM operator proposed in Section 4. Consequently, a practical example is provided in Section 6 to verify the validity of the proposed methods and to show their advantages. In Section 7, we give some conclusions of this study.

2. **Preliminaries.**

In this section, some basic concepts related to hesitant fuzzy set and Muirhead mean are recapped, which are the basis of this work.

2.1. **Hesitant fuzzy set.**

Definition 2.1. [1] *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a reference set. A hesitant fuzzy set (HFS) F on X is defined in terms of a function $h_F(x)$ that returns a subset of $[0, 1]$ when it is applied to X, i.e., $F = \{\langle x, h_F(x) | x \in X\rangle\}$ where $h_F(x)$ is a set of some different values in $[0, 1]$, representing the possible membership degrees of the element $x \in X$ to F. $h_F(x)$ is called a hesitant fuzzy element (HFE), a basic unit of HFS.*

Definition 2.2. [12, 17] *Let h_1 and h_2 be two HFEs, and some operations on the h_1 and h_2 are defined as follows:*

(1) $h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;
(2) $h_1 \cap h_2 = \bigcap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$;
(3) $h_1^c = \bigcup_{\gamma_1 \in h_1} \{1 - \gamma_1\}$;
(4) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(5) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$;
(6) $\lambda h_1 = \bigcup_{\gamma_1 \in h_1} \left\{1 - (1 - \gamma_1)^\lambda\right\}$, *where* $\lambda > 0$;
(7) $h_1^\lambda = \bigcup_{\gamma_1 \in h_1} \left\{\gamma_1^\lambda\right\}$, *where* $\lambda > 0$.

Definition 2.3. [12] *Let h be an HFE, and*

$$s(h) = \frac{1}{n(h)} \sum_{\gamma \in h} \gamma \tag{1}$$

is called the score function of h, where $n(h)$ is the number of values of h.

For any two HFEs h_1 and h_2, if
$s(h_1) > s(h_2)$, then $h_1 > h_2$;
$s(h_1) = s(h_2)$, then $h_1 = h_2$.

2.2. **Muirhead mean operator.**

The Muirhead mean (MM) operator [18] is a general aggregation function and firstly proposed by Muirhead in 1902, and it is defined as follows.

Definition 2.4. [18] *Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers, $A = \{a_1, a_2, \ldots, a_n\}$ and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector, if*

$$MM^P(a_1, \ldots, a_n) = \left(\frac{1}{n!}\left(\sum_{\theta \in S_n}\left(\prod_{j=1}^{n} a_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{2}$$

then we call MM^P the Muirhead mean (MM), where $\theta(j)$ $(j = 1, 2, \ldots, n)$ is any a permutation of $(1, 2, \ldots, n)$ and S_n is the collection of all permutation of $\theta(j)$ $(j = 1, 2, \ldots, n)$.

There are some special cases when the parameter vector is assessed of different values.

(1) If $P = (1, 0, \ldots, 0)$, MM operator will reduce to arithmetic averaging operator

$$MM^{(1,0,\ldots,0)}(a_1, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^{n} a_j. \tag{3}$$

(2) If $P = (\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})$, MM operator will reduce to Maclaurin symmetric mean (MSM) operator

$$MM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k})}(a_1, \ldots, a_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le n}\prod_{j=1}^{k} a_j}{C_n^k}\right)^{\frac{1}{k}}. \tag{4}$$

(3) If $P = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, MM operator will reduce to geometric averaging operator

$$MM^{\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)}(a_1, \ldots, a_n) = \prod_{j=1}^{n} a_j^{\frac{1}{n}}. \tag{5}$$

From the above discussion we can see that the advantage of the MM operator is that it can capture the interrelationships among the multiple aggregated arguments and it is a generalization of most existing aggregation operators.

3. Hesitant Fuzzy Muirhead Mean Operators.

Because the traditional MM can only process the crisp number, and HFEs can easily express the fuzzy information, it is necessary and significant to extend MM to process HFEs. In this section, we propose the hesitant fuzzy Muirhead mean (HFMM) operator, and discuss its properties.

Definition 3.1. *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs, $A = \{h_1, h_2, \ldots, h_n\}$ and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a hesitant fuzzy Muirhead mean operator is a function HFMM: $A^n \to A$, and*

$$HFMM^P(h_1, \ldots, h_n) = \left(\frac{1}{n!}\left(\oplus_{\theta \in S_n}\left(\otimes_{j=1}^{n} h_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{6}$$

where $\theta(j)$ $(j = 1, 2, \ldots, n)$ is any a permutation of $(1, 2, \ldots, n)$ and S_n is the collection of all permutation of $\theta(j)$ $(j = 1, 2, \ldots, n)$.

Theorem 3.1. *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $HFMM^P(h_1, \ldots, h_n)$ is still an HFE and*

$$HFMM^P(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n}\left\{\left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \gamma_{\theta(j)}^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right\}. \tag{7}$$

Proof: Firstly, we prove Equation (7). According to the operational law of HFEs, we obtain

$$\left(h_{\theta(j)}\right)^{p_j} = \bigcup_{\gamma_{\theta(j)} \in h_{\theta(j)}} \left\{\gamma_{\theta(j)}^{p_j}\right\},$$

and

$$\otimes_{j=1}^{n} h_{\theta(j)}^{p_j} = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{\prod_{j}^{n} \gamma_{\theta(j)}^{p_j}\right\},$$

and then we get

$$\oplus_{\theta \in S_n} \otimes_{j=1}^{n} h_{\theta(j)}^{p_j} = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{1 - \prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \gamma_{\theta(j)}^{p_j}\right)\right\},$$

and

$$\frac{1}{n!} \oplus_{\theta \in S_n} \otimes_{j=1}^{n} a_{\theta(j)^{p_j}} = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \mu_{\theta(j)}^{p_j}\right)\right)^{\frac{1}{n!}}\right\}.$$

Therefore,

$$\left(\frac{1}{n!} \oplus_{\theta \in S_n} \otimes_{j=1}^{n} a_{\theta(j)^{p_j}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$$
$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{\left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \mu_{\theta(j)}^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right\}.$$

In the process of decision making, the aggregation results would be more reliable if the selected operator is monotonic, and the lack of monotonicity may debase the reliability and dependability of the final decision-making results. We can prove $HFMM^P(h_1, \ldots, h_n)$ are idempotent, bounded, and monotonic.

Property 3.1. *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs, $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector and $h_i = h = \{\gamma\}$ $(i = 1, 2, \ldots, n)$, and then*

$$HFMM^P(h_1, \ldots, h_n) = h.$$

Proof: Since

$$HFMM^P(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{\left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \gamma_{\theta(j)}^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right\},$$

we have

$$\bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{\left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \gamma_{\theta(j)}^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right\}$$
$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{\left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n} \gamma^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}\right\}$$

$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \gamma^{\sum_{i=1}^n p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\}$$

$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \left(1 - \gamma^{\sum_{i=1}^n p_j} \right)^{\frac{n!}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\}$$

$$= \left\{ \left(\gamma^{\sum_{i=1}^n p_j} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\} = \{\gamma\} = h.$$

Therefore, $HFMM^P(h_1, \ldots, h_n) = h$.

Property 3.2. (Monotonicity) *Let* $h_a = \{h_{a_1}, h_{a_2}, \ldots, h_{a_n}\}$ *and* $h_b = \{h_{b_1}, h_{b_2}, \ldots, h_{b_n}\}$ *be two collections of HFEs, and* $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ *be a parameter vector. If for any* $\gamma_{a_i} \in h_{a_i}$ *and* $\gamma_{b_i} \in h_{b_i}$*, we have* $\gamma_{a_i} \le \gamma_{b_i}$ *for any* i $(i = 1, 2, \ldots, n)$*, then*

$$HFMM^P(h_{a_1}, h_{a_2}, \ldots, h_{a_n}) \le HFMM^P(h_{b_1}, h_{b_2}, \ldots, h_{b_n}).$$

Proof: Let

$$HFMM^P(h_{a_1}, h_{a_2}, \ldots, h_{a_n}) = \bigcup_{\gamma_{a_i} \in h_{a_i}, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \prod_{j=1}^n \gamma_{a_{\theta(j)}}^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\}.$$

Since $\gamma_{a_i} \le \gamma_{b_i}$ for any i $(i = 1, 2, \ldots, n)$, we have

$$1 - \prod_{j}^{n} \gamma_{a_{\theta(j)}} \ge 1 - \prod_{j}^{n} \gamma_{b_{\theta(j)}}.$$

So we have

$$\left(1 - \left(\prod_{\theta \in S_n} \left(1 - \prod_{j=1}^n \gamma_{a_{\theta(j)}}^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \le \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \prod_{j=1}^n \gamma_{b_{\theta(j)}}^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}}.$$

According to Theorem 3.1 and Definition 3.1, we have

$$\begin{aligned} & HFMM^P(h_{a_1}, h_{a_2}, \ldots, h_{a_n}) \\ &= \bigcup_{\gamma_{a_i} \in h_{a_i}, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \prod_{j=1}^n \gamma_{a_{\theta(j)}}^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\} \\ &\le \bigcup_{\gamma_{b_i} \in h_{b_i}, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \prod_{j=1}^n \gamma_{b_{\theta(j)}}^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\} \\ &= HFMM^P(h_{b_1}, h_{b_2}, \ldots, h_{b_n}). \end{aligned}$$

Property 3.3. (Boundedness) *Let* h_i $(i = 1, 2, \ldots, n)$ *be a collection of HFEs,* $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ *be a parameter vector,*

$$h^- = \min_i \left\{ h_i^- | h_i^- = \min\{\gamma_i \in h_i\} \right\},$$
$$h^+ = \max_i \left\{ h_i^+ | h_i^+ = \max\{\gamma_i \in h_i\} \right\},$$

and then

$$h^- \le HFMM^P(h_1, \ldots, h_n) \le h^+.$$

Proof: Since $h^- \le h_i^- \le \gamma_i \le h_i^+ \le h^+$, we have $(h^-)^{p_j} \le (\gamma_i)^{p_j} \le (h^+)^{p_j}$ and then

$$(h^-)^{\sum_{j=1}^n p_j} \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ \prod_{j=1}^n \gamma_{\theta(j)}^{p_j} \right\} \le ((h^+)^n)^{\sum_{j=1}^n p_j}.$$

And so,

$$1-(h^+)^{\sum_{j=1}^n p_j} \quad \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ 1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right) \right\} \le 1-(h^-)^{\sum_{j=1}^n p_j}$$

and

$$\left(1-(h^+)^{\sum_{j=1}^n p_j}\right)^{n!\frac{1}{n!}} \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ \left(\prod_{\theta \in S_n} \left(1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{n!}} \right\}$$
$$\le \left(1-(h^-)^{\sum_{j=1}^n p_j}\right)^{n!\frac{1}{n!}},$$

that is,

$$1-(h^+)^{\sum_{j=1}^n p_j} \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ \left(\prod_{\theta \in S_n} \left(1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{n!}} \right\} \le 1-(h^-)^{\sum_{j=1}^n p_j}.$$

Therefore,

$$1-\left(1-(h^-)^{\sum_{j=1}^n p_j}\right) \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ 1-\left(\prod_{\theta \in S_n} \left(1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{n!}} \right\}$$
$$\le 1-\left(1-(h^+)^{\sum_{j=1}^n p_j}\right),$$

that is,

$$(h^-)^{\sum_{j=1}^n p_j} \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ 1-\left(\prod_{\theta \in S_n} \left(1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{n!}} \right\} \le (h^+)^{\sum_{j=1}^n p_j}.$$

And so

$$h^- = \left((h^-)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}} \le \bigcup_{\gamma_i \in h_i, i=1,2,\dots,n} \left\{ \left(1-\left(\prod_{\theta \in S_n} \left(1-\prod_{j=1}^n \left(\gamma_{\theta(j)}^{p_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}} \right\}$$
$$\le \left((h^+)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}} = h^+.$$

Therefore,

$$h^- \le HFMM^P(h_1,\dots,h_n) \le h^+.$$

It is easy to obtain that hesitant fuzzy Murihead mean operator is commutative according to Definition 3.1.

Property 3.4. (Commutativity) *Let h_i $(i=1,2,\dots,n)$ be a collection of HFEs, $P=(p_1,p_2,\dots,p_n) \in \mathbf{R}^n$ be a parameter vector, and $\left(h_1^{'},\dots,h_n^{'}\right)$ be any permutation of $(h_1,\dots,h_n)$, then*

$$HFMM^P(h_1,\dots,h_n) = HFMM^P\left(h_1^{'},\dots,h_n^{'}\right).$$

Now, we will develop some special cases of HFMM operator with respect to different parameter vectors. Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector.

(1) If $P = (1, 0, \ldots, 0)$, HFMM operator will reduce to hesitant fuzzy arithmetic averaging (HFA) operator

$$HFMM^{(1,0,\ldots,0)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{\frac{1}{n}} \right\}. \tag{8}$$

(2) If $P = (\lambda, 0, \ldots, 0)$, HFMM operator will reduce to generalized fuzzy arithmetic averaging (GFAA) operator

$$HFMM^{(\lambda,0,\ldots,0)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j^{\lambda}\right)^{\frac{1}{n}} \right)^{\frac{1}{\lambda}} \right\}. \tag{9}$$

(3) If $P = (1, 1, 0, \ldots, 0)$, HFMM operator will reduce to hesitant fuzzy Bonferroni Mean (HFBM) operator

$$HFMM^{(1,1,0,\ldots,0)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_J, i \neq j} \left\{ \left(1 - \prod_{i,j=1, i\neq j}^{n} (1 - \gamma_i \gamma_j)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right\}. \tag{10}$$

(4) If $P = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{n-k})$, HFMM operator will reduce to hesitant fuzzy Maclaurin symmetric mean (HFMSM) operator

$$\begin{aligned} &HFMM^{(\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{n-k})}(h_1, \ldots, h_n) \\ &= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \gamma_{i_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\}. \end{aligned} \tag{11}$$

(5) If $P = (1, 1, \ldots, 1)$, HFMM operator will reduce to hesitant fuzzy geometric averaging (HFGA) operator

$$HFMM^{(1,1,\ldots,1)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(\prod_{j=1}^{n} \gamma_j \right)^{\frac{1}{n}} \right\}. \tag{12}$$

(6) If $P = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, HFMM operator will reduce to hesitant fuzzy geometric averaging (HFGA) operator

$$HFMM^{\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(\prod_{j=1}^{n} \gamma_j \right)^{\frac{1}{n}} \right\}. \tag{13}$$

In section, the HFMM aggregation operator was investigated along with its properties and some special cases. However, the weight of attributes was not considered in HFMM. We will consider the hesitant fuzzy weighted MM operator in Section 4.

4. **Hesitant Fuzzy Weighted Muirhead Mean Operators.** Weights of attributes play a vital role in decision making and will directly reflect the results of decision making results. In Section 3, we proposed the HFMM aggregation operators which cannot consider the weights of attributes, so it is very important to consider weights of attributes in the process of information aggregation.

Definition 4.1. *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs, $A = \{h_1, h_2, \ldots, h_n\}$, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_i $(i = 1, 2, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a hesitant fuzzy weighted Muirhead mean (HFWMM) operator is a function HFWMM: $A^n \to A$, and*

$$HFWMM^P(h_1, \ldots, h_n) = \left(\frac{1}{n!}\left(\oplus_{\theta \in S_n}\left(\otimes_{j=1}^n\left(w_{\theta(j)}h_{\theta(j)}\right)^{p_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^n p_j}}, \tag{14}$$

where $\theta(j)$ $(j = 1, 2, \ldots, n)$ is any a permutation of $(1, 2, \ldots, n)$ and S_n is the collection of all permutation of $\theta(j)$ $(j = 1, 2, \ldots, n)$.

Theorem 4.1. *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_i $(i = 1, 2, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $HFWMM^P(h_1, \ldots, h_n)$ is still an HFE and*

$$\begin{aligned} &HFWMM^P(h_1, \ldots, h_n) \\ &= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n}\left\{\left(1-\left(\prod_{\theta \in S_n}\left(1-\left(\prod_{j=1}^n\left(1-\left(1-\gamma_{\theta(j)}\right)^{w_{\theta(j)}}\right)^{p_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right\}. \end{aligned} \tag{15}$$

Proof: Since $h_{\theta(j)}$ is an HFE, we have $w_{\theta(j)}h_{\theta(j)}$ is also an HFE. By the operation of HFEs, we have $w_{\theta(j)}h_{\theta(j)} = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n}\left\{1-(1-\gamma_{\theta(j)})^{w_{\theta(j)}}\right\}$. Therefore, we can directly obtain the result according to Theorem 3.1.

Similar to Property 3.3 and Property 3.4, we can prove $HFWMM^P(h_1, \ldots, h_n)$ are bounded, and monotonic.

Property 4.1. (Monotonicity) *Let $h_a = \{h_{a_1}, h_{a_2}, \ldots, h_{a_n}\}$ and $h_b = \{h_{b_1}, h_{b_2}, \ldots, h_{b_n}\}$ be two collections of HFEs, and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector. If for any $\gamma_{a_i} \in h_{a_i}$ and $\gamma_{b_i} \in h_{b_i}$, we have $\gamma_{a_i} \le \gamma_{b_i}$ for any i $(i = 1, 2, \ldots, n)$, then*

$$HFWMM^P(h_{a_1}, h_{a_2}, \ldots, h_{a_n}) \le HFWMM^P(h_{b_1}, h_{b_2}, \ldots, h_{b_n}).$$

Property 4.2. (Boundedness) *Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs, $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector,*

$$h^- = \min_i\left\{h_i^- | h_i^- = \min\{\gamma_i \in h_i\}\right\},$$
$$h^+ = \max_i\left\{h_i^+ | h_i^+ = \max\{\gamma_i \in h_i\}\right\},$$

and then

$$h^- \le HFWMM^P(h_1, \ldots, h_n) \le h^+.$$

Now, we will develop some special cases of HFWMM operator with respect to the parameter vector. Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_i $(i = 1, 2, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and $P = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ be a parameter vector.

(1) If $P = (1, 0, \ldots, 0)$, HFWMM operator will reduce to

$$HFWMM^{(1,0,\ldots,0)}(h_1, \ldots, h_n) = \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ 1 - \prod_{j=1}^{n} (1-\gamma_j)^{\frac{w_j}{n}} \right\}. \tag{16}$$

(2) If $P = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{n-k})$, HFWMM operator will reduce to hesitant fuzzy weighted Maclaurin symmetric mean (HFWMSM) operator

$$HFWMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k})}(h_1, \ldots, h_n)$$
$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} (1-\gamma_{i_j})^{w_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\}. \tag{17}$$

5. **Model for Multiple Attribute Decision Making with Hesitant Fuzzy Information.** As an important extension of fuzzy sets, hesitant fuzzy sets (HFSs) permit the membership degree of an element to a set to be represented as several possible values between 0 and 1, human beings hesitate among a set of membership degrees and they need to represent such a hesitation. There are many decision making problems in which decision makers (DMs) to give their assessments on attributes by not several possible values between 0 and 1 not crisp numbers. In current hesitant aggregations, the interrelationships of the attributes are not considered. However, these interrelationships of attributes should be considered in the process of decision making. To do so, in this section, we develop a novel MADM method with hesitant fuzzy information based on the proposed HFWMM operator. The following assumptions or notations are used to represent the MADM problems for potential evaluation of emerging technology commercialization with hesitant fuzzy information.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of m alternatives, $G = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, and $w = \{w_1, \ldots, w_n\}$ be the weight vector of attributes with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Suppose that $A = (h_{ij})_{m \times n}$ is the decision making matrix, where h_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are in the form of HFEs.

In the following, a novel MADM method is developed with hesitant fuzzy information based on HFWMM operator, which is shown in the following.

Step 1. Construct the hesitant fuzzy decision matrix $H = (h_{ij})_{m \times n}$ according to the decision making information provided by the decision makers. If there are cost attributes in decision making problems, then we need to transform the decision matrix H into a normalization matrix $P = (p_{ij})_{m \times n}$, where

$$P_{ij} = \begin{cases} h_{ij} & \text{for benefit attribute } G_{ij}, \\ h_{ij}^c & \text{for cost attribute } G_{ij}, \end{cases}$$

where $h_{ij}^c = \bigcup_{\gamma_{ij} \in h_{ij}} \{1 - \gamma_{ij}\}$.

Step 2. Aggregate all assessment values h_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ of the alternative A_i $(i = 1, 2, \ldots, m)$ on all attributes G_j $(j = 1, 2, \ldots, n)$ into the overall assessment h_i $(i = 1, 2, \ldots, m)$ based on the

$$HFWMM^P(h_1, \ldots, h_n)$$

$$= \bigcup_{\gamma_i \in h_i, i=1,2,\ldots,n} \left\{ \left(1 - \left(\prod_{\theta \in S_n} \left(1 - \left(\prod_{j=1}^{n} \left(1 - \left(1 - \gamma_{\theta(j)}\right)^{w_{\theta(j)}}\right)^{p_j} \right) \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \right\}. \quad (18)$$

Step 3. Calculate the score values $S(h_i)$ of all collective overall values to rank the all alternatives A_i $(i = 1, 2, \ldots, m)$, the bigger the $S(a_i)$ is, the better the A_i is.

Step 4. Rank all alternatives A_i $(i = 1, 2, \ldots, m)$ and determine the desirable alternative according to $S(h_i)$.

Step 5. End.

6. Numerical Example and Comparative Analysis.

6.1. Numerical example.

In this section, we will use HFWMM operator to show the applications of the proposed approach in Section 5.

Example 6.1. *This illustrative example is cited and adapted from* [25]*, which is an evaluation on the emergency response capabilities of relevant department when some disasters occur. There is a panel with five emerging departments A_i $(i = 1, 2, 3, 4, 5)$ that should be considered that have taken part in the rescue work. A_1 is the transportation department, A_2 is the health departments, A_3 is the telecommunications department, A_4 is the supplies department and A_5 is the other departments except the above four departments. The government needs to give an evaluation according to four attributes: (1) G_1 is the emergency forecasting capability; (2) G_2 is the emergency process capability; (3) G_3 is the after-disaster loss evaluation capability; (4) G_4 is the after-disaster reconstruction capability, and $w = (0.1, 0.4, 0.2, 0.3)$ is the weight vector of them. The five possible alternatives $\{A_1, A_2, A_3, A_4, A_5\}$ are evaluated by using the hesitant fuzzy information, and the hesitant fuzzy linguistic decision matrix $A = (a_{ij})_{4\times5}$ is shown in Table 1.*

Table 1. Hesitant fuzzy decision matrix

	G_1	G_2	G_3	G_4
A_1	$\{0.6, 0.8\}$	$\{0.2, 0.6, 0.8\}$	$\{0.6\}$	$\{0.4, 0.5\}$
A_2	$\{0.4, 0.7, 0.9\}$	$\{0.2, 0.4\}$	$\{0.6, 0.9\}$	$\{0.5\}$
A_3	$\{0.5\}$	$\{0.7, 0.8\}$	$\{0.3, 0.5, 0.7\}$	$\{0.5, 0.7\}$
A_4	$\{0.4, 0.5, 0.6\}$	$\{0.1, 0.3\}$	$\{0.4, 0.9\}$	$\{0.3\}$
A_5	$\{0.4, 0.7\}$	$\{0.2, 0.3\}$	$\{0.8\}$	$\{0.3, 0.4, 0.8\}$

Now, we utilize the proposed method based on HFWMM operator to drive the collective overall value when parameter $P = (1, 1, 1, 1)$ (P reflects the interrelationships of the four attributes. We take $P = (1, 1, 1, 1)$ just as an example to show the proposed MADM method. Of course, P can take any real vector, and the influence of the parameter vector P on the decision making results will be discussed in Section 6.2), we obtain the following.

Step 1. *Consider all attributes G_j $(j = 1, 2, 3, 4)$ are the benefit attributes; therefore, the attribute values of the alternatives do not need to be normalized.*

Step 2. *Based on Equation (18),*

$$h_i = HFWMM^P(h_{i1}, \ldots, h_{i5})$$

$$= \bigcup_{\gamma_i \in h_{ij}, i=1,2,\ldots,4} \left\{ \left(1 - \left(\prod_{\theta \in S_4} \left(1 - \left(\prod_{j=1}^{5} \left(1 - \left(1 - \gamma_{\theta(j)}\right)^{w_{\theta(j)}}\right)^{p_j} \right) \right) \right)^{\frac{1}{4!}} \right)^{\frac{1}{\sum_{j=1}^{5} p_j}} \right\}.$$

we have

$$h_1 = \{0.1155, 0.1238, 0.1590, 0.1705, 0.1773, 0.1901, 0.1318, 0.1413, 0.1815, 0.1946, 0.2024, 0.2170\};$$
$$h_2 = \{0.1075, 0.1310, 0.1304, 0.1589, 0.1321, 0.1610, 0.1602, 0.1952, 0.1533, 0.1868, 0.1859, 0.2265\};$$
$$h_3 = \{0.1349, 0.1520, 0.1579, 0.1780, 0.1791, 0.2019, 0.1424, 0.1605, 0.1667, 0.1879, 0.1890, 0.2131\};$$
$$h_4 = \{0.0671, 0.0937, 0.0899, 0.1255, 0.0722, 0.1009, 0.0968, 0.1351, 0.0772, 0.1079, 0.1035, 0.1445\};$$
$$h_5 = \{0.1044, 0.1136, 0.1455, 0.1166, 0.1269, 0.1625, 0.1282, 0.1395, 0.1787, 0.1433, 0.1558, 0.1997\}.$$

Step 3. *We utilize the score function to calculate the score values of collective overall assessment values a_i $(i = 1, 2, 3, 4)$,*

$$S(h_1) = 0.1671,\ S(h_2) = 0.1607,\ S(h_3) = 0.1720,\ S(h_4) = 0.1012,\ S(h_5) = 0.1429.$$

Step 4. *According to the score values of h_i $(i = 1, 2, 3, 4, 5)$ calculated in Step 3, all feasible alternative A_i $(i = 1, 2, 3, 4, 5)$ are ranked as follows:*

$$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3.$$

Therefore, the desirable alternative is A_3.

Example 6.2. *This example is adopted from* [26]. *The following practical example involves a supplier selection problem in a supply chain. The authorized decision makers in a small enterprise attempt to reduce the supply chain risk and uncertainty to improve customer service, inventory levels, and cycle times, which results in increased competitiveness and profitability. The decision makers consider various criteria involving (i) C_1: performance (e.g., delivery, quality, price); (ii) C_2: technology (e.g., manufacturing capability, design capability, ability to cope with technology changes); (iii) C_3: organizational culture and strategy (e.g., feeling of trust, internal and external integration of suppliers, compatibility across levels and functions of the buyer and supplier), and $w = (0.3, 0.5, 0.2)$ is the weight vector of them. Using the supplier rating system, the decision maker evaluates three suppliers: S_1, S_2 and S_3. and the hesitant fuzzy linguistic decision matrix $A = (a_{ij})_{3\times3}$ is shown in Table 2.*

Table 2. Hesitant fuzzy decision matrix

	C_1	C_2	C_3
S_1	$\{0.5\}$	$\{0.3, 0.4\}$	$\{0.6\}$
S_2	$\{0.7, 0.9\}$	$\{0.8\}$	$\{0.5, 0.6\}$
S_3	$\{0.3, 0.4\}$	$\{0.4, 0.5\}$	$\{0.8\}$

Now, we utilize the proposed method based on HFWMM operator to drive the collective overall value when parameter $P = (1, 1, 1)$ (of course, the P can take any real vector), and we obtain the following.

Step 1. *Consider all attributes G_j $(j = 1, 2, 3, 4)$ are the benefit attributes; therefore, the attribute values of the alternatives do not need to be normalized.*

Step 2. *Based on Equation (18), we have*

$$h_1 = \{0.1725, 0.1921\};$$

$$h_2 = \{0.2789, 0.3039, 0.3292, 0.3587\};$$
$$h_3 = \{0.1846, 0.2015, 0.2066, 0.2254\}.$$

Step 3. *We utilize the score function to calculate the score values of collective overall assessment values a_i $(i = 1, 2, 3)$,*

$$S(h_1) = 0.1823, \quad S(h_2) = 0.3177, \quad S(h_3) = 0.2024.$$

Step 4. *According to the score values of h_i $(i = 1, 2, 3)$ calculated in Step 3, all feasible alternative S_i $(i = 1, 2, 3)$ are ranked as follows:*

$$A_1 \prec A_3 \prec A_2.$$

Therefore, the desirable alternative is A_2.

6.2. **The influence of the parameter vector P on the decision making results.** In order to show the influence of the parameter vector P on the decision making results, we use different parameter vector P in our proposed method based on HFWMM operator to rank the alternatives in Example 6.1. The ranking results are shown in Table 3.

TABLE 3. Ranking results by using different parameter vector P in HFWMM operator

Parameter vector P	The score values of A_i $(i = 1, 2, 3, 4)$	Ranking results
$(1, 0, 0, 0)$	$S(h_1) = 0.1914$, $S(h_2) = 0.1827$, $S(h_3) = 0.2337$, $S(h_4) = 0.1278$, $S(h_5) = 0.1746$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$(1, 1, 0, 0)$	$S(h_1) = 0.1780$, $S(h_2) = 0.1730$, $S(h_3) = 0.2051$, $S(h_4) = 0.1130$, $S(h_5) = 0.1603$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$(1, 1, 1, 0)$	$S(h_1) = 0.1720$, $S(h_2) = 0.1669$, $S(h_3) = 0.1874$, $S(h_4) = 0.1063$, $S(h_5) = 0.1510$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$(1, 1, 1, 1)$	$S(h_1) = 0.1671$, $S(h_2) = 0.1607$, $S(h_3) = 0.1720$, $S(h_4) = 0.1012$, $S(h_5) = 0.1429$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$S(h_1) = 0.1671$, $S(h_2) = 0.1607$, $S(h_3) = 0.1720$, $S(h_4) = 0.1012$, $S(h_5) = 0.1429$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$(2, 0, 0, 0)$	$S(h_1) = 0.2079$, $S(h_2) = 0.1962$, $S(h_3) = 0.2674$, $S(h_4) = 0.31484$, $S(h_5) = 0.1951$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
$(3, 0, 0, 0)$	$S(h_1) = 0.2244$, $S(h_2) = 0.2086$, $S(h_3) = 0.2951$, $S(h_4) = 0.1663$, $S(h_5) = 0.2137$	$A_4 \prec A_2 \prec A_5 \prec A_1 \prec A_3$

We explain the following aspects to illustrate the influence of parameter vector P on the decision making results.

(1) We see from Section 3 that our method is more general. Specially, when $P = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, 0, \ldots, 0}^{k})$, the HFWMM operator will become hesitant fuzzy weighted Maclaurin mean, which is also family aggregation operators when the parameter k takes different values.

(2) Parameter vector P can capture interrelationship between the individual arguments that can be fully taken into account. As far as the HFWMM operator is concerned, we can find from Table 3 that the more interrelationships of attributes which we consider, the smaller value of score functions, that is, the parameter vector P have greater control ability, the values of score function will become greater. So, different parameter vector P can be regarded as the decision makers' risk preference.

6.3. **Comparisons and analyses.** In order to verify the effectiveness of the proposed methods with HFWMM operator, we compare our proposed method with other existing methods including the HFWA operator, HFGA operator and HFMSM operator. The results are shown in Table 4, which indicates that four methods have the same desirable alternative, which further verifies the validity of the method proposed in this paper with HFWMM operator.

TABLE 4. Ranking results by using different methods

Aggregation operator	Parameter vector	Ranking results
HFWA	No	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
HFGA	No	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
HFMSM	$(1,1,1,0)$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$
HFWMM in this paper	$(1,1,1,1)$	$A_4 \prec A_5 \prec A_2 \prec A_1 \prec A_3$

In the following, we will give some comparisons of the three methods and our proposed methods with respect to some characteristics, which are listed in Table 5.

TABLE 5. Comparisons of different methods

Methods	Capture interrelationship of MAs	Make method flexible by PV
HFWA	$\times$	$\times$
HFGA	$\times$	$\times$
HFMSM	$\surd$	$\surd$
HFWMM in this paper	$\surd$	$\surd$

where MA means multiple attributes and PV means parameter vector.

HFWA and HFGA are special cases of HFWMM operator. Compared with the methods based on the HFWA operator and HFGA operator, there are two limitations: (1) the method based on HFWA and HFGA operator thinks that the input arguments are independent; (2) the method based on HFWA and HFGA operator does not consider the interrelationship among input arguments. However, the new proposed operator in this paper can also consider the interrelationship among all input arguments and it is also a generalization of most existing aggregation operators. Therefore, the proposed method is more general and flexible to solve MADM problems than HFWA and HFGA. Therefore, we extend HFMSM to HFWSMM which is special cases of *HFWMM* operators when parameter vector $P = (\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{k})$. Thus, the new method proposed in this paper can make the hesitant fuzzy information aggregation process more flexible by the parameter vector P.

7. **Conclusions.** In recent years, aggregation operators play a vital role in decision making and many aggregation operators under different environment have been developed. However, they still have some limitations in solving some practical problems. Some traditional Maclaurin symmetric mean (MSM) operator fails in dealing with the hesitant fuzzy information. In this paper, we have investigated the MADM problems with the hesitant fuzzy information based on new aggregation operator which can capture interrelationships of multiple attributes among any number of attributes by a parameter vector P. To begin with, we presented some new hesitant fuzzy MM aggregation operators to deal with MADM problems with hesitant fuzzy information, including the hesitant fuzzy Muirhead mean (HFMM) operator, the hesitant fuzzy weighted Muirhead mean (HFWMM) operator. In addition, some properties of these new aggregation operators were proved and some special cases were discussed. Moreover, we presented a new method to solve the MADM problems with hesitant fuzzy information. Finally, we used an illustrative example to show the feasibility and validity of the new methods by comparing with the other existing methods.

In further research, it is necessary to solve the real decision making problems by applying these operators. In addition, we can develop some new aggregation operators on the basis of Muirhead mean operator by considering that MM operator has the superiority of compatibility.

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