International Journal of Innovative
Computing, Information and Control
Volume **14**, Number **5**, October **2018**

ICIC International ©2018 ISSN 1349-4198
pp. 1681–1707

ON THE STABILITY OF TWO NEW RECURSIVE 2D-ROESSER-BASED DISCRETE MODELS WITH SPATIAL AND TIME DELAYS

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Received January 2018; revised May 2018

Abstract. *This paper presents and discusses from the stability point of view two discrete Roesser-based models whose state evolution has a recursive nature which can be combined in the spatial horizontal and vertical substates and time evolution. A first set of stability results is derived in the absence of controls based on the stability/convergence properties of the matrices defining the dynamics. Later on, two linear control laws are given and the stability properties are reformulated under control feedback. Also, some controllability conditions of certain relevant pairs defining the closed-loop dynamics what allows to prescribe the spectral radii to convenient values to guarantee the stability by means of the synthesis of the controller gains.*

Keywords: Delays, Roesser models, Spectral radius, Stability

1. **Introduction.** Roesser models have been widely studied in the literature. Their main characteristic is that they are bidimensional in the sense that they have an horizontal substate and a vertical substate. They are of interest in certain processes where the information propagates in two independent directions such as thermal processes, gas absorption or water stream heating. More recently, they have been used in some applications like digital image processing or multidimensional digital filter synthesis [1,2]. The positivity of those models under dynamic output-feedback controllers has been investigated in [3] under a switched mechanism and a subsequent decomposition into the linear form to the light of Takagi-Sugeno rules. Note that positivity is a very relevant property in dynamic systems modeling of biological and epidemic models. See, for instance [25,26] and references therein. The differences between the concepts of asymptotic stability and structural stability are examined in [4] through the equivalence between models. In [5], a novel elementary operation approach for order reduction for the Roesser state-space model of multidimensional systems by introducing the so-called Jordan transformation, which guarantees the establishment of an objective matrix with more general structure than the existing one. The standard multidimenmsional discrete Roesser models cannot be described by recursive evolution equations because of their structure. If the matrix of dynamics is triangularizable with diagonal blocks equalizing the diemensionalities of the

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DOI: 10.24507/ijicic.14.05.1681

horizontal and vertical substates, then the evolution equations can be organized recursively. In that context, the subspace identification of a CRSD (that is, a causal, recursive, and separable-in-denominator model) two-dimensional Roesser model is performed in [6] subject to deterministic and stochastic inputs. Such a kind of model has the main attractive characteristic that it is recursively separable in the denominator and the matrix describing its dynamics has a triangular structure. The stability of Roesser models is discussed in [7-11] and references therein and the stabilization of such models has been discussed in [12] and some references therein. Some stability problems for certain classes of two-dimensional problems are studied in [19] and the optimality of a perturbation method is studied in [20] for characterizing solutions of differential equations. In particular, the discrete Lyapunov equation guarantees the global asymptotic stability in the time-invariant case under a block diagonal positive definite matrix defining a quadratic Lyapunov function. The positive definite block matrices correspond to the horizontal and vertical substates. However, it is conjectured in [7], without necessity proof, that such a discrete Lyapunov equation gives a necessary and sufficient stability condition while it is proved in [9] that such condition is just sufficient. Positive real control laws for such systems are proposed in [13] while state-space models and some structural properties of doubly indexed dynamic systems are investigated in [14]. It is well-known the fact that (in general, unnecessary) sufficiency-type stability conditions of systems subject to point delayed dynamics can be obtained from conditions for multidimensional systems by treating the exponential Laplace transform arguments associated with delays as an extra operator to the Laplace operator which describes the delay-free dynamics in the frequency domain. Necessary conditions are then obtained by considering the dependence of the exponential delay argument with the Laplace argument. See [15,16] and references therein. This paper first discusses a sufficiency-type stability condition for the standard 2D-Roesser model under any given summable absolute initial conditions on the whole horizontal and vertical axes without using a Lyapunov function. The stability conditions are based in the characterizations of the spectral radius or spectral norm of the matrix of dynamics guaranteeing the convergence (for the achievement of the global asymptotic stability of the system), or at least the uniform boundedness of the sequence of its positive integer powers (for the achievement of just its global stability). The "global" stability concept is constrained to summable initial conditions but note that this is the case of interest when just one non-zero initial condition is used in each spatial axis characterizing the spatial substates. On the other hand, the found conditions are easy to test in practice. This study is of practical interest since the existing global sufficient and necessary and sufficient stability conditions [7-9] are difficult to test in both time and frequency domains contrarily to those proposed in this paper. Later on, two kinds of linear discrete Roesser-based models which have a state recursive evolution description are proposed and their stability properties are investigated. The mechanism which allows the recursive description of the state evolution is a state constraint evolution formalized through the existence of a linear transformation in-between the state at points where both horizontal/vertical substate arguments jointly shift in advance and its value at the preceding value when only one of the arguments is shifted for each substate. Both models incorporate point delays in the vertical and horizontal dynamics. The first model is stated for horizontal and vertical substates without the presence of a discrete time argument while the second one incorporates furthermore a time-delay evolution with eventual point time internal and external delays for both spatial substates while it does not incorporate a new substate under temporal discrete delay governance. In this sense, both models are still two-dimensional models. The stability properties are investigated by using perturbation theory of matrices by assuming that a certain matrix describing the evolution from one state to the next one where all arguments increase by one is convergent while the remaining coupling resulting eventually delayed dynamics is considered a perturbation. Later on, the stability properties are re-formulated by the use of state-feedback linear control laws which allow to decrease the stability radii of certain matrices being relevant to the dynamics via feedback under certain controllability assumptions. Some simulated examples are then presented and, finally, conclusions end the paper.

Notation

$$\boldsymbol{Z}_{0+} = \boldsymbol{Z}_{+} \cup \{0\}; \quad \boldsymbol{Z}_{+} = \{z \in \boldsymbol{Z} : z > 0\}; \quad \bar{n} = \{1, 2, \ldots, n\}.$$

$$\boldsymbol{R}_{0+} = \boldsymbol{R}_{+} \cup \{0\}; \quad \boldsymbol{R}_{+} = \{r \in \boldsymbol{R} : r > 0\}.$$

I_n is the n-th identity matrix.

The ℓ_2(or spectral)-norm is denoted by $\|.\|_2$ and $\lambda_{\max}(.)$ is the maximum eigenvalue of the (.)-symmetric or Hermitian matrix and $\lambda_{\min}(.)$ is its minimum eigenvalue. The ℓ_1-norm is denoted by $\|.\|_1$ and the ℓ_∞-norm is denoted by $\|.\|_\infty$.

The spectral radius of a complex square matrix X is $r_X = \max\limits_{\lambda \in sp(X)} |\lambda| = \inf \|X\|$, where $sp(X)$ is the spectrum of X, the infimum is taken over the whole set of matrix vector-induced norms and $\|X\|_2 = r^{1/2}_{X^TX} = \lambda^{1/2}_{\max}\left(X^TX\right)$. If X is Hermitian, then $\|X\|_2 = r_X$.

2. **The Basic 2D-Roesser Model.**

The standard two-dimensional Roesser model is the following one:

$$x_h(n_1+1, n_2) = A_{hh}x_h(n_1, n_2) + A_{hv}x_v(n_1, n_2) + B_h u(n_1, n_2) \tag{1}$$

$$x_v(n_1, n_2+1) = A_{vh}x_h(n_1, n_2) + A_{vv}x_v(n_1, n_2) + B_v u(n_1, n_2) \tag{2}$$

$$y(n_1, n_2) = C_h x_h(n_1, n_2) + C_v x_v(n_1, n_2) + D u(n_1, n_2) \tag{3}$$

$\forall n_1, n_2 \in \boldsymbol{Z}_{0+}$, where $x_h(n_1, n_2) \in \boldsymbol{R}^{n_h}$, $x_v(n_1, n_2) \in \boldsymbol{R}^{n_v}$, $x_h(n_1, n_2) \in \boldsymbol{R}^m$ and $y(n_1, n_2) \in \boldsymbol{R}^p$ are the horizontal substate, vertical substate, input and output, respectively, subject to infinitely many initial conditions $x_h(0, n) \in \boldsymbol{R}^{n_h}$, $x_v(n, 0) \in \boldsymbol{R}^{n_v}$, and all the matrices of parameters are of corresponding compatible orders with the corresponding vectors. The whole state of (1)-(3) is $x(n_1, n_2; n_3, n_4) = \left(x_h^T(n_1, n_2), x_v^T(n_3, n_4)\right)^T$ of order $n = n_h + n_v$.

The following stability result is proved in Appendix A.

Theorem 2.1. *Assume that A is a convergent matrix and assume also that the initial conditions $\{x_h(0, n_1), x_v(n_2, 0); n_1, n_2 \in \boldsymbol{Z}_{0+}\}$ of (1)-(3) are absolutely summable for any given $n_1, n_2 \in \boldsymbol{Z}_{0+}$. Then, the state and output sequences of the unforced systems (1)-(3) are bounded and* $\begin{bmatrix} x_h(n_1+1, n_2) \\ x_v(n_1, n_2+1) \end{bmatrix} \to 0$, *$y(n_1, n_2) \to 0$ as $\max(n_1, n_2) \to \infty$.*

The above result is an asymptotic stability result for any initial conditions $\{x_h(n, 0), x_v(0, n); n \in \boldsymbol{Z}_{0+}\}$ which are absolutely summable. This condition includes the case where only a finite number of them is nonzero and finite and the case of infinitely many nonzero initial (or boundary) conditions which are absolutely summable. A case which is covered by the last condition is the one when all of them are selected as elements of some arithmetic of geometric summable series. Note that those conditions imply that the ℓ_2-norm of the boundary conditions is finite. A 2D-Roesser model with internal point delays $d_1, d_2 \in \boldsymbol{Z}_{0+}$ for the states $x_h(.,.)$ and $x_v(.,.)$, respectively, with $\max(d_1, d_2) \geq 1$ generalized from (1)-(3) is the following one:

$$\begin{aligned} x_h(n_1+1, n_2) = {} & A_{hh}x_h(n_1, n_2) + A_{hv}x_v(n_1, n_2) + A_{hhd}x_h(n_1 - d_1, n_2) \\ & + A_{hvd}x_v(n_1, n_2 - d_2) + B_h u(n_1, n_2) \end{aligned} \tag{4}$$

$$\begin{aligned} x_v(n_1,n_2+1) &= A_{vh}x_h(n_1,n_2) + A_{vv}x_v(n_1,n_2) + A_{vhd}x_h(n_1-d_1,n_2) \\ &\quad + A_{vvd}x_v(n_1,n_2-d_2) + B_v u(n_1,k_2) \end{aligned} \tag{5}$$

$$y(n_1,n_2) = C_h x_h(n_1,n_2) + C_v x_v(n_1,n_2) + Du(n_1,n_2) \tag{6}$$

with appropriate dimensionalities of the matrices taking account of the delay contributions. The following stability result is a direct extension of Theorem 2.1. A sketch of its proof is given in Appendix A.

Theorem 2.2. *Assume that* A *is a convergent matrix with convergence abscissa* ρ *and that* $\rho + \|A_d\|_2 < 1$, *where* $A_d = \begin{bmatrix} A_{hhd} & A_{hvd} \\ A_{vhd} & A_{vvd} \end{bmatrix} \in \boldsymbol{R}^{n\times n}$. *Assume also that the initial conditions* $\{x_h(0,n_1+d_1), x_v(n_2+d_2,0); n_1,n_2 \in \boldsymbol{Z}_{0+}\}$ *of (4)-(6) are absolutely summable for any given* $n_1,n_2 \in \boldsymbol{Z}_{0+}$. *Then, the state and output sequences of the unforced system (1)-(3) are bounded and,* $y(n_1,n_2) \to 0$ *as* $\begin{bmatrix} x_h(n_1+1,n_2) \\ x_v(n_1,n_2+1) \end{bmatrix} \to 0$, $\max(n_1,n_2) \to \infty$.

Note that the 2D-Roesser models are difficult to deal with in terms of computational effort and memory consuming expenses since they have not a recursive structure while having multiple boundary conditions which increase in number as the computation progresses.

3. Two Linear Time-Invariant Recursive 2D-Roesser-Based Models with Spatial/Time Delays.

Two linear time-invariant recursive models, which are based in the standard 2D-Roesser model, are now stated with different complexity degrees and eventually incorporating point delays in the spatial substates and/or incorporating temporal evolution eventually subject or not to point delays as well. The first model (Model 1) incorporates a delay d to the evolution of the spatial horizontal and vertical substates. It can be pointed out that some existing background literature supports the presence of delays in the spatial substates. See, for instance, [23,24]. The second proposed mode (Model 2) incorporates, in addition, a discrete temporal contribution to the dynamics eventually subject to discrete-time delays. Again, there is some available background literature on Roesser models backing the introduction of temporal delays to generalize such models. See, for instance, [2]. On the other hand, note that some inherent two-dimensional processes as, for instance, repetitive processes need updating information on former values of the states and eventual resetting. Some of those processes can be linked to different physical modelling requirements, like the generation of spatial or plane images with updated information based on values of neighboring points. Other processes can be influenced by a temporal delayed dynamics like, for instance, the tracking of vehicle paths under surveillance or processes of diffusion, like those of gas absortion, water stream, heating and air drying modelled by Darboux partial differential equations. The integration of both ideas is a justification basis to build models which combine spatial type and temporal-type delays.

a) Model 1: *Recursive model with delayed dynamics*

The following model incorporates internal and external delays $d, d' \in \boldsymbol{Z}_+$ to the 2D-Roesser model as well as evolution constraints for the transition $\left[x_h^T(n_1+1,n_2), x_v^T(n_1, n_2+1)\right]^T$ to $\left[x_h^T(n_1+1,n_2+1), x_v^T(n_1+1,n_2+1)\right]^T$ resulting in the following variant of Model 1:

$$\begin{aligned} x_h(n_1+1,n_2) &= M_{hh}x_h(n_1,n_2) + M_{hv}x_v(n_1,n_2) + M_{hhd}x_h(n_1-d,n_2-d) \\ &\quad + M_{hvd}x_v(n_1-d,n_2-d) + N_h u(n_1,n_2) + N_{hd'}u(n_1-d',n_2-d') \end{aligned} \tag{7}$$

$$\begin{aligned} x_v\left(n_1,n_2+1\right) = {} & M_{vh}x_h\left(n_1,n_2\right) + M_{vv}x_v\left(n_1,n_2\right) + M_{vhd}x_h\left(n_1-d,n_2-d\right) \\ & + M_{vvd}x_v\left(n_1-d,n_2-d\right) + N_v u\left(n_1,n_2\right) + N_{vd'}u\left(n_1-d',n_2-d'\right) \end{aligned} \tag{8}$$

$$\begin{aligned} & x_h\left(n_1+1,n_2+1\right) \\ & = P_{hh}x_h\left(n_1+1,n_2\right) + P_{hv}x_v\left(n_1,n_2+1\right) + P_{hhd}x_h\left(n_1+1-d,n_2-d\right) \\ & + P_{hvd}x_v\left(n_1-d,n_2+1-d\right) + Q_{h1}u\left(n_1+1,n_2\right) + Q_{h2}u\left(n_1,n_2+1\right) \\ & + Q_{h1d'}u\left(n_1+1-d',n_2-d'\right) + Q_{h2d'}u\left(n_1-d',n_2+1-d'\right) \end{aligned} \tag{9}$$

$$\begin{aligned} & x_v\left(n_1+1,n_2+1\right) \\ & = P_{vh}x_h\left(n_1+1,n_2\right) + P_{vv}x_v\left(n_1,n_2+1\right) + P_{vhd}x_h\left(n_1+1-d,n_2-d\right) \\ & + P_{vvd}x_v\left(n_1-d,\ n_2+1-d\right) + Q_{v1}u\left(n_1+1,n_2\right) + Q_{v2}u\left(n_1,n_2+1\right) \\ & + Q_{v1d'}u\left(n_1+1-d',n_2-d'\right) + Q_{v2d'}u\left(n_1-d',n_2+1-d'\right) \end{aligned} \tag{10}$$

$$y\left(n_1,n_2\right) = C_h x_h\left(n_1,n_2\right) + C_v x_v\left(n_1,n_2\right) + Du\left(n_1,n_2\right) \tag{11}$$

$\forall n_1,\ n_2 \in \boldsymbol{Z}_{0+}$, subject to zero delayed initial controls $u(-i,-i)=0$; $\forall i \in \bar{d}'$ and to initial conditions $x_h\left(-i,-i\right) \in \boldsymbol{R}^{n_h}$, $x_v(-i,-i) \in \boldsymbol{R}^{n_v}$; $\forall i \in \bar{d} \cup \{0\}$, where all the matrices of parameters are of corresponding compatible orders with the corresponding vectors. A definition of the whole state composed by the horizontal and vertical substates and the extended control for Model 1 is:

$$\begin{aligned} x\left(n_1,n_2;n_3,n_4\right) &= \left(x_h^T(n_1,n_2), x_v^T(n_3,n_4)\right)^T; \\ u(n_1,n_2;n_3,n_4) &= \left(u^T(n_1,n_2), u^T(n_3,n_4)\right)^T \end{aligned} \tag{12}$$

In order to simplify the above notation, if $n_3=n_1$ and $n_4=n_2$, we can suppress identical arguments and simplify the notation with the replacement $x\left(n_1,n_2;n_1,n_2\right) \to x\left(n_1,n_2\right)$. Thus, one gets for Model 1, the following compacted description from (4) to (8):

$$\begin{aligned} & x\left(n_1+1,n_2+1\right) \\ & = Ax\left(n_1,n_2\right) + A_d x\left(n_1-d,n_2-d\right) + P_d x\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d\right) \\ & \quad + Bu\left(n_1,n_2\right) + B_{d'}u\left(n_1-d',n_2-d'\right) + Qu\left(n_1+1,n_2;n_1,n_2+1\right) \\ & \quad + Q_{d'}u\left(n_1+1-d',n_2-d';n_1-d',n_2+1-d'\right) \end{aligned} \tag{13}$$

where

$$\begin{aligned} & A = \begin{bmatrix} A_{hh} & A_{hv} \\ A_{vh} & A_{vv} \end{bmatrix} = PM \in \boldsymbol{R}^{n\times n};\ A_d = \begin{bmatrix} A_{hhd} & A_{hvd} \\ A_{vhd} & A_{vvd} \end{bmatrix} = PM_d \in \boldsymbol{R}^{n\times n}; \\ & B = \begin{bmatrix} B_h \\ B_v \end{bmatrix} = PN \in \boldsymbol{R}^{n\times m};\ B_{d'} = \begin{bmatrix} B_{hd'} \\ B_{vd'} \end{bmatrix} = PN_{d'} \in \boldsymbol{R}^{n\times m}; \\ & C = \begin{bmatrix} C_h & C_v \end{bmatrix} \in \boldsymbol{R}^{p\times n};\ P = \begin{bmatrix} P_{hh} & P_{hv} \\ P_{vh} & P_{vv} \end{bmatrix} \in \boldsymbol{R}^{n\times n}; \\ & M = \begin{bmatrix} M_{hh} & M_{hv} \\ M_{vh} & M_{vv} \end{bmatrix} \in \boldsymbol{R}^{n\times n};\ M_d = \begin{bmatrix} M_{hhd} & M_{hvd} \\ M_{vhd} & M_{vvd} \end{bmatrix} \in \boldsymbol{R}^{n\times n}; \\ & P_d = \begin{bmatrix} P_{hhd} & P_{hvd} \\ P_{vhd} & P_{vvd} \end{bmatrix} \in \boldsymbol{R}^{n\times n};\ N = \begin{bmatrix} N_h \\ N_v \end{bmatrix} \in \boldsymbol{R}^{n\times m};\ N_{d'} = \begin{bmatrix} N_{hd'} \\ N_{vd'} \end{bmatrix} \in \boldsymbol{R}^{n\times m}; \\ & Q = \begin{bmatrix} Q_{h1} & Q_{h2} \\ Q_{v1} & Q_{v2} \end{bmatrix} \in \boldsymbol{R}^{n\times 2m};\ Q_{d'} = \begin{bmatrix} Q_{h1d'} & Q_{h2d'} \\ Q_{v1d} & Q_{v2d} \end{bmatrix} \in \boldsymbol{R}^{n\times 2m} \end{aligned} \tag{14}$$

b) Model 2: Recursive model with delayed space and time dynamics

A more general model including the property that the output can be defined even if the horizontal and vertical substates are evaluated at different arguments. Furthermore, time evolution in the previous model is allowed subject to eventual delayed dynamics of temporal discrete delay $r \in \boldsymbol{Z}_+$ and a delayed control of temporal discrete delay $r' \in \boldsymbol{Z}_+$ is got from (13) and (14) under the form:

$$\begin{aligned}
&x(n_1+1, n_2+1; k+1) \\
&= \Psi(Ax(n_1, n_2; k) + A_d x\left(n_1 - d, n_2 - d; k\right) \\
&\quad + P_d x\left(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d; k\right)) \\
&\quad + \Psi_r(Ax(n_1, n_2; k - r) + A_d x\left(n_1 - d, n_2 - d; k - r\right) \\
&\quad + P_d x\left(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d; k - r\right)) \\
&\quad + \Gamma(Bu(n_1, n_2; k) + B_{d'} u(n_1 - d', n_2 - d'; k) + Qu(n_1 + 1, n_2; n_1, n_2 + 1; k) \\
&\quad + Q_{d'} u\left(n_1 + 1 - d', n_2 - d'; n_1 - d', n_2 + 1 - d'; k\right)) \\
&\quad + \Gamma_{r'}(Bu(n_1, n_2; k - r') + B_{d'} u(n_1 - d', n_2 - d'; k - r') \\
&\quad + Qu(n_1 + 1, n_2; n_1, n_2 + 1; k - r') \\
&\quad + Q_{d'} u(n_1 + 1 - d', n_2 - d'; n_1 - d', n_2 + 1 - d'; k - r'))
\end{aligned} \tag{15}$$

$$y\left(n_1, n_2; k\right) = C_h x_h\left(n_1, n_2; k\right) + C_v x_v\left(n_1, n_2; k\right) + Du\left(n_1, n_2; k\right) \tag{16}$$

$\forall n_1, n_2, n_3, n_4, k \in \boldsymbol{Z}_{0+}$, where $x\left(n_1, n_2; n_3, n_4; k\right) = \left(x_h^T\left(n_1, n_2; k\right), x_v^T\left(n_3, n_4; k\right)\right)^T$ (abbreviated by $x\left(n_1, n_2; k\right)$ if $n_3 = n_1$ and $n_4 = n_2$), of order $n = n_h + n_v$, subject to zero delayed initial controls $u\left(-i, -i, -j\right) = 0$; $\forall i \in \bar{d}'$; $\forall j \in \bar{r}$ and to initial conditions $x_h\left(-i, -i, -j\right) \in \boldsymbol{R}^{n_h}$, $x_v\left(-i, -i, -j\right) \in \boldsymbol{R}^{n_v}$; $\forall i \in \bar{d} \cup \{0\}$; $\forall j \in \bar{r} \cup \{0\}$, where:

$$\Psi = \begin{bmatrix} \Psi_{hh} & \Psi_{hv} \\ \Psi_{vh} & \Psi_{vv} \end{bmatrix} \in \boldsymbol{R}^{n \times n}; \quad \Psi_r = \begin{bmatrix} \Psi_{hhr} & \Psi_{hvr} \\ \Psi_{vhr} & \Psi_{vvr} \end{bmatrix} \in \boldsymbol{R}^{n \times n} \tag{17}$$

$$\Gamma = \begin{bmatrix} \Gamma_{hh} & \Gamma_{hv} \\ \Gamma_{vh} & \Gamma_{vv} \end{bmatrix} \in \boldsymbol{R}^{n \times n}; \quad \Gamma_{r'} = \begin{bmatrix} \Gamma_{hhr'} & \Gamma_{hvr'} \\ \Gamma_{vhr'} & \Gamma_{vvr'} \end{bmatrix} \in \boldsymbol{R}^{n \times n} \tag{18}$$

The integer arguments n_1 and n_2 govern the spatial horizontal and vertical substates and the discrete argument k stands for the time evolution. Note that the dynamics can eventually have space coordinates memory, associated with "d" and internal (i.e., in the state) and external (i.e., in the input) discrete delays r and r', respectively. However, Model 2 has to be properly considered a two-dimensional one since it cannot be considered a three-dimensional model because the time evolution does not generate a new substate while it governs the two spatial substates. Related to the literature note that each spatial node is defined by a pair of arguments (n_1, n_2). Furthermore in Model 2, a third argument for the sampling instants is considered. Note that in the standard Fornasini-Marchesini model, which is a special case of 2D-model, the nodes are just characterized by only one argument (say, n_1) while the second argument (say, n_2) is interpreted as a sampling instant in the illustrative example of a vehicle path under surveillance [21]. See also [22] for control laws with a well-defined physical basis. We can keep the above interpretation concerning Model 1 for the vehicle path following surveillance with the incorporated eventual contribution of the "spatial" delay d. Alternatively, we can consider that both arguments (n_1, n_2) of the node jointly describe, respectively, a position for one spatial coordinate (say $n_1 = m$) and neighboring positions of the second spatial coordinate (say $n_2 \in \{m + \ell : \ell \in \{m - m_0, m + m_0\}\}$) for some positive integer m_0 of interest in the problem. However, in other problems like, for instance, image processing both arguments can describe generated points in the plane with eventual contribution of neighboring delayed points under spatial delay d (Model 2), while the temporal delays, in the event they contribute to the dynamics, are governed under the temporal arguments r and r'.

Remark 3.1. *Note that, if* $\Psi = \Psi_r = \Gamma = \boldsymbol{I}_n$ *and* $\Gamma_{r'} = 0$, *then the third argument of the various signals is fixed to* $k = 0$ *and can be omitted so that Model 2 becomes, in particular, Model 1 where only the spatial substates are relevant. Note also that all possible particular cases by deleting delays of the set of delays* d, d', r *and* r' *are also included in Model 2 by zeroing the appropriate matrices which parameterize the corresponding neglected dynamics.*

4. **Stability Properties of the Recursive Two-Dimensional Models.** Sufficient conditions of stability of the unforced Model 2, Equations (15) and (16), subject to (14) and (17) and (18), follow.

Theorem 4.1. *Let non-negative real constants* ρ_M, ρ_P, ρ_Ψ *be such that* $\rho_M \geq \|M\|_2$, $\rho_P \geq \|P\|_2$, $\rho_\Psi \geq \|\Psi\|_2$ *and* $\rho = \rho_M \rho_P \rho_\Psi$. *Assume that* ΨPM *is a convergent matrix, then satisfying* $\|\Psi PM\|_2 \leq \rho \in [0,1)$, *and assume also that* $\|A_d\|_2$, $\|P_d\|_2$ *and* $\|\Psi_r\|_2$ *are small enough related to* ρ *satisfying an explicit condition given in the proof. Then, the unforced Model 2 is globally asymptotically stable. Also, the unforced Model 1 is globally asymptotically stable under a parallel sufficiently smallness condition of* $\|A_d\|_2$, $\|P_d\|_2$ *and* $\Psi = \Psi_r = I_n$ *if a redefined* $\rho = \rho_M \rho_P$ *satisfies* $\rho \in (0,1)$ *and* $\|A_d\|_2$, $\|P_d\|_2$ *are small enough related to such a* ρ.

Proof: Since the unforced Model 2 includes the unforced Model 1 as particular case for $A_d = 0$, it suffices to discuss the global asymptotic stability for Model 2. Proceeding recursively with the unforced Equation (15) yields:

$$
\begin{aligned}
& x(i+\ell, i+\ell; s+k) \\
={}& (\Psi A)^i x(\ell,\ell;k) + \sum_{j=0}^{i-1} (\Psi A)^j A_d x(i-1+\ell-j-d, i-1+\ell-j-d; s+k-1) \\
& + \sum_{j=0}^{i-1} (\Psi A)^j P_d x(i+\ell-j-d, i-1+\ell-j-d; i-1+\ell-j-d, \\
& i+\ell-j-d; s+k-1) \\
& + \sum_{q=0}^{i-1}\sum_{j=0}^{r-1} (\Psi A)^{q+j} \Psi_r A x(q-1+\ell, q-1+\ell; s+k-j-1-r) \\
& + \sum_{q=0}^{i-1}\sum_{j=0}^{r-1} (\Psi A)^{q+j} \Psi_r A_d x(q-1+\ell-d, q-1+\ell-d; s+k-j-1-r) \\
& + \sum_{q=0}^{i-1}\sum_{j=0}^{r-1} (\Psi A)^{q+j} \Psi_r P_d x(q+\ell-d, q-1+\ell-d; q-1+\ell-d, \\
& q+\ell-d; s+k-j-1-r); \\
& \forall i,\ell,k \in \boldsymbol{Z}_{0+},\ \forall i,s \in \boldsymbol{Z}_{+}
\end{aligned}
\tag{19}
$$

Since $\Psi A = \Psi PM$, $\rho_M \geq \|M\|_2$, $\rho_P \geq \|P\|_2$ and $\rho_\Psi \geq \|\Psi\|_2$ and $\rho \in (0,1)$ then $\sum_{j=0}^{\ell-1} \rho^j = \frac{1-\rho^\ell}{1-\rho}$; $\forall \ell \in \boldsymbol{Z}_{0+}$, and

$$(\Psi A)^i \leq (\|\Psi\|_2 \|P\|_2 \|M\|_2)^i \leq (\rho_M \rho_P \rho_\Psi)^i = \rho^i;\ \forall i \in \boldsymbol{Z}_{0+}, \left\{\left\|(\Psi A)^i\right\|_2\right\} \to 0 \text{ as } i \to \infty$$

so that one gets from (19) that

$$\begin{aligned} &\|x(i+\ell,i+\ell;s+k)\|_2 \\ &\leq \rho^i \|x(\ell,\ell;k)\|_2 + K(i,r) \sup_{\substack{j_1=0,1;j_2=\overline{i-1}\cup\{0\};\\ j_3\in\overline{r-1}\cup\{0\}}} \big\|x(j_2-d,j_2-j_1-d; \\ &\quad j_2-j_1-d,j_2-d;s+k-j_3-r)\big\|_2 \end{aligned} \tag{20}$$

where

$$\begin{aligned} K(i,r) &= \frac{1}{1-\rho}\left[(1+\|\Psi_r\|_2)\left(\|A_d\|_2+\|P_d\|_2\right)\left(2-\rho^i-\rho^{r+i}\right)\right. \\ &\quad \left. +\|A\|_2\|\Psi_r\|_2\left(1-\rho^{r+i}\right)\right] \\ &\leq K_s = 2\left(1+\|\Psi_r\|_2\right)\left(\|A_d\|_2+\|P_d\|_2\right)+\|A\|_2\|\Psi_r\|_2 \end{aligned} \tag{21}$$

If $K_s < 1-\rho$, then it follows that $\rho^i + K(i,r) \leq \rho^i + K_s < 1$ for any $i \in \boldsymbol{Z}_+$ since $\rho < 1$. Thus, one has from (21). As a result, one gets:

$$\begin{aligned} &\|x(i+\ell,i+\ell;s+k)\|_2 \\ &< x_s(i,\ell,s,k) \\ &= \sup_{\substack{j_1=0,1;j_2=\overline{i-1}\cup\{0\};\\ j_3\in\overline{r-1}\cup\{0\}}} \|x(j_2-d,j_2-j_1-d;j_2-j_1-d,j_2-d;s+k-j_3-r)\|_2 \end{aligned} \tag{22}$$

$\forall i\,(\geq i_0) \in \boldsymbol{Z}_{0+}$. Since i_0 is finite $\|x(i+\ell,i+\ell;s+k)\|_2 < +\infty$ from (21) for any given $\ell, k \in \boldsymbol{Z}_{0+}$ if the initial conditions are finite and then $\|x(i+\ell,i+1+\ell;i+1+\ell,i+\ell;s+k)\|_2 < +\infty$. Also, for this boundedness property and Equations (7)-(9) which govern the generation of the state-trajectory solution, the system is globally stable if $\rho \in (0,1)$ and $K_s < 1-\rho$. Furthermore, from the strict inequality in (22), there exists a monotonically strictly decreasing sequence of supreme $\{x_s(\ell_{k+1}-\ell_k,\ell_k,s_{k+1}-s_k,s_k)\}_{k\in\boldsymbol{Z}_{0+}}$, where $\{\ell_k\}_{k\in\boldsymbol{Z}_{0+}}$ and $\{s_k\}_{k\in\boldsymbol{Z}_{0+}}$ are monotonically strictly increasing sequences of non-negative integers, then $\{x_s(\ell_{k+1}-\ell_k,\ell_k,s_{k+1}-s_k,s_k)\}_{k\in\boldsymbol{Z}_{0+}} \to 0$, $\{\|x(i+\ell,i+\ell;s+k)\|_2\} \to 0$ and $\{\|x(i+\ell,i+1+\ell;i+1+\ell,i+\ell;s+k)\|_2\} \to 0$. The output is also bounded and converges asymptotically to zero, as a result. Then, Model 2 is, in fact globally asymptotically stable if $\rho \in (0,1)$ and $K_s < 1-\rho$. The conditions for global asymptotic stability of Model 1 are modified as follows:

$$K_s = 2\left(\|A_d\|_2+\|P_d\|_2\right) < 1-\rho_M\rho_P \leq 1 \tag{23}$$

under a redefinition of the convergence abscissa of PM as $\rho = \rho_M\rho_P$ such that $\|PM\|_2 \leq \rho_M\rho_P \in [0,1)$ since $\Psi_r = 0$, $\Psi = I_n$. □

The following result extends Theorem 4.1 to the case of state linear time-invariant delay-free feedback control in the absence of external delay d'.

Theorem 4.2. *Assume that the external delay is zero, so that* $Q_{h1} = Q_{h2} = 0$, $N_{hd'} = Q_{h1d'} = Q_{h2d'} = 0$, $Q_{v1} = Q_{v2} = 0$ *and* $N_{vd'} = Q_{v1d'} = Q_{v2d'} = 0$ *and assume also that* P *commutes with* Γ, *so that there exists a matrix* Γ_1 *such that* $\Gamma P = P\Gamma_1$. *Assume, furthermore, that the subsequent linear time-invariant state-feedback law is applied:*

$$u(n_1,n_2;k) = Kx(n_1,n_2;k) \tag{24}$$

with a controller gain $K = \begin{bmatrix} K_h & K_v \end{bmatrix} \in \boldsymbol{R}^{m\times n}$. *Let the non-negative real constants* ρ_P, ρ_Ψ *fulfil the conditions of Theorem 4.1 and consider* $M_c = M + \Gamma_1 NK$ *with* $\rho_{M_c} \geq \|M+\Gamma_1 NK\|_2$, *define* $\rho_c = \rho_{M_c}\rho_P\rho_\Psi$ *and assume that* $\Psi P M_c$ *is a convergent matrix, then satisfying* $\|\Psi P M_c\|_2 \leq \rho_c \in [0,1)$. *Thus, the following properties hold:*

(i) *Model 2 is globally asymptotically stable if* $\|A_d\|_2$, $\|P_d\|_2$ *and* $\|\Psi_r\|_2$ *are small enough related to* ρ_c. *Also, Model 1 subject to the same control law is globally asymptotically stable under a parallel sufficiently smallness condition of* $\|A_d\|_2$, $\|P_d\|_2$ *if* ρ_c, *redefined as* $\rho_c = \rho_{M_c}\rho_P$, *satisfies* $\rho_c \in (0,1)$ *and* $\|A_d\|_2$, $\|P_d\|_2$ *are small enough related to such a* ρ_c.

(ii) *Assume that the pair* $(M, \Gamma_1 N)$ *is completely controllable. Then, there exists a control law (24) such that the Model 2 is globally asymptotically stable if*

$$2\left(1+\|\Psi_r\|_2\right)\left(\|A_d\|_2+\|P_d\|_2\right)+\|P\left(M+\Gamma_1 NK\right)\|_2\|\Psi_r\|_2<1 \tag{25}$$

For Model 1, the constraint that P *commutes with* $\Gamma = I_n$ *has nonsense and it is then deleted, and the parallel sufficiency-type condition of global asymptotic stability to the above one becomes:*

$$2\left(\|A_d\|_2+\|P_d\|_2\right)<1 \tag{26}$$

(iii) *Assume, furthermore, that: a)* $D = 0$, *b) the triple* $(M, \Gamma_1 N, C)$ *is stabilizable and detectable, c) the pair* $(M, \Gamma_1 N)$ *has* n_f *stable fixed modes via linear output feedback, and d)* $\max(p,m) \geq n - n_f$. *Then, there exists a linear time-invariant output-feedback law:*

$$u(n_1,n_2;k)=K_o y(n_1,n_2;k) \tag{27}$$

of controller gain $K_o \in \boldsymbol{R}^{m\times p}$ *such that the Model 2 is globally asymptotically stable if*

$$K_{so}=2\left(1+\|\Psi_r\|_2\right)\left(\|A_d\|_2+\|P_d\|_2\right)+\|P\left(M+\Gamma_1 NK_oC\right)\|_2\|\Psi_r\|_2<1-\rho_m\rho_P\rho_\Psi \tag{28}$$

with $\rho_m < 1/\rho_P\rho_\Psi$ *being the maximum modulus of the stable fixed modes of the pair* $(M, \Gamma_1 N)$. *For Model 1, the condition (28) is replaced with*

$$K'_{so}=2\left(\|A_d\|_2+\|P_d\|_2\right)<1-\rho_m\rho_P \tag{29}$$

with $\rho_m < 1/\rho_P$.

Proof: The closed-loop system obtained from (13) is parameterized with $B_{d'} = N_{d'} = 0 \in \boldsymbol{R}^{n\times m}$; $Q = Q_{d'} = 0 \in \boldsymbol{R}^{n\times 2m}$ in (14) under the following replacement:

$$\begin{aligned}A=PM\to A_c&=A+\Gamma BK=PM+\Gamma PNK=P\left(M+\Gamma_1 NK\right)\\&=\begin{bmatrix}M_{hh}+B_hK_h & A_{hv}+B_hK_v\\ A_{vh}+B_vK_h & A_{vv}+B_vK_v\end{bmatrix}=PM_c\end{aligned} \tag{30}$$

where $M_c = M + \Gamma_1 NK$. The proof of Property (i) for the controlled Model 2 follows in a close way as that of Theorem 4.1 since the convergence abscissa of $\Psi A = \Psi PM_c$ is at most $\rho_c = \rho_{M_c}\rho_P\rho_\Psi$, where ρ_M is that of M_c. For Model 1, the proof is close to that for Model 2 with $\rho_c = \rho_{M_c}\rho_P$. To prove Property (ii), note that if (M, N), for each given set of complex values $\Lambda = \{z_1, z_2, \ldots, z_n\}$ there is a controller gain matrix $K = \begin{bmatrix} K_h & K_v \end{bmatrix} \in \boldsymbol{R}^{m\times n}$ such that all the eigenvalues of $M_c = M + \Gamma_1 NK$ are located at the set Λ since controllability is equivalent to pole-assignment in the time-invariant case. Thus, the eigenvalues of M_c can be fixed arbitrarily and the global asymptotic stability follows if (25) holds via (21). In particular, ρ_{M_c} and then ρ_c, can be fixed arbitrarily close to zero so that M_c is also arbitrarily close to zero and the global asymptotic stability follows if (26) from (25). For Model 1, the condition is (26) by fixing ρ_c closely to zero by the choice of the controller gain matrix K according to (23). To prove Property (iii), note from (27) that $u\left(n_1,n_2;k\right) = K_oCx\left(n_1,n_2;k\right)$ since $D = 0$. Since $(M, \Gamma_1 N, C)$ is stabilizable and detectable and $(M, \Gamma_1 N)$ has n_f stable fixed modes which cannot be reallocated by linear output feedback and, since $\max\left(p,m\right) \geq n - n_f$, then there are at least $(n - n_f)$ eigenvalues of $M_{co} = M + \Gamma_1 NK_oC$ which can be fixed to arbitrary stable positions. So such allocations might be chosen with at least as identical stability degree as the stable fixed modes by the choice of an output feedback gain K_o. Then, the convergence abscissa of $A_{co} = PM_{co}$ is $\rho_m \rho_P \rho_\Psi < 1$ and the proof follows for Model 2 if (28) holds. For Model 1, the counterpart condition is (29) with $\rho_c = \rho_m \rho_P < 1$. □

Remark 4.1. *Assume that all the assumptions of Theorem 4.2 (iii) hold except that the interconnection input-output matrix D is zero. Then the output feedback control law (27) is replaced with*

$$u(n_1, n_2; k) = (I_m - K_o D)^{-1} K_o C y(n_1, n_2; k) \tag{31}$$

provided that $(I_m - K_o D)$ is nonsingular. Thus, there exists K_o satisfying the identity $X = (I_m - K_o D)^{-1} K_o C$, equivalently, $K_o C = (I_m - K_o D) X$, for a given real $m \times n$ matrix X if there is a solution to the subsequent equivalent vectorized algebraic equation:

$$vec(X) = \left(vec\left(I_m \otimes C^T\right) + vec\left(I_m \otimes X^T D^T\right)\right) vec(K_o) \tag{32}$$

provided from the Rouché-Froebenius theorem that

$$\begin{aligned} &rank\left(vec\left(I_m \otimes C^T\right) + vec\left(I_m \otimes X^T D^T\right)\right) \\ &= rank\left(vec\left(I_m \otimes C^T\right) + vec\left(I_m \otimes X^T D^T\right), vec(X)\right) \end{aligned} \tag{33}$$

where the symbol "$\otimes$" stands for the Kronecker product of matrices and $vec(X)$ is the vector associated to X whose components are the entries of X written component-wise ordered by its row vectors. Thus, Theorem 4.2(iii) holds for Model 2 if there is some solution for control gain K_o from (32), under the necessary condition (33), for some $X \in \boldsymbol{R}^{m \times n}$ which prefixes the non-fixed modes of M_c to a value being non larger than the convergence abscissa $\rho_m < 1/\rho_P \rho_\Psi$ of the stable fixed modes. For Model 1, a similar condition can be got. Note that it is not always guaranteed under the given conditions that for any nonzero D, there exists some controller gain matrix K_o such that (32) is solvable for some X of appropriate order which prefixes the convergence abscissa of the achievable non-fixed modes of M_c to a value being non larger than the convergence abscissa of its fixed modes. The property is guaranteed if for some such a matrix X (33) holds so that (32) is a compatible algebraic system.

It turns out that Theorem 4.2(ii) can be formulated under more restrictive conditions if $(M, \Gamma_1 N)$ is stabilizable (but not controllable). That means that if M possesses unstable and/or critically stable modes, there exists controller gain matrix K such that $M_c = M + \Gamma_1 N K$ is a convergence matrix but its spectral radius cannot be necessarily prefixed for some K although an M_c can be achieved being convergent by such a control law. It turns out that if $rank\,[M - zI_n \;\; \Gamma_1 N] = n$ for any complex z then $(M, \Gamma_1 N)$ is controllable and conversely and also stabilizable as a result. However, if the above rank condition only holds for $|z| \geq 1$ then the pair $(M, \Gamma_1 N)$ is just stabilizable but not controllable (Popov-Belevitch-Hutus rank test). The following related result can be proved.

Corollary 4.1. *Assume that the conditions of Theorem 4.2 hold and, furthermore, the pair $(M, \Gamma_1 N)$ is stabilizable. Then, there exists a linear time-invariant state-feedback control law (24) of controller gain matrix $K = \begin{bmatrix} K_h & K_v \end{bmatrix} \in \boldsymbol{R}^{m \times n}$ such that the Model 2 is globally asymptotically stable with $\rho_{M_c} < 1$ if*

$$2\left(1 + \|\Psi_r\|_2\right)\left(\|A_d\|_2 + \|P_d\|_2\right) + \|P\left(M + \Gamma_1 N K\right)\|_2 \|\Psi_r\|_2 < 1 - \rho_{M_c} \rho_P \rho_\Psi \tag{34}$$

For Model 1, the parallel sufficiency-type condition to the above one is:

$$2\left(\|A_d\|_2 + \|P_d\|_2\right) < 1 - \rho_{M_c} \rho_P \tag{35}$$

for some $\rho_{M_c} < 1$.

5. **Feedback Control Laws.** This section considers Models 1 and 2 under linear feedback control laws. It is assumed that $d \neq d'$ and $r \neq r'$. In the case that those delays are identical, the model can be rewritten grouping terms with similar delays in the open-loop version and eventually under the incorporation of the control laws. Consider Model 1 (13) and (14) under the control law

$$u(n_1, n_2) = K(n_1, n_2)\, x(n_1, n_2) + K_d(n_1 - d, n_2 - d)\, x(n_1 - d, n_2 - d) \tag{36.a}$$

$$\begin{aligned} & u(n_1 + 1, n_2; n_1, n_2 + 1) \\ &= K(n_1 + 1, n_2; n_1, n_2 + 1)\, x(n_1, n_2) + K_d(n_1 + 1 - d, n_2 - d; n_1 - d, \\ &\quad n_2 + 1 - d) x(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \end{aligned} \tag{36.b}$$

The state evolution of the closed-loop system obtained by replacing (36) in (13) is:

$$\begin{aligned} & x(n_1 + 1, n_2 + 1) \\ &= A_c(n_1, n_2) x(n_1, n_2) + A_{dc}(n_1 - d, n_2 - d)\, x(n_1 - d, n_2 - d) \\ &\quad + A_{dco}(n_1 + 1, n_2; n_1, n_2 + 1)\, x(n_1 + 1, n_2; n_1, n_2 + 1 - d) \\ &\quad + A_{dcd} x(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \\ &\quad + B_{d'}(K(n_1 - d', n_2 - d')\, x(n_1 - d', n_2 - d') \\ &\quad + K_d(n_1 - d - d', n_2 - d - d')\, x(n_1 - d - d', n_2 - d' - d)) \\ &\quad + Q_{d'} K(n_1 + 1 - d', n_2 - d'; n_1 - d, n_2 + 1 - d')\, x(n_1 + 1 - d', \\ &\quad n_2 - d'; n_1 - d', n_2 + 1 - d') + Q_{d'} K_d(n_1 + 1 - d' - d, n_2 - d' - d; n_1 - d' - d, \\ &\quad n_2 + 1 - d' - d) x(n_1 + 1 - d' - d, n_2 - d' - d; n_1 - d' - d, n_2 + 1 - d' - d) \end{aligned} \tag{37}$$

with

$$\begin{aligned} & A_c(n_1, n_2; k) = A + BK(n_1, n_2) = P(M + NK(n_1, n_2)) = PM_c(n_1, n_2) \\ & A_{dc}(n_1 - d, n_2 - d) = A_d + BK_d(n_1 - d, n_2 - d) \\ & \qquad\qquad = P(M_d + NK_d(n_1 - d, n_2 - d)) = PM_{dc}(n_1 - d, n_2 - d) \\ & A_{dco}(n_1 + 1, n_2; n_1, n_2 + 1) = QK(n_1 + 1, n_2; n_1, n_2 + 1) \\ & \quad A_{dcd}(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \\ & = P_d + QK_d(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \end{aligned} \tag{38}$$

since $B = PN$ and $B_{d'} = PN_{d'}$ from (14). If $d = d'$ then the term $Q_d K(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d')$ can be incorporated to $A_{dcd}(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d)$ in (38) which becomes modified as follows:

$$\begin{aligned} & A_{dcd}(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \\ &= P_d + QK_d(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \\ &\quad + Q_d K(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d) \end{aligned} \tag{39}$$

The following controllability constraints for Model 1 are then used to establish the stability result.

Assumption 5.1. *The pairs* (M, N), (M_d, N) *and* (P_d, Q) *are completely controllable.*

A generic result for eigenvalue assignment, which has been invoked in the proof of Theorem 4.2, and which then allows setting the spectral radius of a closed-loop matrix by the choice of the control gain is as follows.

Assertion 5.1. *If* (X, Y) *is a completely controllable pair of real matrices, then there is some matrix* Z *with the same number or rows as that of columns of* Y *such that the spectrum of* $(X + YZ)$ *can be assigned to any set of values.*

The subsequent result follows directly from Assumption 5.1, Assertion 5.1 and (37) and (38) by prefixing the spectral radii of $A_c(.)$ and $A_{dc}(.)$ as well in arbitrarily close to zero allocations by fixing ρ_{M_c} and $\rho_{M_{dc}}$ by choosing the controller gains $K(.)$ and $K_d(.)$ since (M, N) and (M_d, N_d) are controllable. The proof is direct from (14), the proof of Theorem 4.1 with the inequality (23) concerning the smallness of a part of the parameterization and that of Theorem 4.2 versus Assertion 5.1 to prefix the spectra, and then their spectral radii, of the above mentioned matrix through the controller gains.

Theorem 5.1. *If Assumption 5.1 holds, then there exists a linear state-feedback control law (36) such that Model 1 is globally asymptotically stable provided that*

$$2\left(\|Q\|_2 \overline{K} + \left(\|PN_{d'}\|_2 + \|Q_{d'}\|_2\right)\left(\overline{K} + \overline{K}_d\right)\right) < 1 - \varepsilon \tag{40}$$

for any given $\varepsilon \in (0, 1)$, where $\overline{K}$ and $\overline{K}_d$ are the supreme on the definition domain of the ℓ_2-norms of the corresponding controller gains $K(.)$ and $K_d(.)$.

Proof: From the controllability hypotheses of Assumption 5.1, the spectral radii of $M_c(.)$, $M_{dc}(.)$, then those of $A_c(.)$ and $A_{dc}(.)$, and $A_{dcd}(.)$ in Formula (28) can be made arbitrarily small by the synthesis of appropriate controller gains $K(.)$ and $K_d(.)$. Then from (37) and Theorem 4.1, the result holds if (40) holds for ε being dependent on the (eligible closely to zero) fixed above spectral radii and that of P. □

Note that (40) can be fulfilled under any controller gains if the norms of Q, $Q_{d'}$ and $N_{d'}$ are correspondingly sufficiently small.

Remark 5.1. *If the assignable spectra in Theorem 5.1 are not fixed as closely to zero locations compatible with (40) in Theorem 5.1, or if some of the controllability constraints of Assumption 5.1 eventually fail to hold, then Theorem 5.1 is directly extendable by modifying the left hand sides of the stability constraints of (40) by adding the norms (or the spectral radii) of the eventually unassigned spectra of $A_c(.)$, $A_{dc}(.)$ and/or $A_{dcd}(.)$ to its left-hand-side and, if $d = d'$, eventually that of $A_{dcd}(.)$ if unassigned and by replacing $1 \to (1 - \rho_M \rho_P)$ in the respective right-hand-side.*

The following controllability constraints for Model 2 are then used to establish the stability result.

Assumption 5.2. *There exist non-necessarily unique matrices Ψ_1 and Γ_1 such that the commuting conditions $\Psi P = P\Psi_1$ and $\Gamma P = P\Gamma_1$ hold (with respective necessary and sufficient conditions being $rank(\Psi) = rank(\Psi P)$ and $rank(\Gamma) = rank(\Gamma P)$), and, furthermore, the pairs (P_d, Q), $(\Psi_1 M, \Gamma_1 N)$ and $(\Psi_1 M_d, \Gamma_1 N)$ are completely controllable.*

Assumption 5.3. *There exist non-necessarily unique matrices Ψ_{1r} and Γ_{1r} such that the commuting conditions $\Psi_r P = P\Psi_{1r}$ and $\Gamma_r P = P\Gamma_{1r}$ hold (with respective necessary and sufficient conditions being $rank\,(\Psi_r) = rank\,(\Psi_r P)$ and $rank\,(\Gamma_r) = rank\,(\Gamma_r P)$), and, furthermore, the pairs $(\Psi_{1r} M, \Gamma_{1r} N)$ and $(\Psi_{1r} M_d, \Gamma_{1r} N)$ are completely controllable.*

The closed-loop Model 2 (15)-(18), under the control law:

$$\begin{aligned} u(n_1, n_2; k) = {} & K(n_1, n_2; k)\, x\,(n_1, n_2; k) \\ & + K_d(n_1 - d, n_2 - d; k)\, x\,(n_1 - d, n_2 - d; k) \end{aligned} \tag{41.a}$$

$$\begin{aligned} u(n_1 + 1, n_2; n_1, n_2 + 1; k) = {} & K(n_1 + 1, n_2; n_1, n_2 + 1; k)\, x\,(n_1, n_2; k) \\ & + K_d(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d; k) \\ & x(n_1 + 1 - d, n_2 - d; n_1 - d, n_2 + 1 - d; k) \end{aligned} \tag{41.b}$$

has a state evolution given by

$$\begin{aligned}
&x\left(n_1+1,n_2+1;k+1\right)\\
=\;&A_{c1}\left(n_1,n_2;k\right)x\left(n_1,n_2;k\right)+A_{dc1}x\left(n_1-d,n_2-d;k\right)\\
&+A_{dco1}\left(n_1+1,n_2;n_1,n_2+1;k\right)x\left(n_1+1,n_2;n_1,n_2+1-d;k\right)\\
&+A_{dcd1}\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k\right)\\
&x\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k\right)\\
&+A_{c1r}\left(n_1,n_2;k-r\right)x\left(n_1,n_2;k-r\right)+A_{dc1r}x\left(n_1-d,n_2-d;k-r\right)\\
&+A_{dcd1r}\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k-r\right)\\
&x\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k-r\right)\\
&+\Gamma B_{d'}u\left(n_1-d',n_2-d';k\right)+\Gamma Q_{d'}u\left(n_1+1-d',n_2-d';n_1-d',n_2+1-d';k\right)\\
&+\Gamma_{r'}\left(Bu\left(n_1,n_2;k-r'\right)+B_{d'}u\left(n_1-d',n_2-d';k-r'\right)\right)\\
&+\Gamma_{r'}(Qu\left(n_1+1,n_2;n_1,n_2+1;k-r'\right)\\
&+Q_{d'}u\left(n_1+1-d',n_2-d';n_1-d',n_2+1-d';k-r'\right))
\end{aligned}\tag{42}$$

with

$$\begin{aligned}
&A_{c1}(n_1,n_2;k)=\Psi PM+\Gamma PNK(n_1,n_2;k)\\
&\qquad\qquad=P\left(\Psi_1M+\Gamma_1NK\left(n_1,n_2;k\right)\right)=PM_{c1}\left(n_1,n_2;k\right)\\
&A_{dc1}\left(n_1,n_2;k\right)=\Psi PM_d+\Gamma PNK\left(n_1,n_2;k\right)\\
&\qquad\qquad=P\left(\Psi_1M_d+\Gamma_1NK_d\left(n_1,n_2;k\right)\right)=PM_{dc1}(n_1,n_2;k)\\
&A_{dco1}\left(n_1+1,n_2;n_1,n_2+1;k\right)=\Gamma QK\left(n_1+1,n_2;n_1,n_2+1;k\right)\\
&\quad A_{dcd1}\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k\right)\\
&=\Psi\left(P_d+QK_d\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k\right)\right)\\
&A_{c1r}(n_1,n_2;k)=\Psi_rPM+\Gamma_rPNK(n_1,n_2;k-r)\\
&\qquad\qquad=P\left(\Psi_{1r}M+\Gamma_{1r}NK\left(n_1,n_2;k-r\right)\right)=PM_{c1r}\left(n_1,n_2;k-r\right)\\
&A_{dc1r}\left(n_1,n_2;k\right)=\Psi_rP\left(M_d+\Gamma_1NK_d\left(n_1,n_2;k\right)\right)=\Psi_rPM_{dc1r}\left(n_1,n_2;k\right)\\
&\quad A_{dcd1r}\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k-r\right)\\
&=\Psi_r\left(P_d+QK_d\left(n_1+1-d,n_2-d;n_1-d,n_2+1-d;k-r\right)\right)\\
&M_{c1}\left(n_1,n_2;k\right)=\Psi_1M+\Gamma_1NK\left(n_1,n_2;k\right)\\
&M_{dc1}\left(n_1,n_2;k\right)=\Psi_1M_d+\Gamma_1NK_d\left(n_1,n_2;k\right)\\
&M_{c1r}\left(n_1,n_2;k-r\right)=\Psi_{1r}M+\Gamma_{1r}NK\left(n_1,n_2;k-r\right)\\
&M_{dc1r}\left(n_1,n_2;k-r\right)=\Psi_{1r}M_d+\Gamma_{1r}NK_d(n_1,n_2;k-r)
\end{aligned}\tag{43}$$

In the same way that Assumption 5.1 leads to Theorem 5.1 for Model 1, Assumptions 5.2 and 5.3 lead to the subsequent related result for Theorem 2.2 by fixing closely to zero some of the spectral radii matrices of (43) being relevant to the dynamics of (42) and re-arranging the rest of matrices in a tolerance to parametrical disturbances smallness condition.

Theorem 5.2. *If Assumptions 5.2 and 5.3 hold, then there exists a linear state-feedback control law (41) such that Model 1 is globally asymptotically stable provided that*

$$\begin{aligned}
&2\left(\|\Gamma\|_2\|Q\|_2\overline{K}+(\|\Gamma\|_2\left(\|PN_{d'}\|_2+\|Q_{d'}\|_2\right)\right.\\
&\left.+\|\Gamma_{r'}\|_2\|P\|_2\left(\|N\|_2+\|N_{d'}\|_2\right))\left(\overline{K}+\overline{K}_d\right)\right)<1-\varepsilon
\end{aligned}\tag{44}$$

for any given $\varepsilon \in (0,1)$, *where* $\overline{K}$ *and* $\overline{K}_d$ *are the supreme on the definition domain of the* ℓ_2*-norms of the corresponding dynamics controller gains* $K(.)$ *and* $K_d(.)$.

In Theorem 5.2, in view of (43), ε can be fixed arbitrarily close to zero for given norms of P and Ψ by choosing arbitrarily close to zero the assignable spectra, via the choice of the controller gains, of the corresponding matrices of dynamics in (43) from the controllability conditions in Assumptions 5.2 and 5.3. A trade-off for the spectra assignation can be modulated by the values of the norms of P and Ψ. For instance, if Ψ is convergent with small spectral radius, then the assigned spectra of $A_{dcd1}(.)$ can be made so that the spectral radius is larger while compatible with (44).

If some of the controllability conditions in Assumptions 5.2 and 5.3 fail, then the condition (44) is modified by incorporating to its contribution via the parametrical matrices to the dynamics in (42) whose spectra are not assigned to prefixed values.

Remark 5.2. *Weaker conditions guaranteeing stability which are weaker than those of the norms can be formulated on the matrices spectral radii in the various obtained results. Although the sufficiency-type conditions are weaker, the price to be paid is the need for the computation of the spectral radius and the translation of the spectral radii tolerance conditions into concrete conditions on the disturbance matrices. Let a complex square matrix be* $X = X_0 + \widetilde{X}$ *where* $\widetilde{X}$ *is a disturbance of a nominal matrix* X_0. *Assume that* $\left\|\widetilde{X}\right\| \le \lambda \left\|X_0\right\|$ *where* λ *is a non-negative scalar independent of the type of induced norm. Thus, one has:*

$$r\left(X_0 + \widetilde{X}\right) = \inf \left\|X_0 + \widetilde{X}\right\| \le (1+\lambda) \inf \left\|X_0\right\| = (1+\lambda)\, r\left(X_0\right) < 1$$

provided that $r\left(X_0\right) < (1+\lambda)^{-1}$. *Thus, if* X_0 *is convergent with spectral radius less than* $(1+\lambda)^{-1}$, *then* $\left(X_0 + \widetilde{X}\right)$ *is convergent if* $\left\|\widetilde{X}\right\| \le \lambda \left\|X_0\right\|$ *since a matrix is convergent if and only if its spectral radius is less than unity. Those conditions can be used for spectral radius of the given results rather than for norms. For instance,* ρ_M, ρ_P, ρ_Ψ *in Theorem 4.1 could be the spectral radii of the corresponding matrices or some available upper-bounds.*

A more restrictive condition can be established with norms as follows as addressed in the obtained results of the paper. If $\left\|X_0\right\| < 1$, *then* X_0 *is convergent, and* $\left\|\widetilde{X}\right\| < 1 - \left\|X_0\right\|$ *then* $\left(X_0 + \widetilde{X}\right)$ *is convergent. If* $\left\|\widetilde{X}\right\| \le \lambda \left\|X_0\right\| < 1 - \left\|X_0\right\|$, *then a sufficient condition for* $\left(X_0 + \widetilde{X}\right)$ *to be convergent is that* $\left\|X_0\right\| < (1+\lambda)^{-1}$.

Remark 5.3. *It is of interest to give guidelines about how to proceed when the state is not available for measuring so that only the output can be used for control feedback. Mathematically, the problem is formulated in similar terms as above through this section by replacing the controller gains* K *and* K_d *with* KC *and* K_dC, *respectively. However, the (state) controllability cannot be invoked for the performed spectra assignments to reduce the spectral radii to guarantee closed-loop stability. On the other hand, a joint controllability and observability condition of a certain* (A, B, C)*-triple does not guarantee exact spectrum assignment but only approximate one for a maximum of modes equalizing the maximum of the input and output components [18]. It is possible to extend Assumptions 5.1, 5.2 and 5.3 and the associated Theorems 5.1 and 5.2 to the case of linear output feedback. It is possible to proceed according to certain guidelines to be next given if the various* (A, B, C)*-triples, whose spectra are being partially assigned via linear output feedback, are:*

1) *Stabilizable and detectable, i.e., all its critically stable and unstable* $n_{uc} \leq n$ *modes are stable (that is convergent in the discrete framework) and all of them are controllable and observable.*
2) *Any* $n_s = n - n_{uc}$ *stable modes fixed modes, i.e., they are allocated independent on the controller gains so that it cannot be re-allocated via output feedback.*
3) $n_{uc} \leq \overline{m} = \max(p, m)$*, i.e., the number of non-stable modes does not exceed the maximum of the dimensions of the input and output.*
4) *The control and output matrices* B *and* C *of the triple* (A, B, C) *are full column and row rank, respectively.*

It has been proved that linear output-feedback allows to prefix $\max(p, m)$ closed-loop eigenvalues as closely as suited to prescribed positions [18]. Thus, if any triple (A, B, C), whose dynamics control pair (A, B), is referred to in Assumptions 5.1, 5.2 and 5.3 fulfils the above Conditions 1)-4), then it is possible to re-allocate the non-stable (that is non-convergent) closed-loop modes via linear output feedback to close positions to any prescribed set to that the corresponding spectral radii of the resulting matrices can be given by the dominant eigenvalue of the remaining stable or convergent (fixed) modes. This allows the extensions of Theorems 5.1 and 5.2 to the case of linear output feedback. In particular, the respective stability constraints (40) and (44) become modified by re-defining $\overline{K}$, $\overline{K}_d$ with the maxima of the products of the respective controller gains by the output matrix and the parameter ε being related to the stability abscissas of the relevant convergent matrices M_c, M_{dc}, M_{c1}, M_{c1r}, M_{dc1}, M_{dc1r}, etc.

6. **Simulation Examples.**

This section contains some simulation examples to illustrate the effectiveness of the proposed approach. Firstly, the stability conditions stated in Theorem 2.2 regarding the delayed basic 2D-Roesser model will be numerically and graphically checked. Afterwards, the asymptotic stability of Model 1 is shown along with the design of state feedback-type control laws. Thus, consider Equations (4) and (5) with dynamic matrices:

$$A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & 0.2 \\ -0.15 & 0.3 \end{bmatrix}$$

delay $d = 5$ and the absolutely summable boundary conditions:

$$x_h(0, i) = \begin{cases} 5 & 0 \leq i \leq 15 \\ 0 & i > 15 \end{cases}, \quad x_v(i, 0) = \begin{cases} 6 & 0 \leq i \leq 15 \\ 0 & i > 15 \end{cases}$$

The dynamic matrices satisfy $\|A\|_2 = \rho = 0.6 < 1$ and $\|A_d\|_2 = 0.37$ so that $\rho + \|A_d\|_2 = 0.97 < 1$. Consequently, Theorem 2.2 guarantees the convergence of the horizontal and vertical substates to zero when $\max(n_1, n_2) \to \infty$, as Figures 1 and 2 display.

Now, consider the unforced Model 1 described by:

$$P = \begin{bmatrix} 0.05 & 0.1 \\ 0.1 & 0.05 \end{bmatrix}, \ P_d = \begin{bmatrix} 0.05 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \ M = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}, \ M_d = \begin{bmatrix} 0.1 & 0.2 \\ -0.15 & 0.3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.05 & 0.04 \\ 0.04 & 0.05 \end{bmatrix}, \ A_d = \begin{bmatrix} -0.01 & 0.04 \\ 0.0025 & 0.035 \end{bmatrix}, \ N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with initial conditions given by $x_h(-i, -i) = 5$ and $x_v(-i, -i) = 6$ while the delay is $d = 5$. According to Theorem 4.1, we can conclude that the unforced Model 1 described by these matrices is globally asymptotically stable since $\rho_M = 0.6$, $\rho_P = 0.15$, $\rho = \rho_M \rho_P = 0.09 \in (0, 1)$ and $K_s = 2\left(\|A_d\|_2 + \|P_d\|_2\right) = 0.212 \leq 1 - \rho = 0.91$ which is the condition arrived in Theorem 4.1's proof of the 2-norms of A_d and P_d being small enough compared to the value of ρ. Consequently, we can conclude that $x_h(n_1, n_2) \to 0$ and $x_v(n_1, n_2) \to 0$ as $\max(n_1, n_2) \to \infty$. Figure 3 shows the behavior of the $x_h(n_1, n_2)$ component of this system

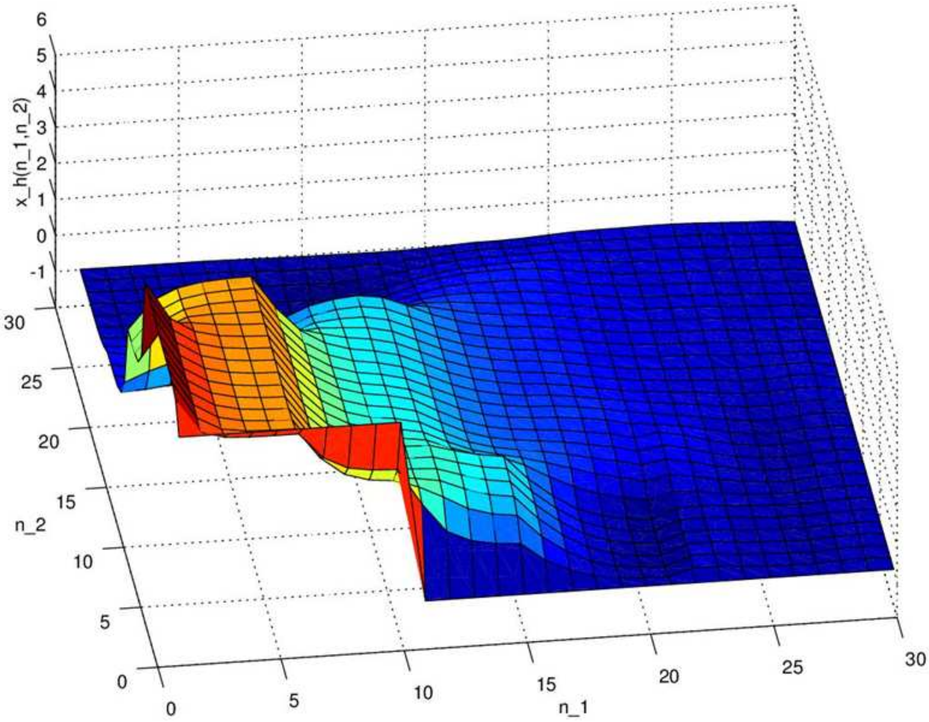


FIGURE 1. Evolution of the $x_h(n_1, n_2)$ substate of the basic delayed Roesser system

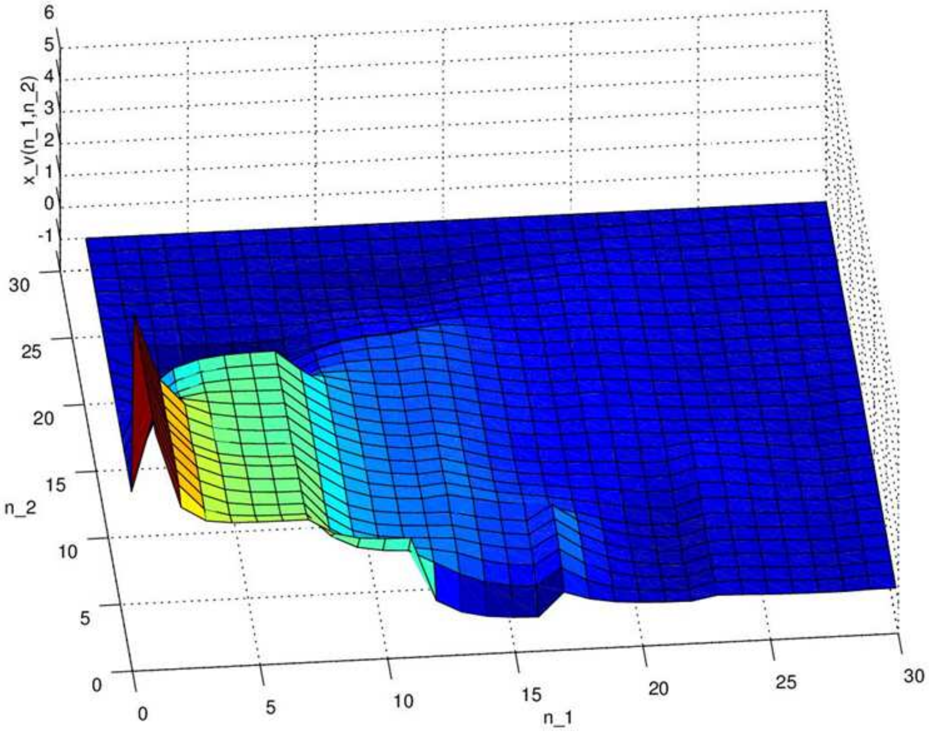


FIGURE 2. Evolution of the $x_v(n_1, n_2)$ substate of the basic delayed Roesser system

while Figure 4 displays the evolution of $x_v(n_1, n_2)$. In both cases, it can be observed that the sub-state converges to zero. It is important to highlight that the stability of the unforced system has been discussed very easily by simply calculating the 2-norm of the system's matrices. This is one of the main advantages of the proposed approach.

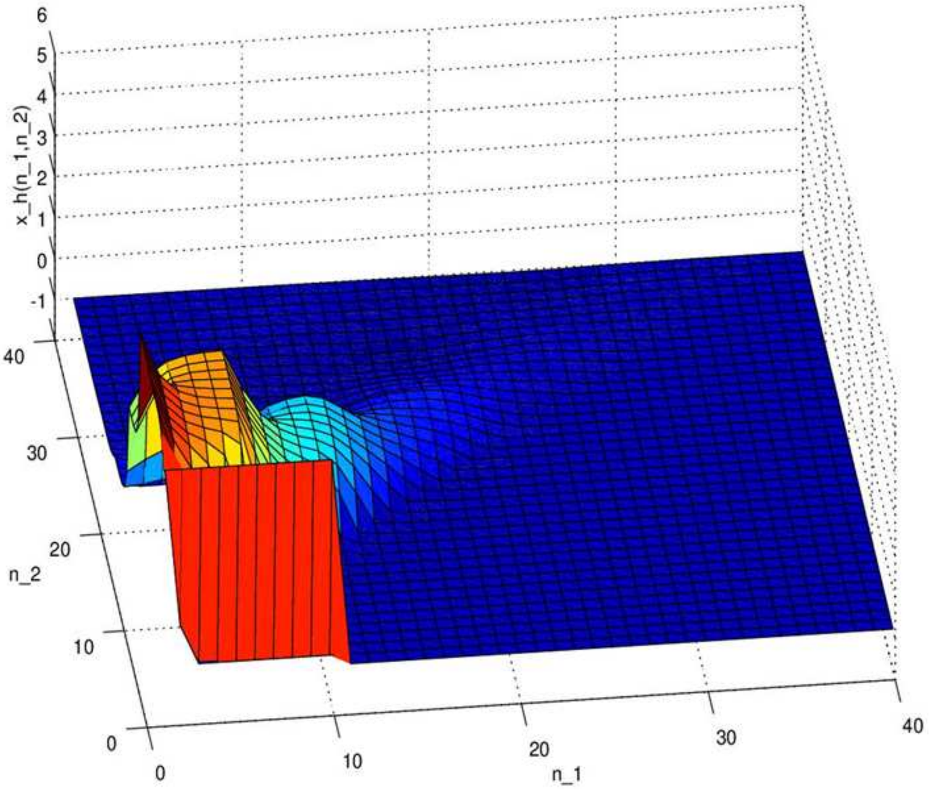


FIGURE 3. Evolution of the $x_h(n_1, n_2)$ substate of the above Model 1 system

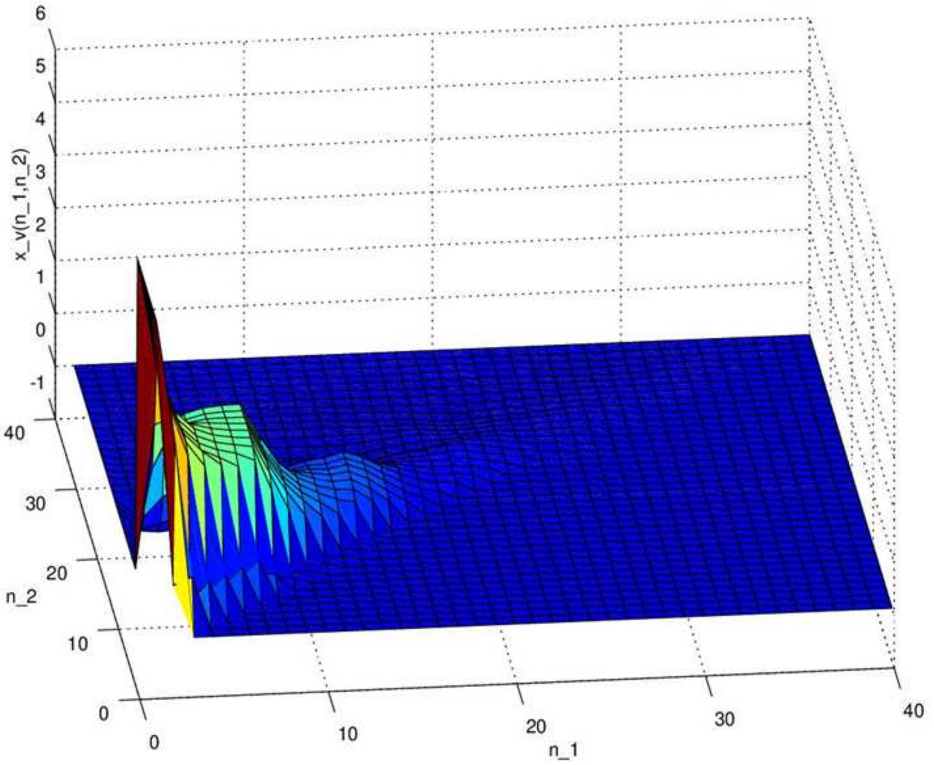


FIGURE 4. Evolution of the $x_v(n_1, n_2)$ substate of the above Model 1 system

Now, the M and P matrices are changed to:

$$M = \begin{bmatrix} 1.2 & 0.9 \\ 0.9 & 0.2 \end{bmatrix}, \quad P = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$$

implying that $\rho_M = 1.73$, $\rho_P = 0.6$, $\rho = \rho_M \rho_P = 1.04 \notin (0, 1)$ while the stability conditions of Theorem 4.1 are no longer satisfied. As a consequence, both components of state diverge as the following Figures 5 and 6 for $x_h(n_1, n_2)$ and $x_v(n_1, n_2)$, respectively, display.

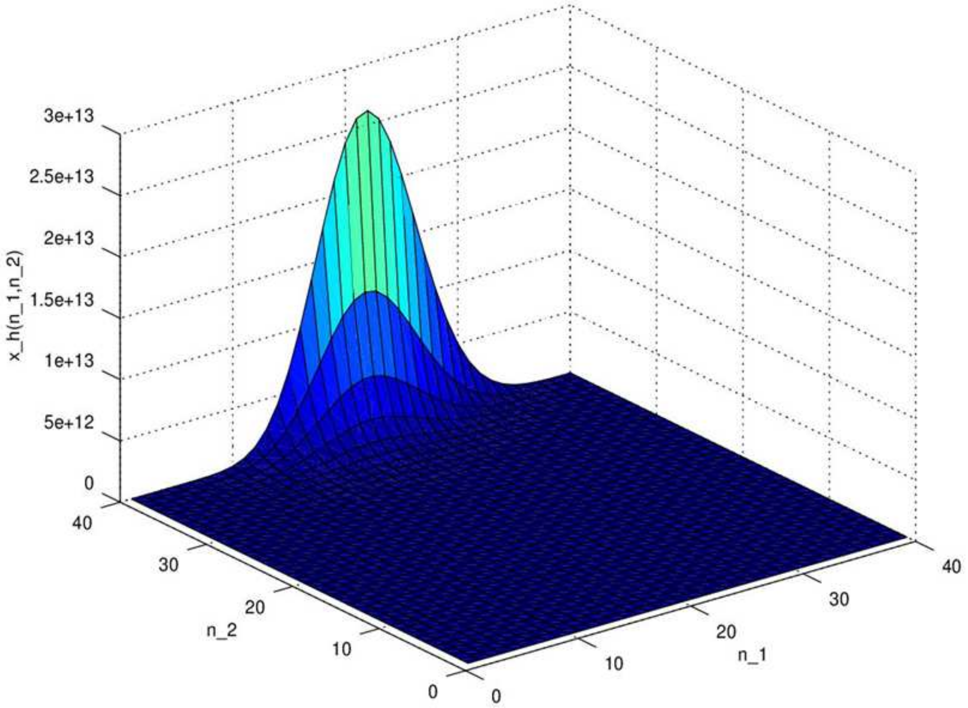


FIGURE 5. Evolution of the $x_h(n_1, n_2)$ substate when the system's matrices change and Theorem 4.1 conditions are not satisfied

Under these circumstances, we can design a state feedback control law in the form of (24) in order to stabilize the system. Thus, we can use any pole-placement algorithm in order to make $M_c = M + NK$ a convergent matrix and satisfy Theorem 4.2 conditions, since the pair (M, N) is controllable. With the choice $K_h = K_v = -1.2$ we have that $\rho_{M_c} = \|M + NK\|_2 = 0.924$, $\rho_c = \rho_{M_c}\rho_P = 0.55 \in (0, 1)$, $\|PM_c\|_2 = 0.46 < \rho_c = 0.55$ and $2(\|A_d\| + \|P_d\|) = 0.54 < 1$. Accordingly, we are in conditions of applying Theorem 4.2 and guaranteeing the convergence to zero of both substates as Figures 7 and 8 show graphically.

The stability of the system is readily checked by the norm-type conditions of Theorem 4.2, which shows the effectiveness of the achieved results. Finally, consider now the Model 2 with the matrices defined originally for Model 1 (4)-(5) and, $r = 5$, $\Psi = I$, $\Psi_r = 0.25I$, $\Gamma = I$ and $\Gamma_{r'} = 0$. In this case we consider the unforced system and will use the stability conditions stated by Theorem 4.2 to check its stability. Therefore, Theorem 4.2 contains as particular case when $K = 0$ the stability conditions of the unforced Model 2. In this way, we have $\|M\|_2 = 0.6$, $\rho_c = \rho_{M_c}\rho_P\rho_\Psi = 0.09 \in (0, 1)$, $\|PM\|_2 = \rho_c$ and condition (25) becomes $2(1 + \|\Psi_r\|_2)(\|A_d\|_2 + \|P_d\|_2) + \|PM\|_2 \|\Psi_r\|_2 = 0.287 < 1$. Consequently, the so defined Model 2 is globally asymptotically stable as can be observed in Figures 9 and 10. Since the third argument of Model 2 represents the time in contrast to the two spatial coordinates represented by n_1 and n_2, each of the substates is represented by a surface

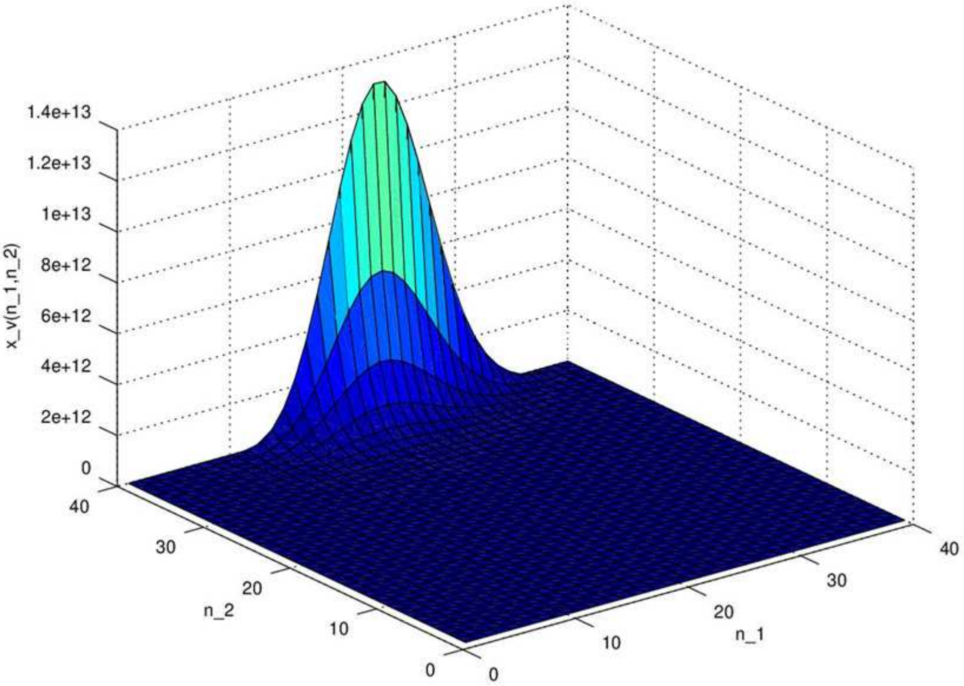


FIGURE 6. Evolution of the $x_v(n_1, n_2)$ substate when the system's matrices change and Theorem 4.1 conditions are not satisfied

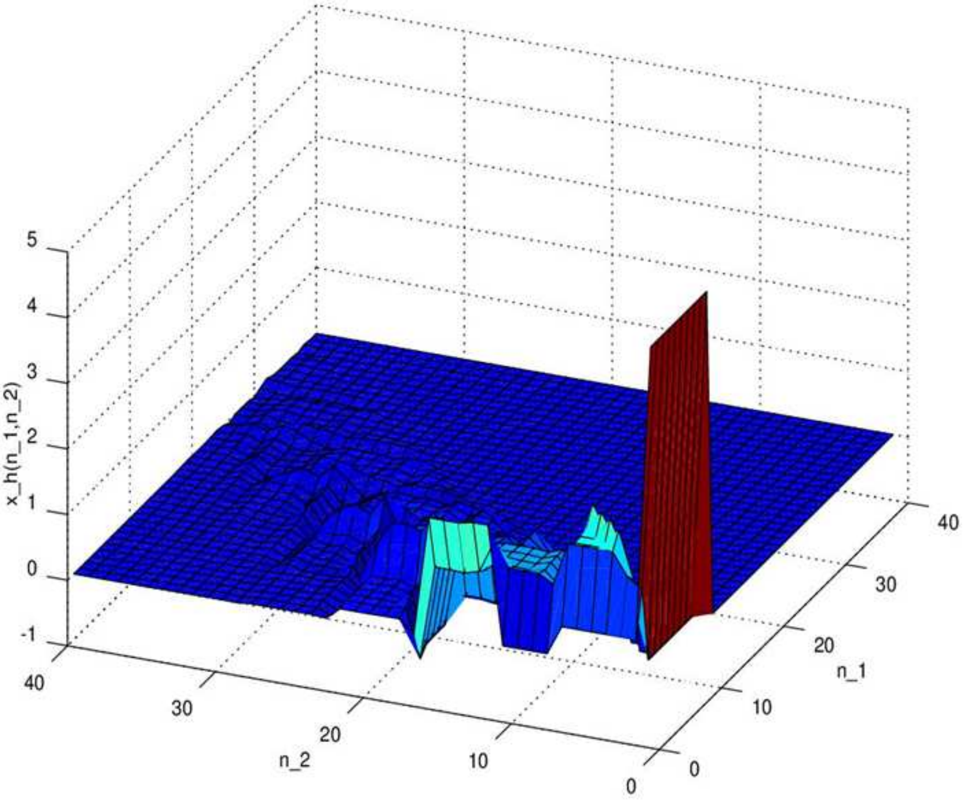


FIGURE 7. Evolution of the $x_h(n_1, n_2)$ substate when the system is under the state feedback control law (24)

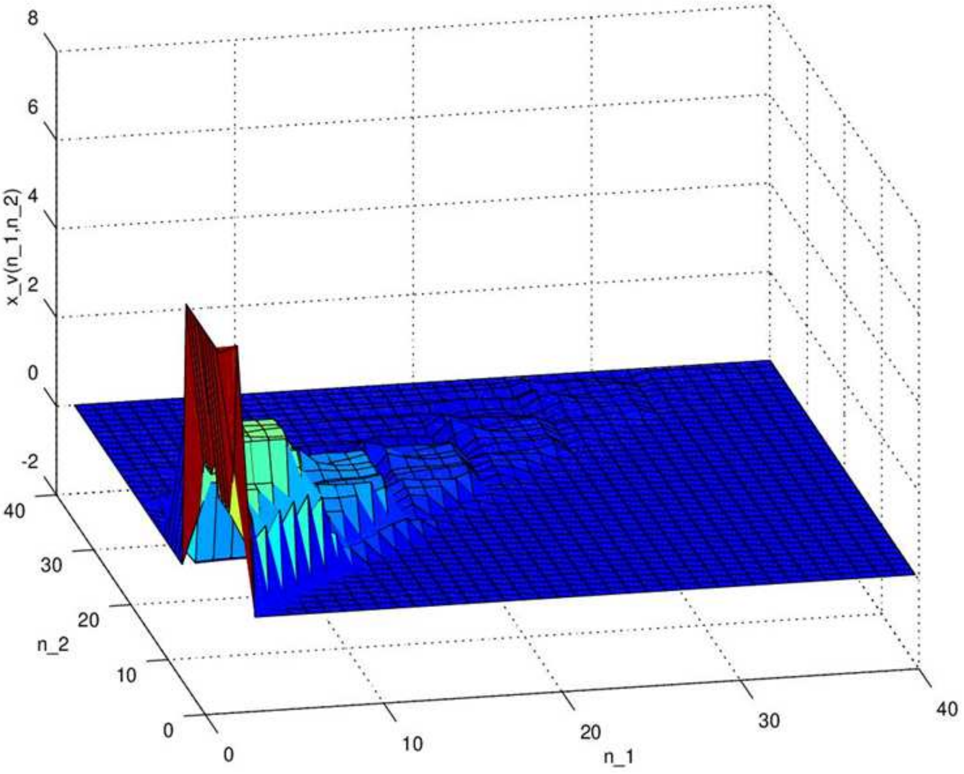


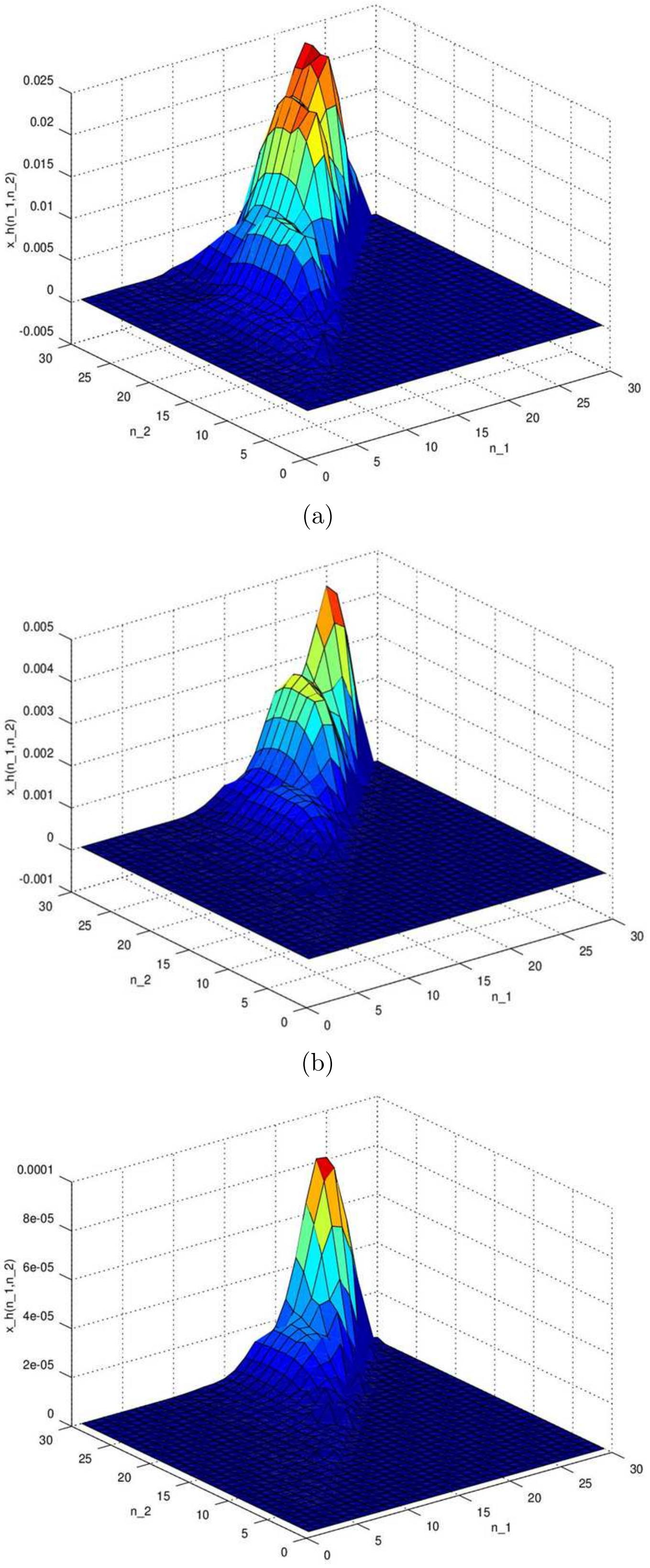
FIGURE 8. Evolution of the $x_v(n_1, n_2)$ substate when the system is under the state feedback control law (24)

at each time k. The following Figures 9 and 10 show the evolution of $x_h(n_1, n_2; k)$ and $x_v(n_1, n_2; k)$ for some particular values of the time k for a total simulation of 30 samples.

It can be concluded from Figures 9 and 10 that, as the time k evolves, (k becomes greater), the peak in the $x_h(n_1, n_2; k)$ and $x_v(n_1, n_2; k)$ components gets smaller (since the scale represented in the vertical axis gets smaller), as it corresponds to an asymptotically stable system.

7. **Conclusions.** This paper has firstly discussed a sufficiency-type stability conditions for the standard 2D-Roesser model under any given summable absolute initial conditions on the whole horizontal and vertical axis without using a Lyapunov function. The stability conditions rely on the characterizations of the spectral radius or spectral norm of the matrix defining the dynamics of the system and the set of summable initial conditions. Later on, two types of linear discrete Roesser-based models which have a state recursive evolution description are proposed and their stability properties are investigated. The mechanism which allows the recursive description of the state evolution is a state constraint evolution formalized through the existence of a linear transformation in-between the state at points where both horizontal/vertical substate arguments jointly shift in advance and its value at the preceding value when only one of the arguments is shifted for each substate. Both models incorporate point delays in the vertical and horizontal dynamics. The stability properties have been characterized via perturbation theory of matrices. Later on, the stability properties have been examined via the use of state-feedback linear control laws focused on decreasing the values of the stability radii of certain matrices being relevant to the dynamics. Some simulated examples have also been discussed.

Acknowledgment. This research is supported by the Spanish Government and by the European Fund of Regional Development FEDER through Grant DPI2015-64766-R and

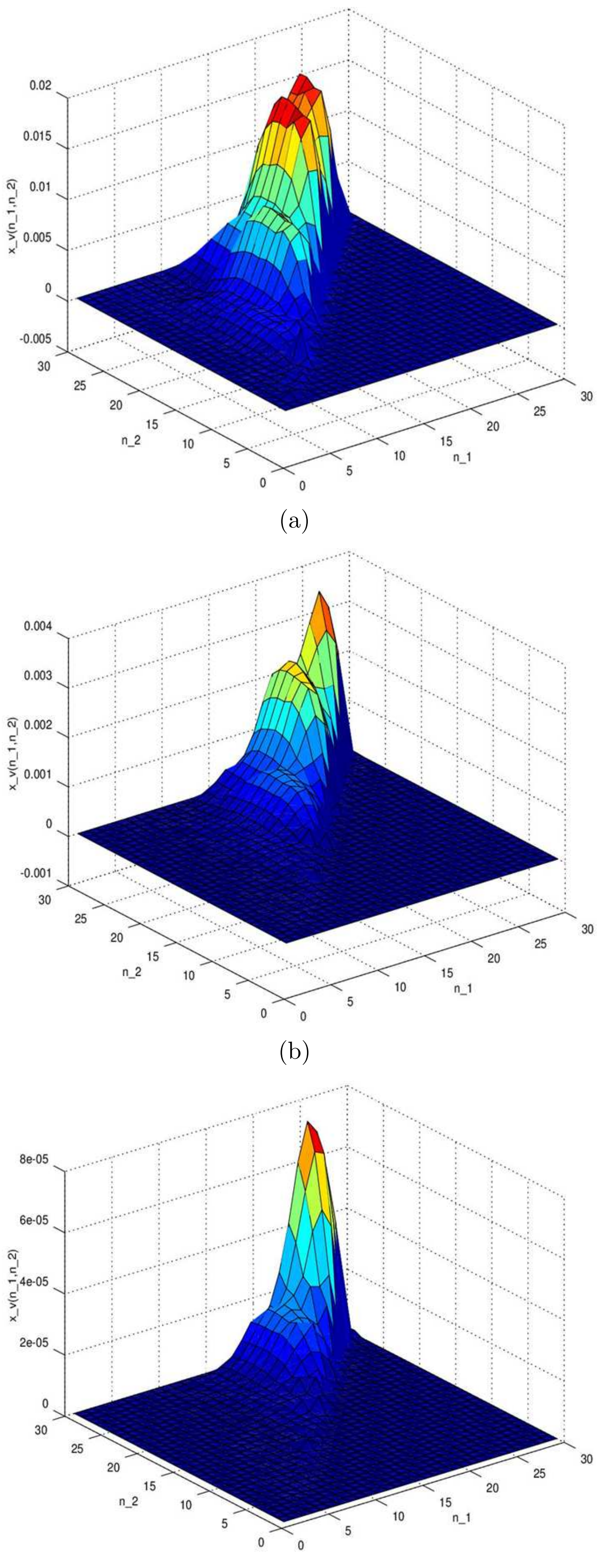


(a)

(b)

(c)

FIGURE 9. Evolution of the $x_h(n_1, n_2; k)$ substate for $k = 15$ (a), $k = 20$ (b) and $k = 28$ (c)



(a)

(b)

(c)

FIGURE 10. Evolution of the $x_v(n_1, n_2; k)$ substate for $k = 15$ (a), $k = 20$ (b) and $k = 28$ (c)

by Grant DPI2016-77271-R and by UPV/EHU by Grant PGC17/33. The authors are also grateful to the reviewers by their interesting suggestions.

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Appendix A. Proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1: We can write from (1) and (2) in the control-free case that:

$$\begin{aligned}
&\begin{bmatrix} x_h(n_1+1,n_2) \\ x_v(n_1,n_2+1) \end{bmatrix} \\
&= A \begin{bmatrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{hh} & A_{hv} \end{bmatrix} \\
&\times \begin{bmatrix} \begin{matrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{matrix} & 0 \\ 0 & \begin{matrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{matrix} \end{bmatrix} \begin{bmatrix} x_h(n_1-2,n_2) \\ x_v(n_1-2,n_2) \\ x_h(n_1-1,n_2-1) \\ x_v(n_1-1,n_2-1) \\ x_h(n_1-1,n_2-1) \\ x_v(n_1-1,n_2-1) \\ x_h(n_1,n_2-2) \\ x_v(n_1,n_2-2) \end{bmatrix}
\end{aligned} \tag{A.1}$$

Define

$$\begin{aligned}
&A^{(0)} = A = \begin{bmatrix} A_{hh} & A_{hv} \\ A_{vh} & A_{vv} \end{bmatrix} = \begin{bmatrix} A_h \\ A_v \end{bmatrix} = \begin{bmatrix} A_h^{(0)} \\ A_v^{(0)} \end{bmatrix}; \\
&A_h^{(0)} = A_h = [A_{hh} \;\; A_{hv}]; \quad A_v^{(0)} = A_v = [A_{vh} \;\; A_{vv}] \\
&A^{(1)} = A^{(0)}\overline{A}^{(0)} = A^{(0)} \begin{bmatrix} A_h^{(0)} & 0 \\ 0 & A_v^{(0)} \end{bmatrix} = \begin{bmatrix} A_{hh} & A_{hv} \\ A_{vh} & A_{vv} \end{bmatrix} \begin{bmatrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{bmatrix} \\
&A^{(2)} = A^{(0)}\overline{A}^{(1)} = A^{(0)}\overline{A}^{(0)} \begin{bmatrix} \begin{matrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{matrix} & 0 \\ 0 & \begin{matrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{matrix} \end{bmatrix} \\
&\overline{A}^{(0)} = \begin{bmatrix} A_h & 0 \\ 0 & A_v \end{bmatrix} = \begin{bmatrix} A_{hh} & A_{hv} & & 0 \\ 0 & & A_{vh} & A_{vv} \end{bmatrix}; \quad \overline{A}^{(1)} = \overline{A}^{(0)} \begin{bmatrix} A_h & 0 & 0 & 0 \\ 0 & A_v & 0 & 0 \\ 0 & 0 & A_h & 0 \\ 0 & 0 & 0 & A_v \end{bmatrix}
\end{aligned} \tag{A.2}$$

Note that $A^{(0)} \in \boldsymbol{R}^{2n\times 2n}$, $\overline{A}^{(0)} \in \boldsymbol{R}^{2n\times 4n}$ and $A^{(1)} \in \boldsymbol{R}^{2n\times 4n}$ with $n = n_h + n_v$. We can write

$$\begin{bmatrix} x_h(n_1+1,n_2) \\ x_v(n_1,n_2+1) \end{bmatrix} = A^{(0)} \begin{bmatrix} x_h(n_1,n_2) \\ x_v(n_1,n_2) \end{bmatrix}$$

$$= A^{(1)} \begin{bmatrix} x_h(n_1-1,n_2) \\ x_v(n_1-1,n_2) \\ x_h(n_1,n_2-1) \\ x_v(n_1,n_2-1) \end{bmatrix} = A^{(2)} \begin{bmatrix} x_h(n_1-2,n_2) \\ x_v(n_1-2,n_2) \\ x_h(n_1-1,n_2-1) \\ x_v(n_1-1,n_2-1) \\ x_h(n_1-1,n_2-1) \\ x_v(n_1-1,n_2-1) \\ x_h(n_1,n_2-2) \\ x_v(n_1,n_2-2) \end{bmatrix} \tag{A.3}$$

Proceeding recursively from initial conditions:

$$\begin{bmatrix} x_h\,(n_1+1,n_2) \\ x_v\,(n_1,n_2+1) \end{bmatrix} = A^{(0)} \begin{bmatrix} x_h\,(n_1,n_2) \\ x_v\,(n_1,n_2) \end{bmatrix}$$

$$= A^{(n_1)} \begin{bmatrix} x_h(0,n_2) \\ x_v(0,n_2) \\ \vdots \\ x_h\,(n_1,n_2-n_1) \\ x_h\,(n_1,n_2-n_1) \end{bmatrix} = A^{(n_1+1)} \begin{bmatrix} x_h\,(-1,n_2) \\ x_v\,(-1,n_2) \\ \vdots \\ x_h\,(0,n_2-n_1) \\ x_v\,(0,n_2-n_1) \\ \vdots \\ x_h\,(0,n_2-n_1) \\ x_v\,(0,n_2-n_1) \\ x_h\,(n_1,n_2-n_1-1) \\ x_v\,(n_1,n_2-n_1-1) \end{bmatrix} \tag{A.4}$$

where

$$A^{(n_1+1)} = A^{(0)}\overline{A}^{(n_1)} = A^{(0)}\overline{A}^{(n_1-1)} Block\ Diag \left[\left(A_h \vdots A_v \right) \cdots \overbrace{\left(A_h \vdots A_v \right)}^{n_1} \right]$$

Similar expressions can be found for $A^{(n_2+1)}$ and for $A^{(\hat{n}+1)}$ for any $n_2 \in \boldsymbol{Z}_{0+}, \hat{n}\,(=n_1+n_2) \in \boldsymbol{Z}_{0+}$. The following auxiliary lemmas, which are used to build the proof of Theorem 2.1, hold.

Lemma A.1. *Assume that there is a finite set of nonzero initial conditions* $x_h(n,0)$, $x_v(0,n)$ *for* $n \in \boldsymbol{Z}_{0+}$ *of the systems (1)-(3). If* $\left\{ \left\| A^{(\hat{n})} \right\|_2 \right\} \to 0$ *as* $\hat{n} \to \infty$, *where* $\hat{n} = \hat{n}\,(n_1,n_2) = \max\,(n_1,n_2)$, *then* $\begin{bmatrix} x_h(n_1+1,n_2) \\ x_v(n_1,n_2+1) \end{bmatrix} \to 0$ *as* $\hat{n} \to \infty$. *The result also holds if the initial conditions of (1)-(3) are absolutely summable, i.e., the sum of all the absolute values of the initial conditions components is finitely bounded.*

Proof: Since the initial conditions are fixed for $x_h(0,0)$ and $x_v(0,0)$ and $x_h(-i,j)$, $x_v(-i,j)$, $x_h(j,-i)$ and $x_v(j,-i)$ are zero for any $i \in \boldsymbol{Z}_+$ and $j \in \boldsymbol{Z}_{0+}$, it follows that fixing there exists a real $\hat{n}$-vector $v^T = v^T\,(\hat{n}) = \left(x_h^T(0,i), x_v^T\,(j,0) : i \in \overline{n}_1 \cup \{0\}, j \in \overline{n}_2 \right) \cup \{0\}$ of $2^{n(\hat{n}+1)}$ components with only a finite number $2\alpha\,(\hat{n})$, depending on $\hat{n}$ but finitely uniformly upper-bounded for any $\hat{n}$, of eventually nonzero repeated components $x_h(0,i)$ and $x_v(j,0)$ for $i \in \overline{n}_1 \cup \{0\}$, $j \in \overline{n}_2$, where $\sup\limits_{n \in Z_{0+}} \alpha\,(\hat{n}) \le M_\alpha < +\infty$ and $\alpha = \alpha\,(\hat{n}) \to \alpha^*$ as $\hat{n} \to \infty$, such that:

$$\left\| \begin{bmatrix} x_h\,(n_1+1,n_2) \\ x_v\,(n_1,n_2+1) \end{bmatrix} \right\|_2 \le 2M_\alpha \max\left(\max_{0 \le i \le n_1} |x_h\,(i,0)|\,, \max_{0 \le i \le n_2} |x_v\,(0,i)| \right) \left\| \, A^{(\hat{n})} \right\|_2 \tag{A.5}$$

The proof follows directly since $2M_\alpha \max\left(\max\limits_{0\leq i\leq n_1}|x_h(i,0)|, \max\limits_{0\leq i\leq n_2}|x_v(0,i)|\right)$ is finite and $\left\{\left\|A^{(\hat{n})}\right\|_2\right\} \to 0$ as $\hat{n} \to \infty$. If the maximum of the right-hand-side of (A.5) is replaced by a finite ℓ_2-norm of infinitely many summable initial conditions (for instance, if all the state absolute boundary conditions at initial points are elements of a geometric series), the result still clearly holds. □

Lemma A.2. *If the matrix* A *is triangular, and convergent, with* $\|A\|_2 \leq \rho < 1$ *with sufficiently small absolute values of the upper-triangular entries compared to* ρ *then* $\left\{\left\|A^{(\hat{n})}\right\|_2\right\} \to 0$ *as* $\hat{n} \to \infty$.

Proof: With no loss in generality, we might assume that A is upper-triangular, that is, $A_{vh} = 0$. Assume also that all the entries of A_{hv} satisfy $\left|(A_{hv})_{ij}\right| \leq \varepsilon$ for some $\varepsilon \in \boldsymbol{R}_+$. Then, note from (A.2) that

$$\left\|A^{(1)}\right\|_2 \leq \left\|A^{(0)}\right\|_2 \left\|\overline{A}^{(0)^T}\overline{A}^{(0)}\right\|_2^{1/2} \leq \rho\left(\rho + \left\|A_{hv}^T A_{hv}\right\|_2^{1/2}\right)$$

since

$$A^{(1)} = A^{(0)}\overline{A}^{(0)}, \quad \overline{A}^{(0)^T}\overline{A}^{(0)} = \begin{bmatrix} A_{hh}^T A_{hh} + A_{hv}^T A_{hv} & 0 \\ 0 & A_{vv}^T A_{vv} \end{bmatrix},$$

$$\rho \geq \max\left(\|A_{hh}\|_2, \|A_{vv}\|_2\right) = \max\left(\left\|A_{hh}^T A_{hh}\right\|_2, \left\|A_{vv}^T A_{vv}\right\|_2\right)^{1/2}$$

$$\left\|\overline{A}^{(0)^T}\overline{A}^{(0)}\right\|_2^{1/2} \leq \rho + \left\|A_{hv}^T A_{hv}\right\|_2^{1/2}$$

In the same way, one has from (A.2), that

$$\left\|A^{(\hat{n})}\right\|_2 \leq \rho\left(\rho + \left\|A_{hv}^T A_{hv}\right\|_2^{1/2}\right)^{\hat{n}}$$

and then the sequence $\left\{\left\|A^{(\hat{n})}\right\|_2\right\}$ converges to zero if $\varepsilon < \frac{1-\rho}{n_h^{3/2}}$ since $\|A_{hv}\|_2 = \left\|A_{hv}^T A_{hv}\right\|_2^{1/2} \leq n_h^{1/2}\|A_{hv}\|_1 \leq n_h^{3/2}\varepsilon$. □

Lemma A.3. (See [17]-Chapter 1, Theorem 2.1 and Theorem 2.2, pp.13-16). *For any given real* $\varepsilon \in \boldsymbol{R}_+$*, there exists a linear transformation with a non-singular associate matrix* $R = R(\varepsilon)$ *such that the similar matrix* $\hat{A}$ *to the square matrix* A*, given by* $\hat{A} = \hat{A}(\varepsilon) = R^{-1}AR$*, is upper-triangular with all its upper-triangular entries having absolute values not larger than* ε.

Lemma A.4. *Assume that the matrix* A *fulfils* $\|A\|_2 \leq \rho < 1$ *and assume also that the initial conditions of (1)-(3) are absolutely summable. Then,* $\left\{\left\|A^{(\hat{n})}\right\|_2\right\} \to 0$ *as* $\hat{n} \to \infty$*, where* $\hat{n} = \hat{n}(n_1, n_2) = \max(n_1, n_2)$*, so that* $\begin{bmatrix} x_h(n_1+1, n_2) \\ x_v(n_1, n_2+1) \end{bmatrix} \to 0$ *as* $\hat{n} \to \infty$.

Proof: Let T be them matrix defining an n-real non-singular linear transformation which allows rewriting the unforced 2D-Roesser model in the transformed coordinates as follows:

$$\begin{aligned} z(n_1+1, n_2, n_1, n_2+1) &= T^{-1}x(n_1+1, n_2, n_1, n_2+1) \\ &= T^{-1}Ax(n_1, n_2, n_1, n_2) = T^{-1}ATz(n_1, n_2, n_1, n_2) \end{aligned}$$

so that $T^{-1}AT$ is upper-triangular. Now, from Lemma A.3, for any given $\varepsilon \in \boldsymbol{R}_+$, there exists a linear transformation with associate matrix $R = R(\varepsilon)$ such that the similar matrix to the square matrix $T^{-1}AT$, given by $\hat{A} = \hat{A}(\varepsilon) = R^{-1}T^{-1}ATR$, is upper-triangular with all its upper-triangular entries having absolute values not exceeding ε. Now if A

is convergent with spectral norm $\rho < 1$, $\hat{A} = \hat{A}(\varepsilon)$ is also convergent with the same eigenvalues and spectral norm as A what is preserved by the linear transformations of associate matrices R and T for any given $\varepsilon \in \boldsymbol{R}_+$. From Lemma A.2, it suffices to fix the arbitrarily eligible ε so as to fulfil $\varepsilon < \frac{1-\rho}{n_{h0}^{3/2}}$ (n_{h0} being the number of rows of the up block matrices of $\hat{A}$) to guarantee that $\left\|\hat{A}^{(\hat{n})}\right\|_2 \to 0$ as $\hat{n} \to \infty$. Since the matrices T and R are non-singular, so that their ℓ_2 condition numbers are finite, it follows that $\left\|A^{(\hat{n})}\right\|_2 \to 0$ as $\hat{n} \to \infty$, so that $\hat{x}(.)$, $z(.)$ and $x(.)$, converge to zero from Lemma A.1 as $\hat{n} \to \infty$ for any given set of absolutely summable initial conditions.

Now, the proof of Theorem 2.1 follows as a direct consequence of Lemma A.4. $\square$

Proof of Theorem 2.2: It is a direct extension of that of Theorem 2.1 so that its details are omitted. A sketch of it follows. Note from the assumptions that $\|A\|_2 = \lambda_{\max}^{1/2}\left(A^T A\right) \le \rho + \|A_d\|_2 < 1$ with the following replacements related to Theorem 2.1:

$$A^{(0)} = A \to \left[\begin{array}{cc} A & A_d \end{array}\right] \in \boldsymbol{R}^{n\times 2n}; \quad \hat{n} \to \max\left(n_1 + d_1, n_2 + d_2\right)$$

$$A_h^{(0)} \to \left[\begin{array}{cccc} A_{hh} & A_{hv} & A_{hhd} & A_{hvd} \end{array}\right]; \quad A_v^{(0)} \to \left[\begin{array}{cccc} A_{vh} & A_{vv} & A_{vhd} & A_{vvd} \end{array}\right]$$

$\square$