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PARAMETER ESTIMATION FOR THE THREE-PARAMETER BURR-XII DISTRIBUTION UNDER ACCELERATED LIFE TESTING WITH TYPE I CENSORING USING PARTICLE SWARM OPTIMIZATION ALGORITHM

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Abstract. *Accelerated life test (ALT) is one widely used method in engineering applications nowadays for evaluating the reliability of highly reliable products. In this study, three-parameter Burr-XII distribution (3pBXIID) is considered to model the lifetime of highly reliable product. The maximum likelihood function and likelihood equations of the 3pBXIID based on ALT type I censored samples are analytically established. The particle swarm optimization algorithm (PSO) is used to obtain maximum likelihood estimates (MLEs), denoted by PSO-MLEs, of parameters in the 3pBXIID. The performance of the proposed PSO-MLE method is evaluated via using Monte Carlo simulations. Simulation results show that the proposed method can obtain reliable MLEs of model parameters for the 3pBXIID with ALT type I censored samples.*
Keywords: Accelerated life test, Hazard rate function, Maximum likelihood estimate, Particle swarm optimization, Survival function

1. **Problem Statement and Literature Review.** Because advanced manufacturing technologies have been widely applied in the production of highly reliable products, collecting failure time samples from highly reliable products for reliability studies becomes difficult. More and more highly reliable products can survive longer than before in life testing. This fact makes that no failure or only few failure times can be observed at the normal-use stress condition even using censoring schemes or truncated schemes in a life test. For overcoming this difficulty to obtain failure information from life tests, ALT is a popular method to speed up the failure of tested product through the use of a stress higher than the normal-use condition in a life test. ALT samples help to cumulate failure information for reliability evaluation of highly reliable products, but the observed failure information from high stress makes the reliability inference for highly reliable products at normal-use condition difficult.

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The 3pBXIID is a flexible model, which contains two shape parameters and one scale parameter in the distribution function. The 3pBXIID includes, overlaps, or has as a limiting case for many well-known lifetime distributions, like gamma distribution, lognormal distribution, log-logistic distribution, bell-shaped distribution and J-shaped beta distribution (but not U-shaped). Hence, the 3pBXIID has plenty of distribution shapes for fitting lifetime data. The two-parameter Burr-XII distribution is firstly introduced by Dr. I. W. Burr [1] in 1942. The 3pBXIID contains two-parameter Burr-XII distribution as special case. Compounding a Weibull distribution with a gamma distribution for its scale parameter yields a two-parameter Burr-XII distribution, and compounding an exponential distribution with a gamma distribution for its rate parameter also generates a two-parameter Burr-XII distribution.

Tadikamalla [2] extended the two-parameter Burr-XII distribution with introducing one more scale parameter into a 3pBXIID. Since then, the applications of Burr-XII distribution received more attention. The 3pBXIID is the asymptotic limiting case for the Weibull and Pareto type I distributions, respectively. It is encouraged for engineers to consider using 3pBXIID as an underlying model to characterize the lifetime of reliable product.

Plenty of reliability studies based on using two-parameter Burr-XII distribution can be found in the literature. Al-Hussaini [3] used a generalized theorem to extend the theorem proposed by Galambos and Kotz [4] to characterize the Burr-XII distribution. Zimmer et al. [5] provided statistical and probabilistic properties of the Burr-XII distribution, and its relationship to other well-known lifetime distributions. Jang et al. [6] proposed a Bayesian estimation method to obtain parameter estimates in the Burr-XII distribution based on general progressive type II censoring data. Thupeng [7] used the Burr-XII distribution to characterize the maximum levels of nitrogen dioxide at a specific site. Moreover, Thupeng did a parallel comparison of his estimation results with using other distribution models. Ismail and Khalid [8] studied using the expectation-maximization (EM) algorithm method for estimating the Burr-XII distribution parameters. Pradhan and Sayyareh [9] studied the predication using type II censored order statistics for the Burr-XII distribution.

Some recent studies about using ALT for the Burr-XII distribution are summarized as follows. Abdel-Hamid [10] constructed constant-partially ALT models for the Burr-XII distribution based on progressively type II censored samples. Srivastava and Mittal [11] dealt with the formulations of optimum multi-objective ramp-stress ALT plans with stress upper bounds for the Burr-XII distribution with type I censoring. Sun et al. [12] presented statistical methods for estimating Burr-XII distribution parameters under a step-stress partially ALT with a tampered random variable model. Zhao et al. [13] proposed a simple constant-stress ALT model for the Burr-XII distribution based on type I progressively hybrid censored samples. Cai et al. [14] considered a step-stress partially ALT for series system model, where component lifetimes are assumed to be independent but follow non-identical Burr-XII distributions. Cai et al. [15] evaluated the reliability of series systems with a step-stress partially ALT and type-I progressive hybrid censoring, in which components are assumed to follow independent Burr-XII distributions. Srivastava and Gupta [16] studied the optimal design of multi-objective modified ramp-stress ALT model with weighted goal programming approach. Rahman et al. [17] studied point and interval estimations for Burr-XII distribution parameters under the step-stress partially ALT with a progressive hybrid censoring scheme. Nassar et al. [18] investigated maximum likelihood estimation for Burr-XII distribution parameters with type-I and adaptive type II progressively hybrid censoring schemes. All these studies focus on using two-parameter Burr-XII distribution with ALT models.

2. **Motivation and Organization.** Collecting lifetime information from a life test for highly reliable products often is time consuming. The long test time makes the use of test equipments less efficient and then increases the test cost. Producers often have high time-pressure to push a new designed product into the market. Hence, how to shorten the life-testing time for collecting lifetime information and evaluate the reliability of highly reliable products in an affordable time period become an important issue for producers. For saving the test cost and time for testing highly reliable products and evaluating reliability, it is very potential to establish an ALT model for 3pBXIID and propose an operational parameter estimation procedure to obtain MLEs. In this study, we consider using ALT type I censoring scheme to develop a generalized constant-stress ALT model for the 3pBXIID based on administrative merits. Limited applications of using the 3pBXIID in reliability applications can be found in the literature. Okasha and Matter [19] applied the 3pBXIID to modeling heavy tailed lifetime data and used a breast cancer data set for illustrating their proposed method. Xin et al. [20] established a reliability evaluation procedure for 3pBXIID lifetime products via using Metropolis-Hastings Markov chain Monte Carlo approach.

Because the target likelihood function for maximization in this study is very complicated, existing gradient computation methods could fail to obtain MLEs of model parameters through maximizing the target likelihood function. We consider using PSO to maximize the target likelihood function in this study. PSO is one evolutionary algorithm, which uses a population of particles to find optimal solutions, see Kennedy and Eberhart [21]. The mechanism of PSO is to mimic swarm behavior in birds flocking and fish schooling to guide the particles for searching global optimal solutions. When implementing a PSO, the velocities and positions of the members of swarm are updated via learning good experiences. The PSO uses stochastic search schemes and can be directly used in continuous real number space. It is noted that no mutation and crossover operators are considered to implement a PSO as implementing a genetic algorithm (GA). Moreover, the PSO does not use the gradient of an objective function. During implementing a PSO, only few parameters need to be adjusted. Hence, the PSO has special merits for saving memory space, compared with other evolutionary algorithms. The PSO can rapidly converge, and the PSO has been proved to be more competitive than the GA in several tasks, mainly in optimization areas, see Esmin and Lambert-Torres [22], Esmin et al. [23], Kiranyaz et al. [24], Silva et al. [25] and Zhao et al. [26] for more detailed discussions. Based on these aforementioned merits, we hence consider using PSO for maximizing the target likelihood function in this study when ALT type I censored samples are used.

The rest of this paper is organized as follows. The constant-stress ALT model with type I censoring for 3pBXIID is analytically established in Section 3. Moreover, we discuss the difficulty why gradient computation methods could fail to obtain MLEs of model parameters in the target likelihood function. The implementation of using PSO to obtain MLEs of model parameters for the 3pBXIID is studied in Section 3. In Section 4, the performance of the proposed PSO-MLE method is evaluated through using Monte Carlo simulations. Moreover, the estimation performance of the PSO-MLE method is compared with the DE-MLE method. Finally, some conclusions are given in Section 5.

3. **The Lifetime Model and Accelerated Life Test Model.** The lifetime model of 3pBXIID is addressed in this section. The likelihood function and likelihood equations for the 3pBXIID under a generalized design of constant-stress ALT with type I censored samples are analytically established. Moreover, the difficulties for searching MLEs of model parameters in this topic are discussed.

3.1. **The lifetime model.** Let the lifetime of highly reliable product, X, follow a 3pB-XIID, whose probability density function (PDF) and cumulative density function (CDF) are respectively defined by

$$f(x;\Theta) = \frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-(k+1)}, \tag{1}$$

and

$$F(x;\Theta) = 1-\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}, \quad c,k,\alpha,x>0, \tag{2}$$

where $\Theta = (c,k,\alpha)$, c is the inner shape parameter, k is the outer shape parameter, and α is the scale parameter. The survival and hazard rate functions of 3pBXIID(Θ) can be presented by

$$S(x;\Theta) = \left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}, \tag{3}$$

and

$$h(x;\Theta) = \frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-1}, \tag{4}$$

respectively. The 3pBXIID defined in Equations (1) and (2) is for complete data set. We need to expand the model definition of 3pBXIID via linking the model parameters with stress variables when an ALT is applied to collecting lifetime information from highly reliable products.

3.2. **The accelerated life test model.** Suppose that a life test is performed with m different stress levels, s_i, $i = 1,2,\ldots,m$. Each stress level contains n lifetime products for life testing. Let x_{ij} denote the j^{th} shortest observed failure time of products under s_i for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Assume that total r_i ($\leq n$) failed products are observed under s_i at the predetermined termination time, denoted by t_i. Hence, we can stop the ADT under stress s_i at $T_i = \min(t_i, x_{in})$. Please note that the r_i in ADT is random due to the fact that the failure number under each stress level is variable. The likelihood function for the BurrXII(Θ) can be obtained and given by

$$\begin{aligned} L(\Theta;\boldsymbol{x},T) &= \prod_{i=1}^{m}\prod_{j=1}^{r_i} f(x_{ij};\Theta)\left[\prod_{i=1}^{m}(S(T_i;\Theta))^{n-r_i}\right] \\ &= \prod_{i=1}^{m}\prod_{j=1}^{r_i}\frac{ck}{\alpha^c}x_{ij}^{c-1}\left(1+\left(\frac{x_{ij}}{\alpha}\right)^{c}\right)^{-(k+1)}\prod_{i=1}^{m}\left(1+\left(\frac{T_i}{\alpha}\right)^{c}\right)^{-k(n-r_i)}, \end{aligned} \tag{5}$$

where $\boldsymbol{x} = \{x_{ij}, i = 1,2,\ldots,m, j = 1,2,\ldots,r_i\}$ denotes the realization of the ALT data set. To cover more scenarios of ALT, we consider that the outer parameter and scale parameter are dependent on the stress and can be characterized by the following two equations:

$$k_i \equiv k(s) = a_0 + a_1 s_i, \tag{6}$$

and

$$\alpha_i \equiv \alpha(s_i) = b_0 + b_1 + b_2 s_i^2, \;\; i = 1,2,\ldots,m. \tag{7}$$

The proposed constant-stress ALT model with Equations (6) and (7) is generalized and covers many existing ALT models, which only consider the link of the outer parameter with stress, as special cases. Denote $\Theta = (c, a_0, a_1, b_0, b_1, b_2)$ hereafter. After algebraic computation, we obtain the first derivatives of the log-likelihood function, $\ell(\Theta) \equiv \log(L(\Theta;\mathbf{x},T))$, respect to the parameters, c, a_0, a_1, b_0, b_1, b_2, by

$$\frac{\partial \ell(\Theta)}{\partial c} = \sum_{i=1}^{m}\left\{\frac{r_i}{c} - r_i\log(\alpha_i) + \sum_{j=1}^{r_i}\left[\log(x_{ij}) - \frac{(k_i+1)x_{ij}^c}{\alpha_i^c + x_{ij}^c}\log\left(\frac{x_{ij}}{\alpha_i}\right)\right.\right.$$

$$-\frac{(n-r_i)k_iT_i^c}{\alpha_i^c+T_i^c}\log\left(\frac{T_i}{\alpha_i}\right)\Bigg]\Bigg\}, \tag{8}$$

$$\frac{\partial\ell(\Theta)}{\partial a_0} = \sum_{i=1}^{m}\left\{\frac{r_i}{k_i}-\sum_{j=1}^{r_i}\left[\log\left(1+\frac{x_{ij}^c}{\alpha_i^c}\right)+(n-r_i)\log\left(1+\frac{T_i^c}{\alpha_i^c}\right)\right]\right\}, \tag{9}$$

$$\frac{\partial\ell(\Theta)}{\partial a_1} = \sum_{i=1}^{m}s_i\left\{\frac{r_i}{k_i}-\sum_{j=1}^{r_i}\left[\log\left(1+\frac{x_{ij}^c}{\alpha_i^c}\right)+(n-r_i)\log\left(1+\frac{T_i^c}{\alpha_i^c}\right)\right]\right\}, \tag{10}$$

$$\frac{\partial\ell(\Theta)}{\partial b_0} = -c\sum_{i=1}^{m}\frac{1}{\alpha_i}\left\{r_i-\sum_{j=1}^{r_i}\left[\frac{(k_i+1)x_{ij}^c}{\alpha_i^c+x_{ij}^c}+\frac{(n-r_i)k_iT_i^c}{\alpha_i^c+T_i^c}\right]\right\}, \tag{11}$$

$$\frac{\partial\ell(\Theta)}{\partial b_1} = -c\sum_{i=1}^{m}\frac{s_i}{\alpha_i}\left\{r_i-\sum_{j=1}^{r_i}\left[\frac{(k_i+1)x_{ij}^c}{\alpha_i^c+x_{ij}^c}+\frac{(n-r_i)k_iT_i^c}{\alpha_i^c+T_i^c}\right]\right\}, \tag{12}$$

and

$$\frac{\partial\ell(\Theta)}{\partial b_2} = -c\sum_{i=1}^{m}\frac{s_i^2}{\alpha_i}\left\{r_i-\sum_{j=1}^{r_i}\left[\frac{(k_i+1)x_{ij}^c}{\alpha_i^c+x_{ij}^c}+\frac{(n-r_i)k_iT_i^c}{\alpha_i^c+T_i^c}\right]\right\}. \tag{13}$$

Because Equations (8)-(13) are very complicated, it is very difficult to obtain MLEs, $\hat{c}$, $\hat{a}_0$, $\hat{a}_1$, $\hat{b}_0$, $\hat{b}_1$ and $\hat{b}_2$, by simultaneously solving likelihood equations, $\partial\ell(\Theta)/\partial c = 0$, $\partial\ell(\Theta)/\partial a_0 = 0$, $\partial\ell(\Theta)/\partial a_1 = 0$, $\partial\ell(\Theta)/\partial b_0 = 0$, $\partial\ell(\Theta)/\partial b_1 = 0$ and $\partial\ell(\Theta)/\partial b_2 = 0$, through using gradient computation methods, for example, the quasi-Newton method, see Byrd et al. [27]. When implementing a gradient computation method, we need good knowledge on setting initial solutions of all model parameters. There are 6 parameters in this study. It is very difficult to well set 6 initial solutions for practitioners to apply gradient computation methods for implementing numerical computation. Hence, gradient based MLEs could have highly biased from true parameters and have big mean square errors (MSEs). In this study, we use PSO to search MLEs of all parameters in the 3pBXIID with ALT type I censored samples.

The PSO can be implemented with four steps as follows.

1) Evaluate the fitness of each particle.
2) Update individual and global bests.
3) Update the velocity and position of each particle
4) Repeat Steps 1)-3) until specific stopping conditions are reached.

The source codes of R, which is a free statistical software, are prepared to implement the PSO for obtaining MLEs of model parameters via maximizing the target likelihood function in (5). To a specific target likelihood function, we can use the R package "**pso**" with default setting for PSO parameters to maximize the target likelihood function. Based on our computation experience, users can adopt the default setting in "**pso**" package to obtain MLEs of model parameters in Equation (5). We will study the implementation of using PSO for obtaining MLEs of model parameters in Section 4.

4. Simulations.

4.1. Performance evaluation.

Let $s'_0 < s'_1 < s'_2 < \cdots < s'_m$ denote the stress levels, in which s'_0 and s'_m denote the normal-use condition and highest stress level to protect the ALT free of over-stress, respectively. Applying normalization transformation for all stress levels by $s_i = (s'_i - s'_0)/(s'_m - s'_0)$ for $i = 1, 2, \ldots, m$, we obtain the transformed stress levels with $s_0 = 0$, $0 < s_i < 1$ for $i = 1, 2, \ldots, (m-1)$ and $s_m = 1$. In practice, we can choose two stress levels, a lower level and a high level, to implement the ALT. That is, an ALT with $m = 2$ and $0 < s_1 < 1$ and $s_2 = 1$.

In this simulation study, we consider $m = 2$, $s_1 = 0.45$ and $s_2 = 1$. Parameters in the 3pBurrXII(Θ) for simulations are considered as $c = 8$, $a_0 = 3$, $a_1 = 4$, $b_0 = 20$, $b_1 = -7$ and $b_2 = -2.5$. For each stress level, $n = 30$ components are initially used for implementing the ALT and the termination time can be $T_i = \min(9, x_{in})$. There are 6 parameters to be estimated based on ALT type I censored samples. We find that gradient computation methods fail to obtain MLEs of model parameters due to the fact that well selecting 6 initial solutions for all model parameters is very difficult.

We generate 20000 ALT type I censored samples from the BurrXII(Θ) with $c = 8$, $a_0 = 3$, $a_1 = 4$, $b_0 = 20$, $b_1 = -7$ and $b_2 = -2.5$. All these samples are used to obtain MLEs of model parameters via using PSO to maximize the target likelihood function in (5). Two criteria are used to evaluate the quality of PSO-MLEs. Let Bias($\hat{\theta}$) and MSE($\hat{\theta}$) denote the bias and MSE of $\hat{\theta}$, which is an estimate of the parameter θ. Let

$$\delta(\hat{\theta}) = \frac{\mathrm{Bias}(\hat{\theta})}{|\theta|}, \tag{14}$$

and

$$\eta(\hat{\theta}) = \frac{\sqrt{\mathrm{MSE}(\hat{\theta})}}{|\theta|}. \tag{15}$$

The $\delta(\hat{\theta})$ is a measure for checking the bias of $\hat{\theta}$ relative to its true parameter, and the $\eta(\hat{\theta})$ is a measure for checking the dispersion of $\hat{\theta}$ around θ relative to the true parameter. In this simulation study, $\hat{\theta}$ can be $\hat{c}$, $\hat{a}_0$, $\hat{a}_1$, $\hat{b}_0$, $\hat{b}_1$ or $\hat{b}_2$. Smaller values of $\delta(\hat{\theta})$ and $\eta(\hat{\theta})$ indicate a better estimation quality of $\hat{\theta}$. It is difficult to narrow the searching range for each parameter because we could have not enough knowledge on domains of model parameters. Hence, we search PSO-MLEs over a wide domain for each parameter in this simulation study. We consider the set $D(\Theta) = \{(c, a_0, a_1, b_0, b_1, b_2)|5 \leq c \leq 15, 1 \leq a_0, a_1 \leq 8, 5 \leq b_0 \leq 50, -10 \leq a_1 \leq -1, -5 \leq c \leq -1\}$ as the domains of the model parameters. We find that the PSO still performs well to obtain reliable MLEs even the domain of each model parameter cannot be accurately set up. The implementation of PSO is based on using R package "**pso**" with default setting. All simulation results are summarized in Table 1. From Table 1 we find that the PSO-MLEs of a_0, b_0 and b_1 are more reliable with small value of $\delta(\hat{\theta})$ and $\eta(\hat{\theta})$. The PSO-MLEs of c and a_1 mildly underestimate their true parameters. The PSO-MLE of $\hat{b}_2$ performs badly due to seriously underestimation. Overall, we find that the MSEs of all PSO-MLEs are smaller.

The density plots of 20000 MLEs for all parameters are presented in Figure 1, in which the dash line indicates the true parameter. The density plots of $\hat{c}$, $\hat{b}_0$ and $\hat{b}_1$ in Figure 1 are unimodal, the density plot of $\hat{a}_1$ has a flat and wide top, and the density plots of $\hat{a}_0$ and $\hat{b}_2$ have an asymptotically unimodal shape. Because the likelihood function is

TABLE 1. The values of mean estimates, $\delta(\hat{\theta})$ and $\eta(\hat{\theta})$ from 20000 simulation tries

Measures	$\hat{c}$	$\hat{a}_0$	$\hat{a}_1$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$
mean	6.6753	3.0903	3.4028	18.3436	−7.0585	−3.7062
$\delta(\hat{\theta})$	−0.1656	0.0301	−0.1493	−0.0828	−0.0084	−0.4825
$\eta(\hat{\theta})$	0.2312	0.4820	0.3928	0.1484	0.2970	0.5901

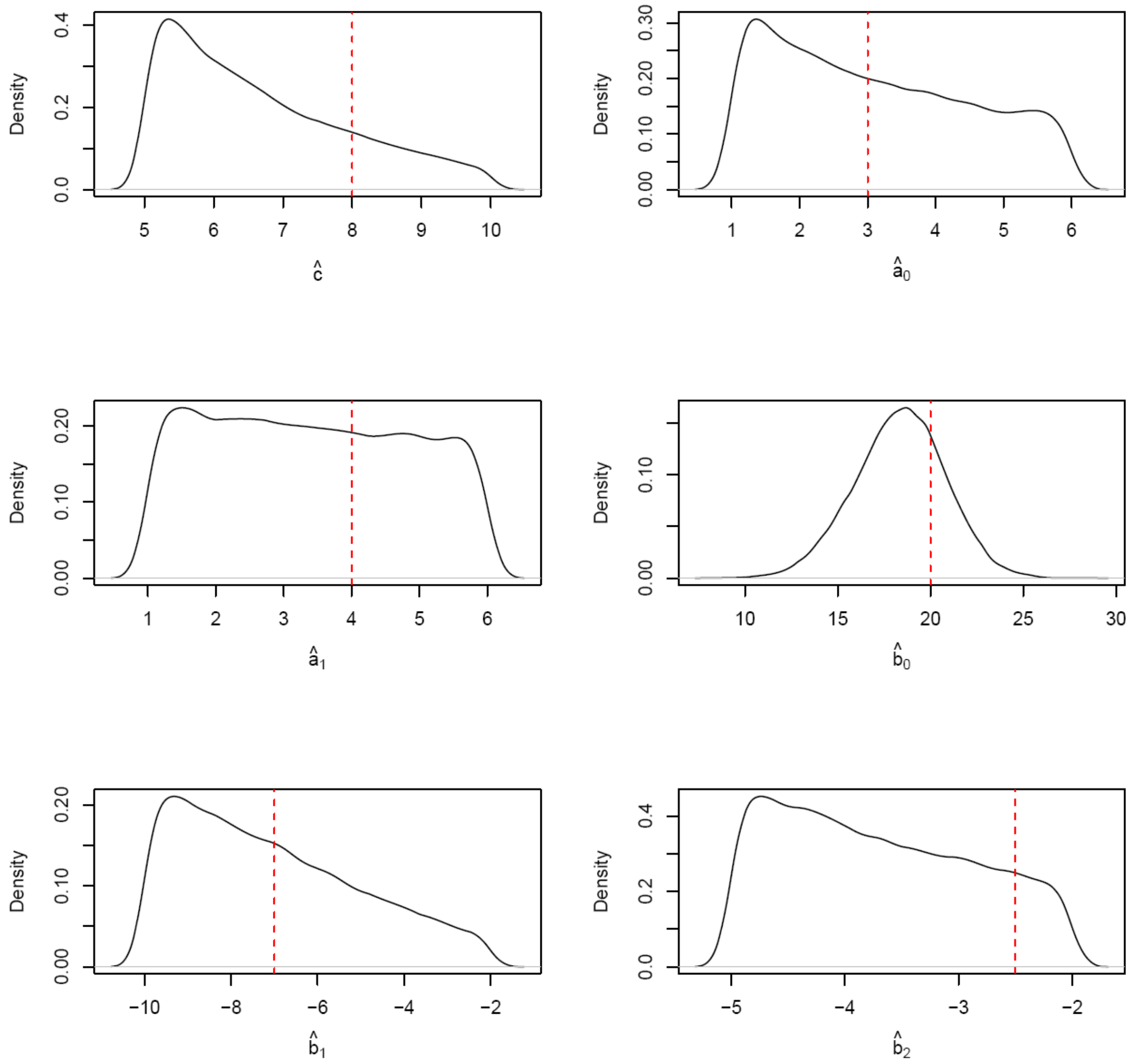

FIGURE 1. The density plots of the PSO-MLEs

complicated with 6 parameters, it is very difficult to obtain reliable MLEs for maximizing the likelihood function if we do not use proper domains for all parameters. Even the working domains are wide in the simulation study, we find that the performance of PSO is still stable to obtain reliable MLEs of model parameters except for estimating b_2.

4.2. **Performance comparison.** In this subsection, we would like to compare the estimation performance of the differential evolution algorithm (DE) and PSO. Zhu et al. [28] have studied the merits of using DE with progressively type I interval-censored samples in reliability applications. The DE could be another efficient algorithm to search the MLEs of the ALT model parameters in this study. However, we found that the DE cannot work well to search the MLEs over the domain $D(\Theta)$ under the studied model in Section 4.1. Some ranges of parameters in the domain $D(\Theta)$ are too wide for DE and make the target function divergence.

For parallel comparison, we consider a narrower domain than $D(\Theta)$ for the model parameters to make both the DE and PSO workable to obtain the DE-MLEs and PSO-MLEs for the ALT model parameters. Let $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = (c, a_0, a_1, b_0, b_1, b_2)$. We consider the range of θ_i with $\theta_{i0} - 3.5 < \theta_i < \theta_{i0} + 3.5$, $i = 1, 2, \ldots, 6$ for parameters and

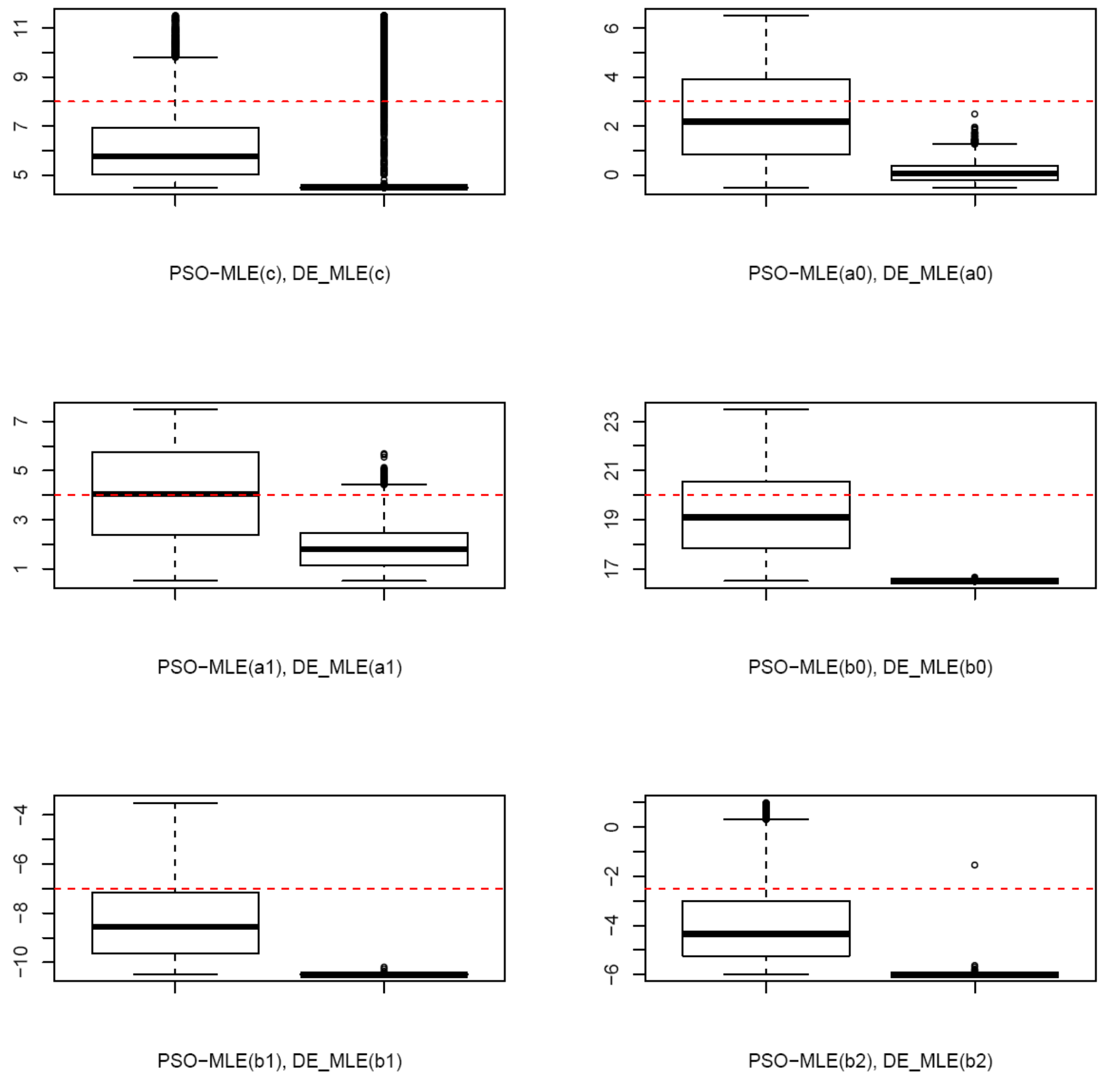

FIGURE 2. The box plots of the PSO-MLEs and DE-MLEs

search the DE-MLEs and PSO-MLEs. The mean estimate, $\delta(\hat{\theta})$ and $\eta(\hat{\theta})$ of the DE-MLEs and PSO-MLEs are evaluated based on 10000 simulation runs. Figure 2 displays the box plots of all MLEs in the simulation results. For each pair of box plots, the left side is the box plot of PSO-MLEs and the right side is the box plot of DE-MLEs. The dash line indicates the true parameter. We can find that the DE-MLEs seriously underestimate their true parameters and perform worse than PSO-MLEs.

5. **Conclusions.** In this study, a generalized ALT model with type I censoring is studied for 3pBXIID. The maximum likelihood function and likelihood equations are analytically established, and we found that both the likelihood function and likelihood equations are quite complicated with 6 parameters. It is almost impossible to use gradient computation methods to obtain reliable MLEs of model parameters in this study. We use PSO instead of using gradient computation methods to search reliable MLEs. The PSO is easy for implementation and does not need to set initial solutions of model parameters for searching MLEs. Monte Carlo simulations were conducted to evaluate the estimation performance of the proposed PSO-MLE method. From simulation results we find that the proposed PSO-MLE method could perform bad for estimating few parameters if their working domains

are too wide. The determination of working domains of model parameters via using evolutionary algorithms is a common issue when searching MLEs of model parameters.

The DE could be another efficient algorithm to search the MLEs of 3pBXIID ALT parameters in this study. In Section 4.2, we conducted a simulation study to compare the estimation performance of the PSO-MLE and DE-MLE methods. We found that the PSO-MLE method outperforms the DE-MLE method to obtain reliable MLEs of the 3pBXIID ALT model parameters. Compared with the PSO, the DE requires a narrower domain to search the MLEs. Moreover, the DE-MLEs seriously underestimate their true parameters in the simulation study. We hence recommend using PSO to obtain reliable MLEs of the model parameters in Equation (5).

The constant-stress ALT model for 3pBXIID in this study is quite generalized. The PSO is applicable for obtaining reliable MLEs of model parameters. However, it could have a room to improve the estimation performance of the proposed PSO-MLE method. How to improve the estimation quality through using other numerical computation methods instead of the PSO is another interesting issue. These two issues are challenged and can be future studies.

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