International Journal of Innovative Computing, Information and Control
Volume **14**, Number **5**, October **2018**
ICIC International ©2018 ISSN 1349-4198
pp. 1617–1631

FINITE-TIME STATE TRACKING CONTROL WITH UNMEASURED STATE AND VARIOUS BOUNDARIES

Dongsheng Chen, Zhiqiang Wang and Jian Li*

School of Automation Engineering
Northeast Electric Power University
No. 169, Changchun Road, Jilin 132012, P. R. China
*Corresponding author: lijian@neepu.edu.cn

Received December 2017; revised April 2018

Abstract. *This paper investigates the problem of observer-based model reference state tracking control over a finite-time interval. Matched and unmatched cases respectively are considered as part of the system. In order to attain the objective of state tracking for an unmeasured state over a finite-time interval, both a state observer and a controller are designed for the system under consideration. Additionally, a switching law is designed via the state tracking error, with the considered tracking error being finite-time bounded and the considered system achieving finite-time weighted H_∞ performance, with average dwell-time. The conservatism of the considered system is investigated, by defining the variable boundaries. Finally, a numerical example is provided as a means of illustrating the effective design method.*

Keywords: State tracking, Finite-time bounded, Average dwell-time, Switched control, Conservative analysis

1. **Introduction.** Model reference control (MRC) has been confirmed as an effective mechanism for engineering systems [1-8]. In [9], an optimal linear quadratic Gaussian controller is devised, according to an internal reference model. Both reference signal and system error are evaluated during the control design. As a means of obtaining the precise tracking performance in [10], a time-varying model-based controller is assessed according to the model principle. A two-stage controller is presented in [11], with the study proposing a control scheme based on the model-based networked predictive output tracking control approach, which enables tracking performance. In [12], a novel output tracking controller is designed according to a specified reference model, which guarantees the tracking error dynamics as robustly stable. Nevertheless, the MRC system can be regarded as a switched system, comprising of both matched and unmatched subsystems. Through this means, the traditional MRC issue can be transformed, achieving the stability offered by switched systems. Based on the multiple Lyapunov function approach of [13,14], an appropriate switching law and a state feedback controller are constructed in order to obtain the desired disturbance attenuation and global stabilization necessary for tracking control. In [15], a priori awareness of the maximum asynchronous duration is assessed. Utilizing the method of average dwell-time (ADT), the state error is converged up to a stated threshold by the proposed laws and switching signals with uncertainties in [16]. In order to match the distinct reference models in [17], controllers are switched through adopting the switched control approach.

Contemporarily, finite-time state tracking has attracted increasing attention in terms of the MRC field. Nevertheless, during various practical applications [18-20], finite-time

DOI: 10.24507/ijicic.14.05.1617

stability (FTS) and finite-time bounded (FTB) have been desirable properties; therefore, one may feel concerned with regard to system trajectories over a fixed time interval. As a means of estimating the reference trajectory in [21], finite-time estimators are constructed, while the finite-time convergence property is analyzed. Applying the method of finite-time convergent observer design, in addition to finite-time output feedback control design, the problem of finite-time output feedback trajectory tracking control is investigated in [22]. In [23], the finite-time attitude control law is proposed, stating that the requisite attitude can be monitored for a rigid body over a finite-time interval. Additionally, the issue of finite-time output tracking control is assessed in relation to matched and unmatched disturbances for n-order multi-agent systems. Furthermore, a finite-time disturbance observer is proposed in [24], resulting in the attainment of finite-time output consensus tracking, regardless of all the disturbances.

It is important to observe that not all states can be measured precisely in engineering systems [25-27]. In [28], an extended state observer is designed as a means of ensuring the tracking error within a given boundary, while stability is analyzed via the Lyapunov theory. In [29], a switched fuzzy state observer is devised as a means of estimating the unmeasured states, while the tracking errors are confined to all times within the prescribed parameters. Through applying the Lyapunov-Krasovskii functional approach as in [30], the Luenberger-type observer is devised for state estimation, while the observer-based controller is developed further. Utilizing a set of linear matrix inequalities in [31], a state observer is developed with bounded unknown inputs and measured disturbances, which is guaranteed to produce a precisely asymptotic state. It should be noted that the scheme of observer-based adaptive iterative learning control is proposed for nonlinear systems with unknown time-varying parameters and delays [32].

Motivated by the above literature, the issue of finite-time MRC for state tracking control over a finite-time interval is investigated through the switched control method. The contribution of this paper can be outlined as follows. Firstly, a state observer has been proposed to resolve the state-unmeasured cases. Secondly, sufficient conditions for the design of a switching signal and a controller are established to guarantee the tracking error is finite-time bounded, while the considered system attained the finite time weighted H_∞ performance. Meanwhile, both matched and unmatched cases are considered within the system, by the error Lyapunov-like function. Furthermore, the conservatism of the considered system is analyzed with various parameters.

The paper is organized as follows. Section 2 introduces the state observer and the considered systems. In addition, the relative controller and switching signal are also given in this section. Section 3 demonstrates the property of finite-time tracking error bounded via average dwell-time approach. Meanwhile, the finite-time weighted H_∞ performance has achieved. An example is given to show the feasibility of the designed approach of state observer, controller and switching signal in Section 4. It is noted that the conservatism of the considered system has studied with different boundaries in this section. Some conclusions of this paper are given in the last section.

Notation: For a matrix A, A^T denotes its transpose. For a symmetric matrix, $A > 0$ ($A \geq 0$) and $A < 0$ ($A \leq 0$) denote positive-definiteness (positive semi-definite matrix) and negative-definiteness (negative semi-definite matrix), respectively. $\lambda(A)$ is used to define the eigenvalues of the matrix A. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional Euclidean space and the set of all real $m \times n$ matrices. $\|A\|$ denotes the absolute value of the matrix A.

2. **Problem Formulation.** Consider the following continuous-time system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_{\sigma(t)}(t) + Ew(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $y(t) \in \mathbb{R}$ is the output, $w(t)$ is the exogenous disturbance, $u_{\sigma(t)}(t) \in \mathbb{R}$ is the control input and the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $E \in \mathbb{R}^{n\times 1}$ are constant matrices.

The control objective for the system (1) is to make the state of $x(t)$ track the reference state $x_m(t)$ exactly over a finite-time interval, which is generated from the reference model:

$$\dot{x}_m(t) = A_m x(t) + B_m u_m(t), \tag{2}$$

where $A_m \in \mathbb{R}^{n\times n}$ is a constant Hurwitz matrix, $B_m \in \mathbb{R}^{n\times 1}$ is a constant matrix, $u_m(t) \in \mathbb{R}$ is a bounded input signal and $x_m(t)$ is the desirable state for $x(t)$ to track.

In some engineering environment, the state of $x(t)$ is difficult to measure, such that the state observer is required. Consequently, a state observer can be obtained as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{\sigma(t)}(t) + G(y - C\hat{x}(t)), \tag{3}$$

where $\hat{x}(t)$ denotes the estimate of $x(t)$, G is the gain matrix of the state observer with appropriate dimension. Therefore, the observer error can be defined:

$$\bar{e}(t) = x(t) - \hat{x}(t) \tag{4}$$

and

$$\begin{aligned} \dot{\bar{e}}(t) &= Ax(t) + Bu_{\sigma(t)}(t) + Ew(t) - (A\hat{x}(t) + Bu_{\sigma(t)}(t) + G(y - C\hat{x}(t))) \\ &= A\bar{e}(t) + Ew(t) - GC\bar{e}(t) = (A - GC)\bar{e}(t) + Ew(t). \end{aligned} \tag{5}$$

Similarly, the tracking error is defined as follows:

$$\tilde{e}(t) = \hat{x}(t) - x_m(t) \tag{6}$$

and

$$\begin{aligned} \dot{\tilde{e}}(t) &= A\hat{x}(t) + Bu_{\sigma(t)}(t) + GC\bar{e}(t) - (A_m x_m(t) + B_m u_m(t)) \\ &= A_m\tilde{e}(t) + (A - A_m)\hat{x}(t) + GC\bar{e}(t) + Bu_{\sigma(t)}(t) - B_m u_m(t). \end{aligned} \tag{7}$$

In order to realize the state tracking, we propose the controller $u_{\sigma(t)}(t)$ as follows:

$$u_{\sigma(t)}(t) = k_1\hat{x}(t) + k_2 u_m(t) + k_3(t), \tag{8}$$

and the parameters of the controller are determined by the following equations:

$$k_1 = MP(A - A_m), \quad k_2 = NB_m, \quad k_3(t) = -\zeta GCE\sqrt{d_w}\; t,$$

where $M \in \mathbb{R}^{1\times n}$ and $N \in \mathbb{R}^{1\times n}$ are the adjust matrices for the controller $u_{\sigma(t)}(t)$, ζ is the adjust matrix and $\sqrt{d_w}$ is the upper bound of the disturbance. The positive definite matrices P and Q satisfy the following inequality

$$A_m^T P + PA_m \le -Q.$$

Before giving our results, we define two kinds of working cases: matched and unmatched cases. Given a threshold value ξ, if the state error exceeds the threshold ξ, we define system (1) is running in the unmatched cases. On the contrary, if the state error is within the threshold ξ, we define system (1) is running in the matched cases, respectively, as follows:

$$\begin{cases} \|\hat{x}(t) - x_m(t)\| \le \xi, & \text{matched cases} \\ \|\hat{x}(t) - x_m(t)\| > \xi, & \text{unmatched cases} \end{cases}$$

Thus, we have designed an appropriate switching signal to solve the problem of state tracking for matched and unmatched cases, as follows:

$$\begin{cases} \|\hat{x}(t) - x_m(t)\| \leq \vartheta, & \text{low level signal} \\ \|\hat{x}(t) - x_m(t)\| > \vartheta, & \text{high level signal} \end{cases}$$

and ϑ is a tolerance of state tracking for system (1).

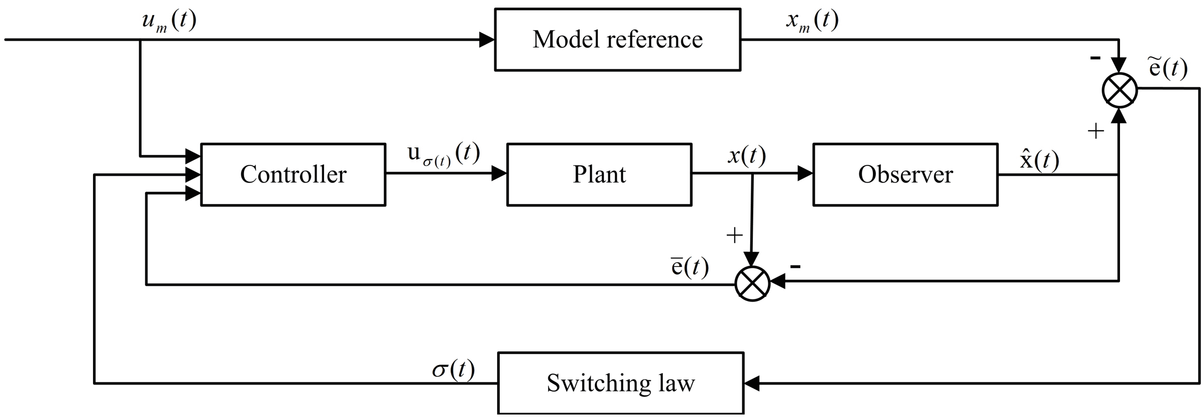

Figure 1. Structure of observer-based MRC control

Remark 2.1. *Based on the above definition, system (1) can be seen as a switched system during the runtime which contains matched and unmatched cases. In order to describe the matched and unmatched cases over a finite-time interval* $T_f[t_0, T_f]$*, we denote* $T_\uparrow[0, t]$ *and* $T_\downarrow[0, t]$ *to stand for the runtime of matched and unmatched cases, respectively.*

Remark 2.2. *Given a positive constant* η *to express the relationship between them. If* $T_{\downarrow[t_0,T_f]} \geq \eta(T_f - t_0)$ *holds for any* $t \in T_f[t_0, T_f]$*, we define that the switching signal* $\sigma(t)$ *has the maximum ratio between the runtime of matched and unmatched cases. And the constant* η *with this property is called MRRT for simplicity.*

Remark 2.3. *In this paper, high level signal and low level signal denote that the controller* $u_{\sigma(t)}(t)$ *works or does not work. Meanwhile, they are expressed by 1 and 0 in Figure 2, respectively.*

Assumption 2.1. *The external disturbance* $w(t)$ *is bounded and satisfies*

$$\int_0^{T_f} w^T(t)w(t)dt \leq d_w, \ d_w \geq 0,$$

where T_f *is the upper bound of a finite-time interval.*

Remark 2.4. *Assumption 2.1 is a standard assumption for studying finite-time boundedness problem, which is presented in [33-35].*

Definition 2.1. [36]*: For a switching signal* $\sigma(t)$ *and each* $t_2 \geq t_1 \geq 0$*, let* $N_{\sigma(t)}(t_2, t_1)$ *denote the number of discontinuities of* $\sigma(t)$ *in the open interval* (t_1, t_2)*. If there exist two positive numbers* N_0 *and* τ_a *that satisfy*

$$N_{\sigma(t)}(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a}, \quad \forall t_2 \geq t_1 \geq 0,$$

then, τ_a *is called average dwell-time and* N_0 *is called a chatter bound.*

Definition 2.2. [34]*: Given a positive definite matrix R, positive constants t_0, T_f, c_1, c_2, and $c_1 < c_2$. Consider a finite interval $[t_0, T_f]$ and a certain switching signal $\sigma(t)$. If $\forall w(t)$ satisfies Assumption 2.1 and*

$$x^T(t_0)Rx(t_0) < c_1 \Rightarrow x^T(t)Rx(t) < c_2, \quad \forall t \in [t_0, T_f],$$

the continuous-time linear system (1) is said to be finite-time bounded with respect to $(c_1, c_2, T_f, d_w, R, \sigma)$.

Remark 2.5. *This paper investigates the state tracking of MRC in a finite-time interval. Tracking error reflects the dynamic performance of state tracking directly. Therefore, we will employ $\tilde{e}^T(t)R\tilde{e}(t)$ rather than $x^T(t)Rx(t)$ in the following researches, particularly.*

Lemma 2.1. [37]*: If there exist functions $\phi(t)$ and $\nu(t)$ satisfying*

$$\dot{\phi}(t) \leq -\xi\phi(t) + \kappa\nu(t)$$

then

$$\phi(t) \leq e^{-\xi(t-t_0)}\phi(t_0) + \kappa \int_0^{t-t_0} e^{-\xi(t-\tau)}\nu(\tau)d\tau.$$

We assume that the trajectory $x(t)$ is continuous at everywhere, that is, the state of switched system does not jump at switching instants and the switching signal $\sigma(t)$ has finite switching number in finite interval time. Meanwhile, t_0 and x_0 represent the initial time and initial state respectively.

3. Main Results.

In this section, we study the tracking error finite-time bounded for the MRC problem, and then we will focus on the finite-time weighted H_∞ performance for the linear system (1).

3.1. Finite-time error bounded for the MRC problem.

The bounded property for the error of state tracking over a finite-time interval is a precondition in MRC problem. Hence, we will analyze the finite-time tracking error bounded property of the continuous-time system (1) in this section.

Theorem 3.1. *For given constants $\alpha > 0$, $\beta > 0$, $P > 0$, $\eta > 0$, $\mu > 1$, $c_2 > c_1 > 0$, $\tau_a > 0$, $N_0 \geq 0$ and a finite-time interval $[t_0, T_f]$. For $\forall t \in [t_0, T_f]$, if the following inequalities hold*

$$T_{\downarrow[t_0,T_f]} \geq \eta(T_f - t_0), \tag{9}$$

$$Re\ \lambda(A - GC) < 0, \tag{10}$$

and

$$\lambda_1 = \max\left(\lambda\left(\widetilde{P}\right)\right) = \max\left(\lambda\left(R^{-\frac{1}{2}}PR^{-\frac{1}{2}}\right)\right),$$

$$\lambda_2 = \min\left(\lambda\left(\widetilde{P}\right)\right) = \min\left(\lambda\left(R^{-\frac{1}{2}}PR^{-\frac{1}{2}}\right)\right),$$

then, for any switching signal σ with average dwell-time satisfying

$$\tau_f > \tau_f^* = (T_f - t_0)\ln\mu/F^*, \tag{11}$$

the tracking error $\tilde{e}(t)$ in (6) is finite-time bounded with respect to $(c_1, c_2, T_f, d_w, R, \sigma)$, where

$$F^* = (\eta(\alpha+\beta) - \beta)(T_f - t_0) - \ln(\lambda_1 c_1) + \ln\left(\lambda_2 c_2 + \int_{t_0}^{T_f} y^T(\tau)y(\tau)d\tau - \gamma^2 d_w\right).$$

Proof: Firstly, choose an error Lyapunov-like function as follows

$$V(t)=V(\tilde{e}(t))=\tilde{e}^T(t)P\tilde{e}(t).$$

a) Error Lyapunov-like function $V(\tilde{e}(t))=\tilde{e}^T(t)P\tilde{e}(t)$ is continuous and its derivative satisfies

$$\dot{V}\left(\tilde{e}(t)\right)\le\begin{cases}-\alpha V\left(\tilde{e}(t)\right)+J, & \forall t\in T_{\downarrow}[t_0,T_f)\\ \beta V\left(\tilde{e}(t)\right)+J, & \forall t\in T_{\uparrow}[t_0,T_f)\end{cases},\tag{12}$$

where $J(t)\triangleq -y(t)^Ty(t)+\gamma^2w(t)^Tw(t)$.

b) $V\left(\tilde{e}(p)\right)\le\mu V\left(\tilde{e}(q)\right)$ holds on all the switching point t_i, $\forall\left(\sigma(t_i)=p,\sigma(t_i^-)=q\right)\in\mathcal{Q}\times\mathcal{Q}$, $p\neq q$, $i\in\mathbb{Z}^+$.

At the initial instant $t=t_0$, we will get

$$V\left(\tilde{e}(t_0)\right)=\tilde{e}^T(t_0)P\tilde{e}(t_0)\le\lambda_{\max}\left(\tilde{P}\right)\tilde{e}^T(t_0)R\tilde{e}(t_0)\le\lambda_1c_1.\tag{13}$$

Meanwhile, $\forall t\in[t_0,T_f]$, one has

$$V\left(\tilde{e}(t)\right)=\tilde{e}^T(t)P\tilde{e}(t)\ge\lambda_{\min}\left(\tilde{P}\right)\tilde{e}^T(t)R\tilde{e}(t)=\lambda_2\tilde{e}^T(t)R\tilde{e}(t).\tag{14}$$

Thus, we can obtain

$$\tilde{e}^T(t)R\tilde{e}(t)\le\frac{V\left(\tilde{e}(t)\right)}{\lambda_2}.\tag{15}$$

According to Lemma 2.1 and (12), we achieve

$$V(\tilde{e}(t))\le\begin{cases}e^{-\alpha(t-t_k)}V\left(\tilde{e}(t_k)\right)+\Gamma_1(t), & \forall t\in T_{\downarrow}[t_k,T_f],\\ e^{\beta(t-t_k)}V\left(\tilde{e}(t_k)\right)+\Gamma_2(t), & \forall t\in T_{\uparrow}[t_k,T_f],\end{cases}\tag{16}$$

where

$$\Gamma_1(t)\triangleq\int_{t_k}^{t}e^{-\alpha(t-\tau)}J(\tau)d\tau,\quad \Gamma_2(t)\triangleq\int_{t_k}^{t}e^{\beta(t-\tau)}J(\tau)d\tau.$$

Therefore, according to the property b and inequality (16), $\forall t\in[t_0,T_f]$, we get

$$\begin{aligned}V(\tilde{e}(t))&\le e^{-\alpha T_{\downarrow[t_k,t]}+\beta T_{\uparrow[t_k,t]}}V(\tilde{e}(t_k))+\int_{t_k}^{t}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\le\mu e^{-\alpha T_{\downarrow[t_k,t]}+\beta T_{\uparrow[t_k,t]}}V(\tilde{e}(t_k^-))+\int_{t_k}^{t}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\le\mu e^{-\alpha T_{\downarrow[t_{k-1},t]}+\beta T_{\uparrow[t_{k-1},t]}}V(\tilde{e}(t_{k-1}))+\int_{t_{k-1}}^{t_k}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\quad+\int_{t_k}^{t}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\le\mu^2e^{-\alpha T_{\downarrow[t_{k-2},t]}+\beta T_{\uparrow[t_{k-2},t]}}V\left(\tilde{e}(t_{k-2})\right)+\mu^2\int_{t_{k-2}}^{t_{k-1}}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\quad+\mu\int_{t_{k-1}}^{t_k}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau+\int_{t_k}^{t}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\le\mu^{N_{[t_0,t]}}e^{-\alpha T_{\downarrow[t_0,t]}+\beta T_{\uparrow[t_0,t]}}V(\tilde{e}(t_0))+\mu^{N_{[t_0,t]}}\int_{t_0}^{t_1}e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau\\
&\quad+\cdots\end{aligned}$$

$$+\mu^2\int_{t_{k-2}}^{t_{k-1}} e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau+\mu\int_{t_{k-1}}^{t_k} e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau$$
$$+\int_{t_k}^{t} e^{-\alpha T_{\downarrow[\tau,t]}+\beta T_{\uparrow[\tau,t]}}J(\tau)d\tau.$$

As $t=T_f$, we have

$$\begin{aligned}
&V(\tilde{e}(t))\\
&\le \mu^{N_{[t_0,T_f]}}e^{-\alpha T_{\downarrow[t_0,T_f]}+\beta T_{\uparrow[t_0,T_f]}}V(\tilde{e}(t_0))+\mu^{N_{[t_0,T_f]}}\int_{t_0}^{t_1} e^{-\alpha T_{\downarrow[\tau,T_f]}+\beta T_{\uparrow[\tau,T_f]}}J(\tau)d\tau\\
&\quad+\cdots\\
&\quad+\mu^2\int_{t_{k-2}}^{t_{k-1}} e^{-\alpha T_{\downarrow[\tau,T_f]}+\beta T_{\uparrow[\tau,T_f]}}J(\tau)d\tau\\
&\quad+\mu\int_{t_{k-1}}^{t_k} e^{-\alpha T_{\downarrow[\tau,T_f]}+\beta T_{\uparrow[\tau,T_f]}}J(\tau)d\tau+\int_{t_k}^{t} e^{-\alpha T_{\downarrow[\tau,T_f]}+\beta T_{\uparrow[\tau,T_f]}}J(\tau)d\tau\\
&\le \mu^{N_0+\frac{T_f-t_0}{\tau_f}}e^{-\alpha T_{\downarrow[t_0,T_f]}+\beta T_{\uparrow[t_0,T_f]}}V(\tilde{e}(t_0))+\int_{t_0}^{T_f} e^{-\alpha T_{\downarrow[\tau,T_f]}+\beta T_{\uparrow[\tau,T_f]}}\mu^{N_{[\tau,T_f]}}J(\tau)d\tau. \qquad (17)
\end{aligned}$$

It should be noted that

$$\begin{aligned}
\tilde{e}(t)&=e^{A_mT_f}\tilde{e}(t_0)+\int_{t_0}^{T_f} e^{A_m(T_f-\tau)}\left[(A-A_m)\hat{x}(\tau)+GC\bar{e}(\tau)+Bu_{\sigma(\tau)}(\tau)-B_mu(\tau)\right]d\tau\\
&=e^{A_mT_f}\tilde{e}(t_0)+\int_{t_0}^{T_f} e^{A_m(T_f-\tau)}\Bigg[(A-A_m)\hat{x}(\tau)+GC\Bigg(e^{(A-GC)T_f}\bar{e}(t_0)\\
&\quad+Bu_{\sigma(\tau)}(\tau)-B_mu(\tau)+\int_{t_0}^{T_f} e^{(A-GC)(T_f-\tau)}Ew(\tau)d\tau\Bigg)\Bigg]d\tau.
\end{aligned}$$

As the initial conditions $\tilde{e}(t_0)=0$ and $\bar{e}(t_0)=0$, we can obtain that

$$\begin{aligned}
\tilde{e}(t)=&\int_{t_0}^{T_f} e^{A_m(T_f-\tau)}\Bigg[(A-A_m)\hat{x}(\tau)+GC\int_{t_0}^{T_f} e^{(A-GC)(T_f-\tau)}Ew(\tau)d\tau\\
&+Bu_{\sigma(\tau)}(\tau)-B_mu(\tau)\Bigg]d\tau. \qquad (18)
\end{aligned}$$

Thus, if matrix $(A-GC)$ satisfies (10), then we can design an appropriate controller $u_{\sigma(t)}(t)$ to satisfy the tracking performance.

Combined with (9), then the inequality (17) can be rewritten as

$$\begin{aligned}
V(\tilde{e}(t))&\le \mu^{N_0+\frac{T_f-t_0}{\tau_f}}e^{(\beta-\eta(\beta+\alpha))(T_f-t_0)}V(\tilde{e}(t_0))+\int_{t_0}^{T_f} -y^T(\tau)y(\tau)+\gamma^2w^T(\tau)w(\tau)d\tau\\
&\le \mu^{N_0+\frac{T_f-t_0}{\tau_f}}e^{(\beta-\eta(\beta+\alpha))(T_f-t_0)}V\left(\tilde{e}(t_0)\right)-\int_{t_0}^{T_f} y^T(\tau)y(\tau)d\tau+\gamma^2d_w. \qquad (19)
\end{aligned}$$

Thus, we can obtain that

$$\begin{aligned}
\tilde{e}^T(t)R\tilde{e}(t)&\le \left(\mu^{N_0+\frac{T_f-t_0}{\tau_f}}e^{(\beta-\eta(\beta+\alpha))(T_f-t_0)}V\left(\tilde{e}(t_0)\right)-\int_{t_0}^{T_f} y^T(\tau)y(\tau)d\tau+\gamma^2d_w\right)\Big/\lambda_2\\
&<\left(\mu^{N_0+\frac{T_f-t_0}{\tau_f}}e^{(\beta-\eta(\beta+\alpha))(T_f-t_0)}\lambda_1c_1-\int_{t_0}^{T_f} y^T(\tau)y(\tau)d\tau+\gamma^2d_w\right)\Big/\lambda_2.
\end{aligned}$$

For any switching signal σ with average dwell-time satisfying the inequality (11), we can achieve that

$$\tilde{e}^T(t)R\tilde{e}(t) < c_2. \tag{20}$$

Thus, the tracking error of MRC is finite-time bounded with respect to $(c_1, c_2, T_f, d_w, R, \sigma)$, which completes the proof.

3.2. **Finite-time weighted H_∞ performance analysis for the linear system (1).** We have studied the problem of state tracking for unmeasured state based on the state observer in a finite-time interval. Furthermore, we know that the tracking error is finite-time bounded. The next content will discuss the weighted H_∞ performance for considered system (1) in a finite-time interval.

Theorem 3.2. *Considering the system (1), let $\alpha > 0$, $\beta > 0$, $\mu > 1$, $\eta > 1$ and a finite-time interval $[t_0, T_f]$. For $\forall t \in [t_0, T_f]$, if the following inequalities hold*

$$T_{\downarrow[t_0,T_f]} \geq \eta(T_f - t_0), \tag{21}$$

$$\int_{t_0}^{T_f} e^{(\beta-\eta(\alpha+\beta))(\tau-t_0)} - e^{-N_{[t_0,\tau]}\ln\mu} d\tau \leq 0, \tag{22}$$

$$\int_{t_0}^{T_f} e^{(\beta-\eta(\alpha+\beta))(\tau-t_0)} e^{-N_{[t_0,\tau]}\ln\mu} d\tau \leq 1, \tag{23}$$

then the system (1) achieves the finite-time weighted H_∞ performance over a finite-time interval.

Proof: In the front section we have obtained (17) and under the initial condition $\tilde{e}(t_0) = 0$, we will get

$$\begin{aligned} V(\tilde{e}(t)) &\leq \int_{t_0}^{T_f} \mu^{N_{[\tau,T_f]}} e^{-\alpha T_{\downarrow[\tau,T_f]} + \beta T_{\uparrow[\tau,T_f]}} J(\tau) d\tau \\ &= \int_{t_0}^{T_f} e^{N_{[\tau,T_f]}\ln\mu} e^{-\alpha T_{\downarrow[\tau,T_f]} + \beta T_{\uparrow[\tau,T_f]}} J(\tau) d\tau. \end{aligned} \tag{24}$$

Both sides of (24) by multiplying $e^{-N_{[t_0,T_f]}\ln\mu}$ yield

$$e^{-N_{[t_0,T_f]}\ln\mu} V(\tilde{e}(t)) \leq \int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{-\alpha T_{\downarrow[\tau,T_f]} + \beta T_{\uparrow[\tau,T_f]}} J(\tau) d\tau. \tag{25}$$

According to (21), the inequality (22) can be simplified as follows

$$e^{-N_{[t_0,T_f]}\ln\mu} V(\tilde{e}(t)) \leq \int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} J(\tau) d\tau,$$

which means

$$\begin{aligned} &\int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} y^T(\tau) y(\tau) d\tau \\ \leq &\int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} \gamma^2 w^T(\tau) w(\tau) d\tau. \end{aligned} \tag{26}$$

In the case of (22), (23) and (26), we will obtain

$$\int_{t_0}^{T_f} e^{(\beta-\eta(\alpha+\beta))(T_f-t_0)} y^T(\tau) y(\tau) d\tau$$

$$
\begin{aligned}
&= \int_{t_0}^{T_f} e^{(\beta-\eta(\alpha+\beta))(\tau-t_0)} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} y^T(\tau)y(\tau)d\tau \\
&\leq \int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} y^T(\tau)y(\tau)d\tau \\
&\leq \int_{t_0}^{T_f} e^{-N_{[t_0,\tau]}\ln\mu} e^{(\beta-\eta(\alpha+\beta))(T_f-\tau)} \gamma^2 w^T(\tau)w(\tau)d\tau \\
&\leq \gamma^2 \int_{t_0}^{T_f} w^T(\tau)w(\tau)d\tau. \qquad (27)
\end{aligned}
$$

Therefore, the proof of finite-time weighted H_∞ performance γ has completed.

Remark 3.1. *In fact, the problem of standard H_∞ performance for switched systems is an unsolved problem with the constraint of ADT, which is introduced in* [38] *and is not solved yet in* [39]*. Therefore, we solve the problem with weighted H_∞ performance over a finite-time interval.*

4. **Example.** To demonstrate the validity of the state tracking over a finite-time interval, an example is given in this section. To obtain an excellent performance for the system (1), the choice of parameters for the reference model is to make the reference model have a good transient performance and a stable performance. Owing to a numerical simulation in this paper, the choice of parameters for the system (1) has a wide range. Our object is to make the state of system (1) track the reference model exactly with designing a switching signal and a controller. In other words, the considered system (1) achieves the same performance as the reference model through designing a switching signal and a controller. The corresponding parameters are given as follows:

$$
A_m = \begin{bmatrix} -2.3 & 1.6 \\ -2.2 & -3.1 \end{bmatrix},\ B_m = \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix},\ A = \begin{bmatrix} -3.5 & 1.8 \\ -3.2 & -3.5 \end{bmatrix},\ B = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix},
$$
$$
G = \begin{bmatrix} 1.6 \\ 2.3 \end{bmatrix},\ E = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},\ C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix},\ \zeta = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Given constants $\alpha = 0.75$, $\beta = 0.8$, $\mu = 1.12$, $\eta = 0.65$, $u_m = 0.03$ and the exogenous disturbance $w = 0.2\sin(2\pi t)$. By solving the corresponding linear matrix inequality, we can obtain the matrix P

$$
P = \begin{bmatrix} 0.1511 & -0.0039 \\ -0.0039 & 0.1045 \end{bmatrix}
$$

and its characteristics root can be obtained $\lambda_1 = 0.1514$ and $\lambda_2 = 0.1042$ by calculation.

To satisfy the tracking performance between matched and unmatched cases, we have designed a switching signal. The result is shown in Figure 2. In the research of this paper, we set $\xi = \vartheta = 0.5$. Thus, Figure 2 shows clearly that the state tracking error is less than the given tolerance ϑ at the initial time. After a brief time, the switching control signal appears high level signal. That means the state tracking error exceeds the given tolerance ϑ, and the controller $u_{\sigma(t)}(t)$ is needed to reduce the tracking error at this time. By means of the controller $u_{\sigma(t)}(t)$, the state error is controlled within the given tolerance ϑ. A finite number jumps later, the switching control signal becomes continued low level signal. Therefore, the state tracking error is controlled within the given tolerance ϑ finally.

Figure 3 shows the output of the controller $u_{\sigma(t)}(t)$. It is obvious that a large amplitude jitter appears at the initial time. Then, the output of controller $u_{\sigma(t)}(t)$ changes to zero after a few minor fluctuations. This implies that the state tracking error is controlled within the given tolerance ϑ and the switching is terminated.

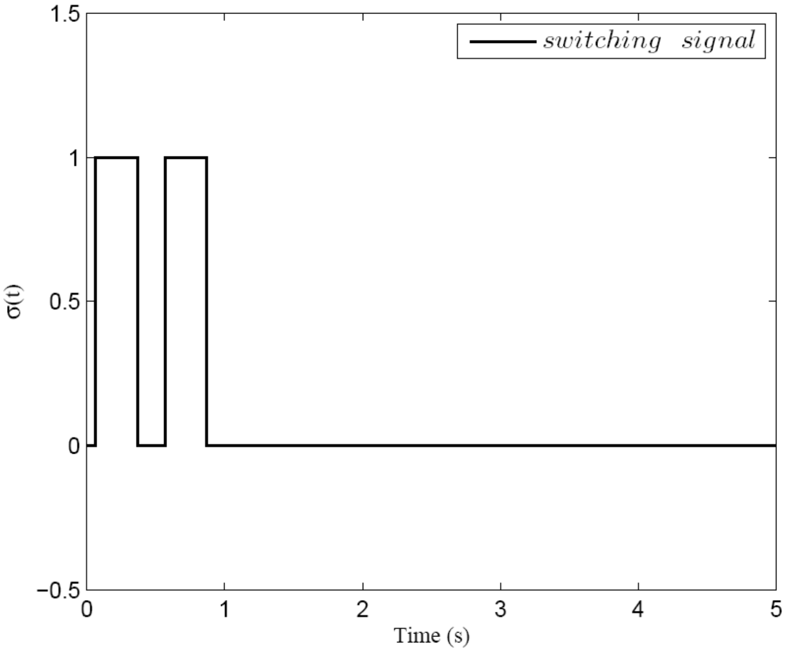

Figure 2. Switching signal

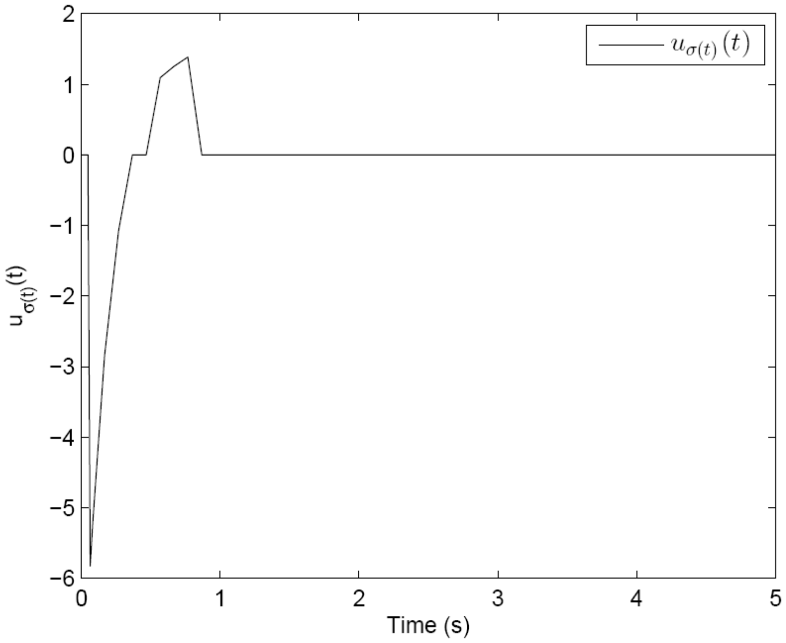

Figure 3. Output of the controller $u_{\sigma(t)}(t)$

Figure 4 shows the state trajectory of model reference system (2). The control object is to make the state of system (1) track the state of model reference (2). The observation error between $\hat{x}(t)$ and $x(t)$ is shown as Figure 5. From Figure 5 we know that the observation error is presented in the form of sine signal, which is consistent with Equation (5).

Figure 6 shows the state tracking error curves between $\hat{x}(t)$ and $x_m(t)$. In the process of state tracking, there is a large tracking error in the first 1.5 seconds. With the help of controller $u_{\sigma(t)}(t)$, the tracking error $\tilde{e}(t)$ reduced to zero gradually. Obviously, the reference state $x_m(t)$ can be tracked by $\hat{x}(t)$ over a finite-time interval.

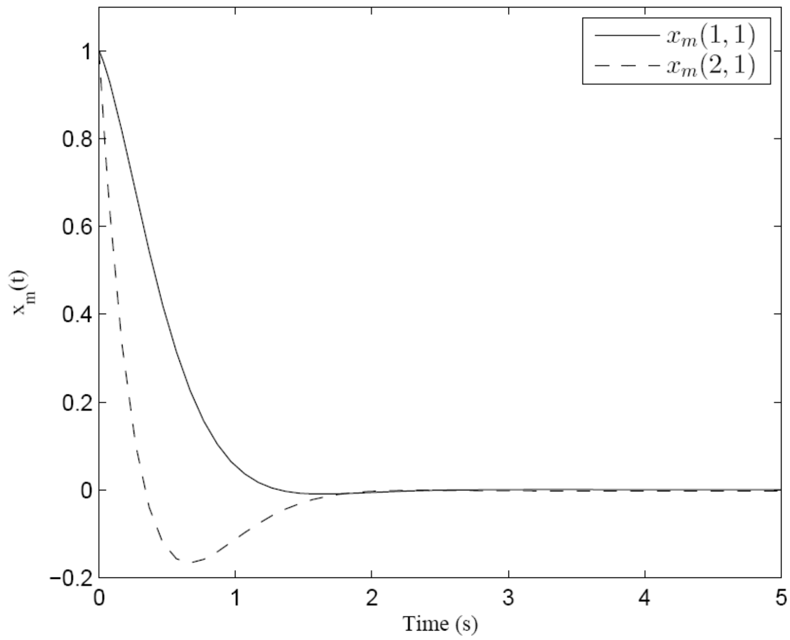

FIGURE 4. Model reference state

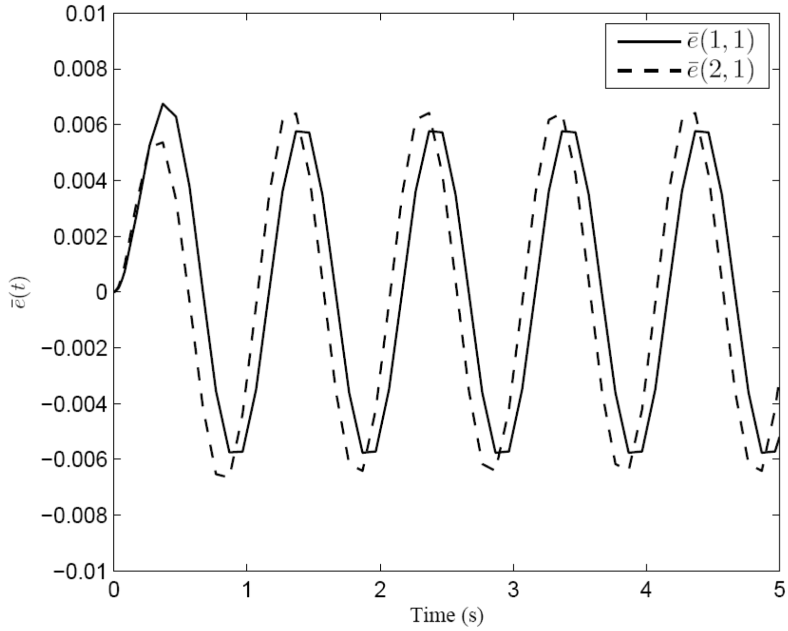

FIGURE 5. Observation error

Set constants

$$c_1 = 1.5,\ c_2 = c_{2i}\ (c_{2i} \text{ as shown in Table 1}),$$
$$t_0 = 0,\ T_f = 5,\ \gamma^2 = 0.08,\ R = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given initial condition $\tilde{e}^T(t_0) = [0\ 0]$, the trajectory of $\tilde{e}^T(t)R\tilde{e}(t)$ can be shown in Figure 7. From Figure 7 we know that the value of $\tilde{e}^T(t)R\tilde{e}(t)$ is less than the given boundary c_{2i} over the finite-time interval. That means system (1) is finite-time bounded with the controller (8).

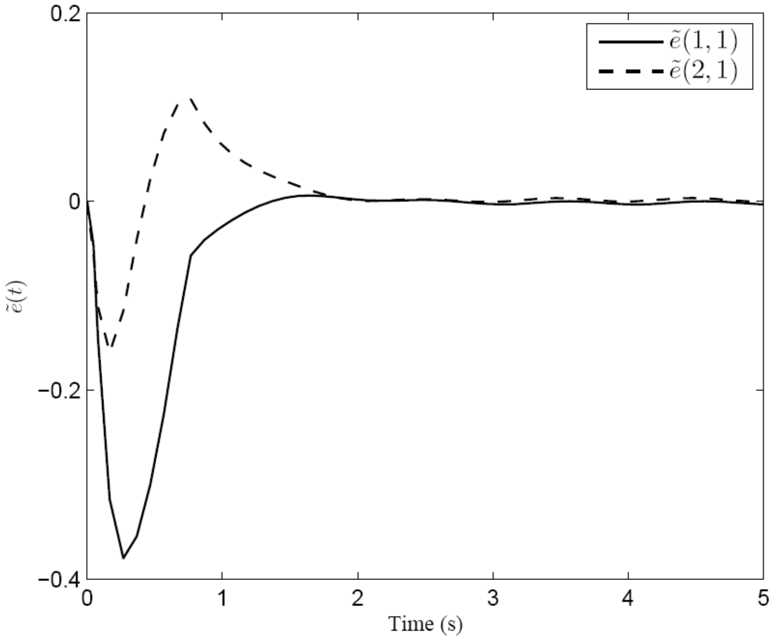

FIGURE 6. Tracking error

TABLE 1. The value of τ_{fi} with different c_{2i}

Para.	Value	Value	Value	Value	Value	Value	Value	Value
T_f	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
c_1	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5
γ^2	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
c_{2i}	3.2	3.5	3.8	4.1	4.4	4.7	5.0	5.3
τ_a	0.546	0.546	0.546	0.546	0.546	0.546	0.546	0.546
τ_f	0.367	0.349	0.334	0.321	0.309	0.300	0.291	0.283

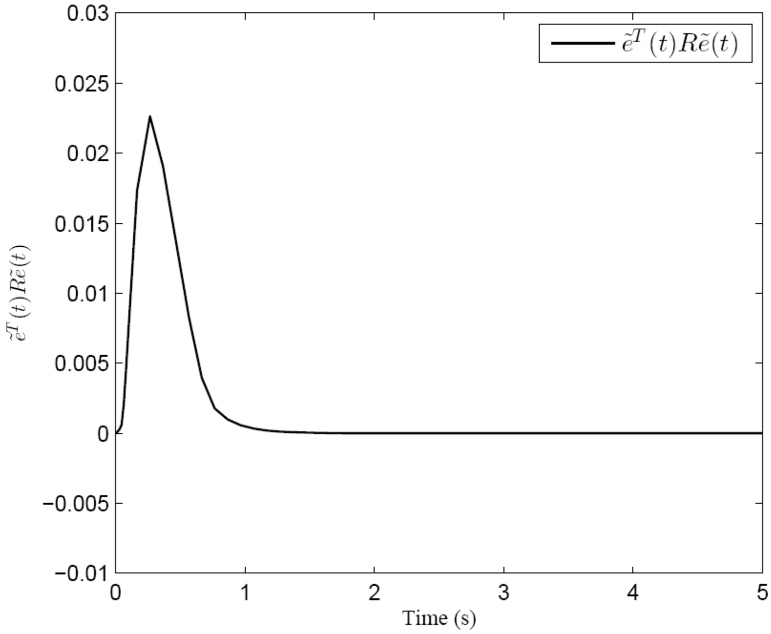

FIGURE 7. Trajectory of $\tilde{e}^T(t)R\tilde{e}(t)$

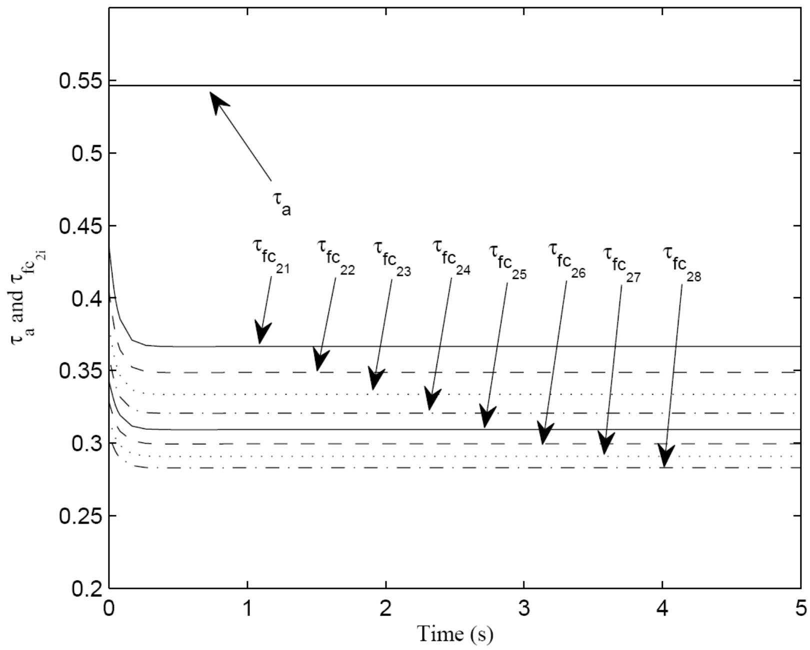

FIGURE 8. ADT of τ_a and $\tau_{fc_{2i}}$

As the constants T_f, c_1 and γ^2 are respectively unified, we can learn from Table 1 and Figure 8 that the larger c_{2i} produces the shorter ADT τ_f for the system (1). It means that with the loss of the system requirements for boundary c_{2i}, the ADT τ_f will decrease accordingly. What is more, there exists a unique and the smallest ADT τ_f for any given boundary c_{2i}. In addition, with the method of finite-time control, the ADT τ_f we have achieved is less than the traditional ADT τ_a, which can be seen from both Table 1 and Figure 7.

5. **Conclusions.** This paper has considered the problem of model reference state tracking control for a continuous-time system with unmeasured state over a finite-time interval. As a means of estimating the unmeasured state of the system under consideration, a state observer was designed initially. Following this, a switching signal and a controller were devised for the continuous-time system with matched and unmatched cases. In order to guarantee that the investigated system state can monitor the state of the reference system over a finite time interval, we have provided the maximum ratio of time occupancy. Subsequently, sufficient conditions were established for the continuous-time system, securing both finite-time tracking error bounded and finite-time weighted H_∞ performance. Through defining the various parameters c_2, we were able to determine the interaction between conservatism and boundary c_2. The future work is to consider the solutions of nonlinear and delayed on the state tracking systems over a finite-time interval.

Acknowledgment. This work was supported by the National Natural Science Foundation of China (61503071), Natural Science Foundation of Jilin Province (20180520211JH) and Science Research of Education Department of Jilin Province (201693, JJKH20170106 KJ).

REFERENCES

[1] M. Uddin, S. Mekhilef and M. Rivera, Experimental validation of minimum cost function-based model predictive converter control with efficient reference tracking, *IET Power Electronics*, vol.8, no.2, pp.278-287, 2015.

[2] R. P. Aguilera, A. Pablo, P. Lezana, G. Konstantinou, B. Wu, S. Bernet and V. G. Agelidis, Selective harmonic elimination model predictive control for multilevel power converters, *IEEE Trans. Power Electronics*, vol.32, no.3, pp.2416-2426, 2017.

[3] A. Gaeta and U. Montanaro, Application of a robust model reference adaptive control algorithm to a nonlinear automotive actuator, *International Journal of Automation and Computing*, vol.11, no.4, pp.377-391, 2014.

[4] M. Xiao, Modeling and adaptive sliding mode control of the catastrophic course of a high-speed underwater vehicle, *International Journal of Automation and Computing*, vol.10, no.3, pp.210-216, 2013.

[5] B. Stellato, T. Geyer and P. J. Goulart, High-speed finite control set model predictive control for power electronics, *IEEE Trans. Power Electronics*, vol.32, no.5, pp.4007-4020, 2017.

[6] L. H. Zhao and G. O. Shi, A model following control system for nonlinear descriptor systems, *Journal of Northeast Dianli University*, vol.32, no.4, pp.87-90, 2012.

[7] H. Gao, B. Wu, D. W. Xu, M. Pande and R. P. Aguilera, Common-mode-voltage-reduced model-predictive control scheme for current-source-converter-fed induction motor drives, *IEEE Trans. Power Electronics*, vol.32, no.6, pp.4891-4904, 2017.

[8] A. A. Ahmed, B. K. Koh, H. S. Park, K. B. Lee and Y. I. Lee, Finite-control set model predictive control method for torque control of induction motors using a state tracking cost index, *IEEE Trans. Industrial Electronics*, vol.64, no.3, pp.1916-1928, 2017.

[9] H. Habibullah, H. R. Pota, I. R. Petersen and M. S. Rana, Tracking of triangular reference signals using LQG controllers for lateral positioning of an AFM scanner stage, *IEEE/ASME Trans. Mechatronics*, vol.19, no.4, pp.1105-1114, 2014.

[10] P. K. Gillella, X. Y. Song and Z. X. Sun, Time-varying internal model-based control of a camless engine valve actuation system, *IEEE Trans. Control Systems Technology*, vol.22, no.4, pp.1498-1510, 2014.

[11] Z. H. Pang, G. P. Liu, D. H. Zhou and M. Y. Chen, Output tracking control for networked systems: A model-based prediction approach, *IEEE Trans. Industrial Electronics*, vol.61, no.9, pp.4867-4877, 2014.

[12] L. G. Wu, X. B. Yang and F. B. Li, Nonfragile output tracking control of hypersonic air-breathing vehicles with an LPV model, *IEEE/ASME Trans. Mechatronics*, vol.18, no.4, pp.1280-1288, 2013.

[13] S. Z. Zhao, Dimirovski, M. Georgi and R. C. Ma, Robust H_∞ control for non-minimum phase switched cascade systems with time delay, *Asian Journal of Control*, vol.17, no.5, pp.1590-1599, 2015.

[14] B. Niu and J. Zhao, Robust stabilization and tracking control for a class of switched nonlinear systems, *Asian Journal of Control*, vol.15, no.5, pp.1496-1502, 2013.

[15] L. X. Zhang and H. J. Gao, Asynchronously switched control of switched linear systems with average dwell time, *Automatica*, vol.46, no.5, pp.953-958, 2010.

[16] J. Xie and J. Zhao, Model reference adaptive control for switched LPV systems and its application, *IET Control Theory & Applications*, vol.10, no.17, pp.2204-2212, 2016.

[17] H. C. Pangborn and A. G. Alleyne, Switched linear control for refrigerant superheat recovery in vapor compression systems, *Control Engineering Practice*, vol.57, pp.142-156, 2016.

[18] S. Ghabraei, H. Moradi and G. Vossoughi, Finite time-Lyapunov based approach for robust adaptive control of wind-induced oscillations in power transmission lines, *Journal of Sound and Vibration*, vol.371, no.9, pp.19-34, 2016.

[19] R. H. Li, W. S. Chen and S. Li, Finite-time stabilization for hyper-chaotic Lorenz system families via adaptive control, *Applied Mathematical Modelling*, vol.37, no.4, pp.1966-1972, 2013.

[20] C. G. Yang, H. B. Ma, M. Y. Fu and A. M. Smith, Optimized model reference adaptive motion control of robot arms with finite time tracking, *Proc. of the 31st Chinese Control Conference*, Hefei, China, pp.25-27, 2012.

[21] Z. J. Yang, Robust consensus tracking control of higher-order uncertain non-linear systems via cascaded finite-time estimation of reference states, *IET Control Theory & Applications*, vol.10, no.11, pp.1231-1239, 2016.

[22] S. Li, X. Wang and L. Zhang, Finite-time output feedback tracking control for autonomous underwater vehicles, *IEEE Journal of Oceanic Engineering*, vol.40, no.3, pp.727-751, 2015.

[23] S. C. Liu, Z. Y. Geng and J. Y. Sun, Finite-time attitude control: A finite-time passivity approach, *IEEE/CAA Journal of Automatica Sinica*, vol.2, no.1, pp.102-108, 2015.

[24] S. H. Yu, X. J. Long and G. Guo, Continuous finite-time output consensus tracking of high-order agents with matched and unmatched disturbances, *IET Control Theory & Applications*, vol.10, no.14, pp.1716-1723, 2016.

[25] H. Dahmani, O. Pagès and A. E. Hajjaji, Observer-based state feedback control for vehicle chassis stability in critical situations, *IEEE Trans. Control Systems Technology*, vol.24, no.2, pp.636-643, 2016.

[26] F. Zhang, X. Wu and J. Shen, Extended state observer based fuzzy model predictive control for ultra-supercritical boiler-turbine unit, *Applied Thermal Engineering*, vol.118, pp.90-100, 2017.

[27] S. H. Lee, H. J. Kang and C. C. Chung, Robust fast seek control of a servo track writer using a state space disturbance observer, *IEEE Trans. Control Systems Technology*, vol.20, no.2, pp.346-355, 2012.

[28] S. B. Wang, X. M. Ren, J. Na and T. Y. Zeng, Extended-state-observer-based funnel control for nonlinear servomechanisms with prescribed tracking performance, *IEEE Trans. Automation Science and Engineering*, vol.14, no.1, pp.98-108, 2017.

[29] S. Sui, Y. M. Li and S. C. Tong, Observer-based adaptive fuzzy control for switched stochastic nonlinear systems with partial tracking errors constrained, *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol.46, no.12, pp.1605-1617, 2016.

[30] L. J. Liu, X. L. Liu, C. T. Man and C. Y. Xu, Delayed observer-based H_∞ control for networked control systems, *Neurocomputing*, vol.179, pp.101-109, 2016.

[31] A. Chakrabarty, R. Ayoub, S. H. Żak and S. Sundaram, Delayed unknown input observers for discrete-time linear systems with guaranteed performance, *Systems & Control Letters*, vol.103, pp.9-15, 2017.

[32] W. S. Chen, R. H. Li and J. Li, Observer-based adaptive iterative learning control for nonlinear systems with time-varying delays, *International Journal of Automation and Computing*, vol.7, no.4, pp.438-446, 2010.

[33] X. Z. Lin, C. X. Lv, S. H. Li and Y. Zou, Finite-time stability and finite-time boundedness for switched systems with sector bounded nonlinearities, *Proc. of the 34th Chinese Control Conference*, Hangzhou, China, pp.651-656, 2015.

[34] X. Z. Lin, H. B. Du, S. H. Li and Y. Zou, Finite-time stability and finite-time weighted L_2-gain analysis for switched systems with time-varying delay, *IET Control Theory & Applications*, vol.7, no.7, pp.1058-1069, 2013.

[35] L. Li, Q. L. Zhang, J. Li and G. L. Wang, Robust finite-time H_∞ control for uncertain singular stochastic Markovian jump systems via proportional differential control law, *IET Control Theory & Applications*, vol.8, no.16, pp.1625-1638, 2014.

[36] X. L. Luan, C. Z. Zhao and F. Liu, Finite-time H_∞ control with average dwell-time constraint for time-delay Markov jump systems governed by deterministic switches, *IET Control Theory & Applications*, vol.8, no.11, pp.968-977, 2014.

[37] C. S. Tseng and B. S. Chen, L_∞-gain fuzzy control for nonlinear dynamic systems with persistent bounded disturbances, *Proc. of IEEE International Conference on Fuzzy Systems*, Budapest, Hungary, pp.783-788, 2004.

[38] V. D. Blondel and A. Megretski, *Unsolved Problems in Mathematical Systems and Control Theory*, Princeton University Press, 2004.

[39] W. M. Xiang and J. Xiao, Discussion on "Stability, L_2-gain and asynchronous H_∞ control of discrete-time switched systems with average dwell time", *IEEE Trans. Automatic Control*, vol.57, no.12, pp.3259-3261, 2012.