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# PROCESS-DELAY MODEL ESTIMATION AND RISK-AVOIDANCE METHOD

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Abstract. *We propose the production process with time delay as the Ornstein-Uhlenbeck (OU) process in finance theory, which is a mathematical model of a mass-production process with a production delay. We also estimate the expected value and variance of the throughput of the whole period by utilizing Kalman filter theory, which is used for state estimation in control theory under the incomplete information available for the whole period of the manufacturing process. Further, we propose an empirical equation, which represents a product value at the exit of the production process. For the theoretical verification, we present a numerical simulation.*
**Keywords:** Ornstein-Uhlenbeck process, Process delay, Stochastic process, Ito's lemma, Kalman filter

1. **Introduction.** Several our previous studies have proposed financial approaches to evaluate a production business including supplier [1, 2, 3]. To evaluate a production process, the lead time of production system in the production stage by using a stochastic differential equation of the log-normal type, which is derived from its dynamic behavior, is modeled [1]. The use of a mathematical model that focuses on the selection process and adaptation mechanism of the production lead time is used [1]. Using this model and risk-neutral integral, the evaluation equation for the compatibility condition of the production lead time is defined and then calculated. Furthermore, it is clarified that the throughput of the production process was reduced [1, 2]. With respect to determine a throughput rate, an expected value and volatility of throughput of the whole process period is estimated by utilizing Kalman filter theory having been used for a state estimation problem in the control theory [2]. With respect to a physical approach, a state in which the production density of each process corresponded to the physical propagation of heat was introduced in our previous study [4]. Using this approach, the diffusion equation, which dominates the production process was shown. Moreover, we clarified that the production process was dominated of a diffusion equation [4]. To improve a production lead time, there are several studies to shorten production throughput (lead times) [5, 6]. From the time of product ordering, the lead time depends on the work required to make ready for production. The several our research results which were mathematical modelings and the evaluation method of the production processes have reported.

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The synchronization method is superior for improving throughput in production processes, which is used by a production flow process [7]. The production flow process is utilized for production of high-mix low-volume equipments, which are produced through several stages in the production process. This method is good for producing specific control equipment such as semiconductor manufacturing equipment in our experience. Then, we have reported that the production flow process has nonlinear characteristics in our previous study [8]. Moreover, a working-time delay is propagated through the stages in the production process. Its delays are due to volatility in the model. Indeed, the actual data indicated that in the production flow process, the delays were propagated to the successive stages [4].

Our production business utilizes the services of outside companies when ordering materials and attending to logistics. In this business environment, we analyze the changes in the lead time. For various reasons, the equipment ordered may be delayed. To evaluate the corporate management strategy when the total production of a business depends on suppliers, we compare our model output with actual rate of return data, which follow a log-normal probability distribution. The results demonstrate the potential applicability of our proposed strategy to the manufacturing industry. We also represent actual throughput data of a company with high productivity and a company not yet adopting a production flow process.

In this study, the following assumptions are made.

1) There is no major change in probability during the production process.
2) The Ornstein-Uhlenbeck (OU) process is derived for the production process with a time delay [9, 10, 11].
3) The production is evaluated continuously; however, upper and lower limits are set for stochastic throughput.

Given control equipment is ordered by a customer and is then manufactured in a manner that is classified into a number of production elements, whereupon the finished product is delivered to the customer. The feature of this study is that a production element in the manufacturing processes is treated as a stochastic production operation. In a company, it is important to determine a proper lead time with which the production can be continued in a state of incomplete information. The OU process is reported in finance theory; OU is a mathematical model of a mass-production process with a production delay. Various environmental changes contribute to the changes in the product delivery date. In cases wherein incomplete information is available for the whole period of the manufacturing process, the expected value and variance of the throughput of the whole period are estimated with a propagation delay by utilizing Kalman filter theory; this theory is used for state estimation in control theory. We propose a risk premium value, which represents a production value and an empirical equation at production process exit, that is, before shipment. Moreover, for the theoretical evaluation, we present a numerical simulation. As a result, we report that we obtained good results. To the best of our knowledge, this is the first analysis of the OU process model based on a production process with a delay and a risk premium value before shipment.

2. **Production Framework in Equipment Manufacturer.** We refer to the production system in manufacturing equipment industry studied in this paper. This is not a special system but "Make-to-order system with version control". Make-to-order system is a system which allows necessary manufacturing after taking orders from clients, resulting in "volatility" according to its delivery date and lead time. In addition, "volatility" occurs in lead time depending on the contents of make-to-order products (production equipment). However, effective utilization of the production forecast information on the

orders may suppress certain amount of "variation", but the complete suppression of variation will be difficult. In other words, "volatility" in monthly cash flow occurs and of course influences a rate of return in these companies. Production management system, suitable for the separate make-to-order system which is managed by numbers assigned to each product upon order, is called as "product number management system" and is widely used. All productions are controlled with numbered products and instructions are given for each numbered product.

Thus, ordering design, logistics and suppliers are conducted for each manufacturer's serial numbers in most cases except for semifinished products (unit incorporated into the final product) and strategic stocks. Therefore, careful management of the lead time or production date may not suppress "volatility" in manufacturing (production). The company in this study is the "supplier" in Figure 1 and "factory" here. Companies are under the assumption that there are $N$ (numbers of) suppliers; however, this study deals with one company because no data is published for the rest of the companies $(N-1)$.

2.1. **Production flow process.** A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the production process. In Figure 2, the processes consist of six stages. In each step S1-S6 of the manufacturing process, materials are being produced. The direction of the arrows represents the direction of the production flow. Production materials are supplied through the inlet and the end-product is shipped from the outlet [7].

2.2. **Propagation of production density.** Figure 3 shows that connection between processes can be treated as diffusive propagation of products [4]. In Figure 3, $u$ and $n$ represent the throughput and production density, respectively [4]. In fluid dynamics, $S$

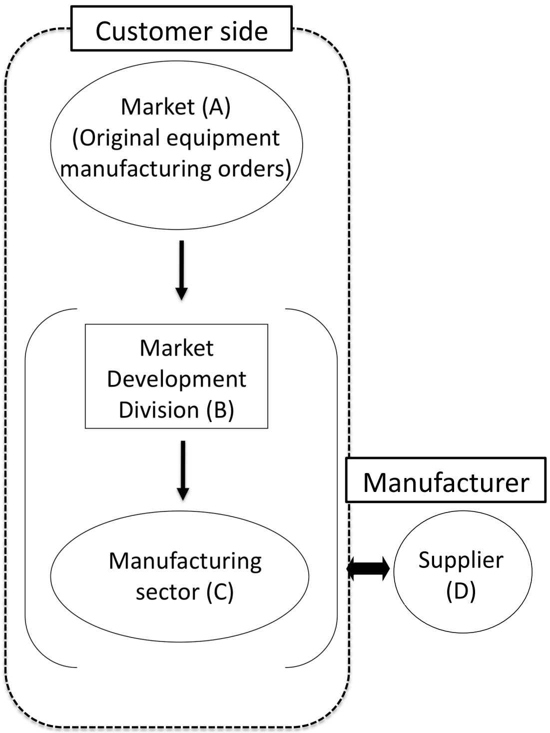


Figure 1. Business association chart of company of research target

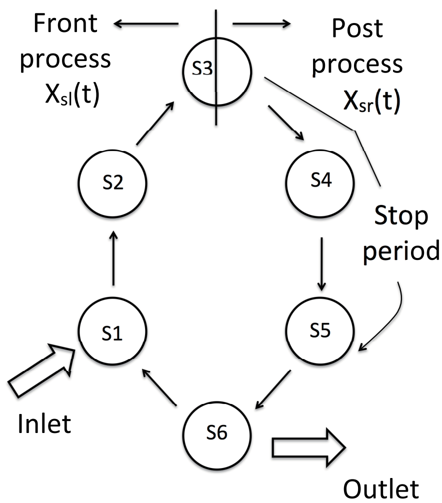


Figure 2. Bottleneck period in production flow processes

represents the cross-sectional area; the number density continuity equation is described as follows [12]:

$$\Delta(nS\Delta x) = n(t,x)u(t,x)\Delta tS - n(t,x+\Delta x)u(t,x+\Delta x)\Delta t \tag{1}$$

$$\left(\frac{\Delta n}{\Delta t}\right)_x = -\frac{n(t,x+\Delta x)u(t,x+\Delta x) - n(t,x)u(t,x)}{\Delta x} \tag{2}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial (nu)}{\partial x} \tag{3}$$

$u$ is the advection term in $\frac{\partial(nu)}{\partial t}$ of Equation (3). Now, let $u = v$ (constant value), we consider the following equation.

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} = 0 \tag{4}$$

Equation (4) denotes a linear wave motion traveling to $+x$ direction at a constant speed $c$. Then, in Figure 4, when the advection speed changes, Figure 4(a) shows that the $A$ part moves quickly to the right, and the distance between $AB$ is shortened gradually because the $B$ part moves slowly. Figure 4(b) shows that the $A$ part catches up with the $B$ part and overtakes it after a certain time has elapsed, following which the wave collapses. Figure 4(c) shows that the dissipation area suppresses processes like the wave until a limited gradient forms when the spatial gradient becomes sharp. The fill area also shows an area where dissipation occurs [15]. Figure 2 depicts a production process that is termed as a production flow process. This production process is employed in the production of control equipment. In this example, the production flow process consists of six stages. Equation (3) is a continuous equation describing the throughput. The bottleneck occurs at some stage of the process in Figure 2.

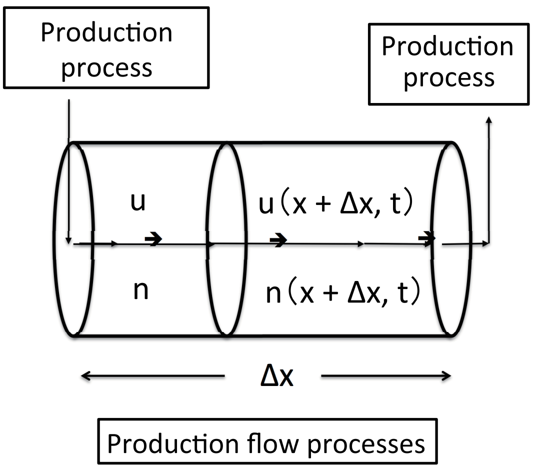


FIGURE 3. Production flow processes

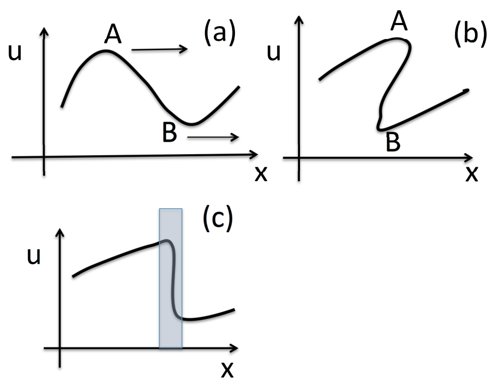


FIGURE 4. Bottleneck phenomenon similar as waves

3. **Mathematical Modeling by Burgers Equation.** We consider the fluctuation characteristics of the turbulent and production lead time of production field by using the Burgers equation. The factors causing fluctuations include the following again [12]:

- Uncertainty of logistics
- Uncertainty of production planning
- Stochastic characteristics of the order and start time series

Linkage of these factors causes the fluctuation; that is, we reported that an on-off intermittency was observed, and then a bottleneck occurs in the production processes.

Figure 5 shows a boundary surface of fluctuation characteristics. In this study, we used the boundary surface characteristics of the fluctuations to develop a solution for Burgers equation. Figure 6 shows the transition between laminar flow and turbulent flow occurs in production processes when an improvement or change of the endogenous parameters is made. A proper understanding of the critical value of the Reynolds number in the vicinity of the turbulence spot is required. This value needs to be defined for each production process; hence, formulating a mathematical model as its foundation is of utmost importance. The turbulence spot represents a fluctuation in free energy. Therefore, a synchronous status can be approached if the turbulence has a reduced spot width and the management person confines the possible production flow to a narrow region between laminar and turbulent flow.

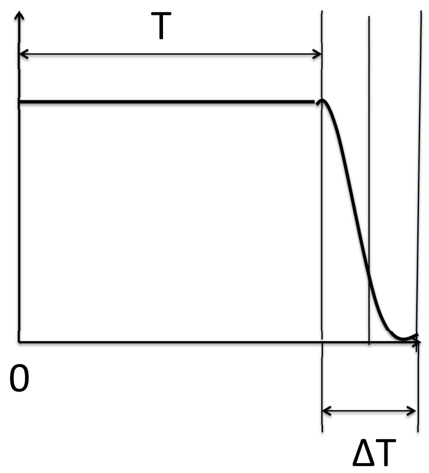


FIGURE 5. Boundary surface of fluctuation characteristics

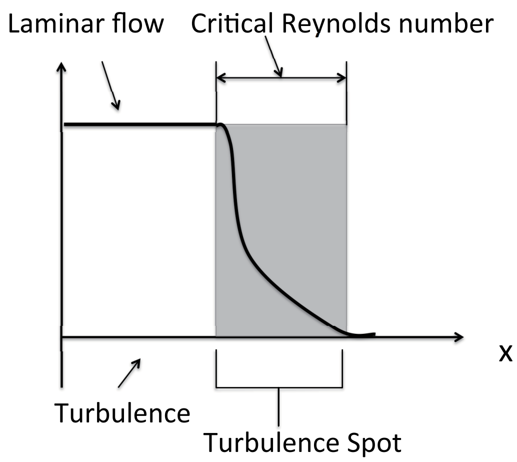


FIGURE 6. Critical Reynolds number (turbulence spot)

Then, the corresponding Burgers equation that ignores the pressure term is as follows [13]:

$$\frac{\partial n}{\partial t} + n\frac{\partial n}{\partial t} = v\frac{\partial^2 n}{\partial x^2} \tag{5}$$

By executing Cole-Hopf transformation [14],

$$n = -v\frac{\partial}{\partial x}\ln\psi \tag{6}$$

where $\psi$ is a production density.

We obtain as

$$\frac{\partial \psi}{\partial t} = v\frac{\partial^2 \psi}{\partial x^2} \tag{7}$$

Here, we considered the model of production processes in detail by the above described model.

**Definition 3.1.** *$C(t,x)$ is a production density.*

A production flow is

$$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial x} \tag{8}$$

Then,

$$J = Cv - D\frac{\partial C}{\partial x} \tag{9}$$

where $D$ is a diffusion coefficient.

Then $D$ is

$$D = \tau v^2 \tag{10}$$

where $v$ is an advection term and $\tau$ is an average parts combination work time.

Further, $v$ is

$$v = v_0\left(1 - \frac{C}{C_s}\right) \tag{11}$$

where $v_0$ is an initial velocity. the production speed is assumed to depend on the production density. $C_s$ is a maximum production density.

From above results,

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)C - 2\left(\frac{v_0}{C_s}\right)C\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0 \tag{12}$$

Then, we introduce the variable $\xi$ for transformation of Equation (12).

$$\xi = -x + v_0 t \tag{13}$$

From Equation (13), Equation (12) is

$$\frac{\partial C}{\partial t} + aC\frac{\partial C}{\partial \xi} = D\frac{\partial^2 C}{\partial \xi^2} \tag{14}$$

where $a = (2v_0/C_s)$.

Equation (14) represents Burgers equation.

Here, assuming that $D = 0$, Equation (14) is derived as follows.

$$\frac{dC_t^L}{dt} = [\eta_t^c - \eta_{c_0}]\left[C_A - C_t^L\right] \tag{15}$$

where $C_t^L$, $C_A$, $\eta_t^c$ and $\eta_{c_0}$ are the production density at $x = L$, the target value of production density at $x = L$, the production capacity and steady state value respectively.

Equation (15) depicts a mean regression equation representing that the state function (throughput) converges to the synchronization constant parameter on average. Considering stochasticity further, it becomes the following.

$$dC_t^L = \eta\left[C_A - C_t^L\right]dt + \sigma dB_t \tag{16}$$

where $\eta = \eta_t^c - \eta_{c_0}$ is a drift value. $\sigma$ and $B_t$ denote a volatility and Wiener process respectively.

Therefore, we propose to estimate the process state by utilizing the Kalman filter as a method to predict uncertainty of $C_t^L$.

4. **Stochastic Analysis of an Ornstein-Uhlenbeck Model for Production Process with Time Delay.** First, the model of the normal process is derived as follows.

**Definition 4.1.** *Normal process* $S_t^n$

$$dS_t^n = \mu_n S_t^n dt + \sigma_n dZ_n \tag{17}$$

where $\mu_n$, $\sigma_n$ and $Z_n$ are average, volatility and Wiener process respectively.

The solution of Equation (17) is derived as follows.

$$S_t^n = S_0^n \exp\left\{\left(\frac{\mu_n - \sigma_n}{2}\right) t + \sigma_n dZ_n\right\} \tag{18}$$

where $S_0^n$ is an initial value.

According to Equation (16), we define a mathematical model with time delay in the production process.

**Definition 4.2.** *Ornstein-Uhlenbeck process $S_t$ for production process with time delay*

$$dS_t = \eta\left(\bar{S} - S_t\right) dt + \sigma_\sigma dZ_t \tag{19}$$

where $\bar{S}$ and $Z_t$ are the target value of production density at $x = L$ and a Wiener process respectively.

Figure 7 depicts a dual problem of this system. Figure 8 indicates an input-output rate control method by a processing line (synchronization of processes). By estimating an expected value and a variance value of throughput of all processes, a manufacturing process is finished with every process finished on time. Here, as a measurement item in a processing line, "average" and "variance" of throughput are made be measurement data. According to process progress of a process, the process makes input request to an input side buffer that is the preceding process. In order to keep lead time (throughput) of the process in question strictly, it controls the "line". With a production rate of the thus controlled processing line, output is performed to an output side buffer.

Regarding control of the output side buffer, by receiving output of the "processing line" that is the preceding process by the buffer, the output side buffer controls, for the subsequent processes, an output rate of its own in order to keep the total throughput strictly. Under such control, it is necessary for a processing line to measure throughput of its own. Therefore, although a throughput function must be obtained by measurement

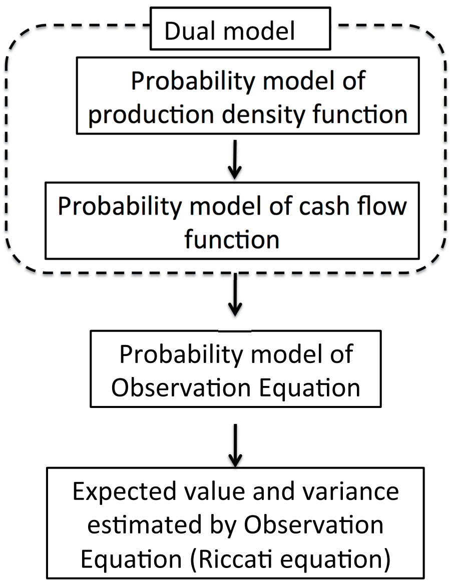


Figure 7. Process of expected value and variance value observed from dual model

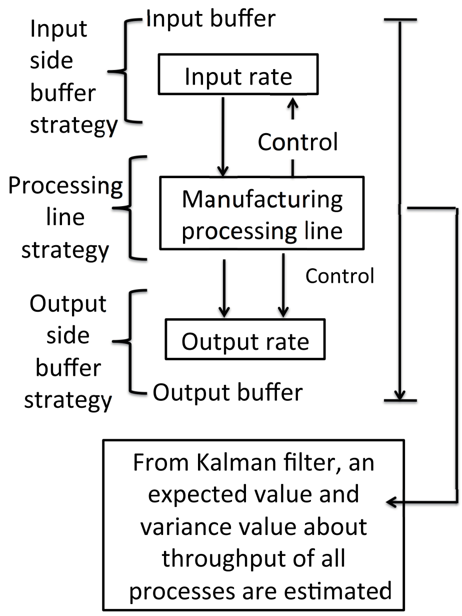


Figure 8. Input-output rate control by a processing line (synchronization of processes) and estimation of expected value and variance of throughput of all processes

in the input side and output side, it is not always entirely observable (complete) over all processes. Therefore, taking an average value and variance of a throughput function as a measurable variable, and using the Kalman filter theory, the average value and variance of the throughput function are estimated.

Next, on the premise of such system model, a method for strictly keeping a manufacturing process in a state where a system is in a non-complete state will be described [16].

## 5. **Option Value of Manufacturing Equipments.**

### 5.1. **State estimation of OU process model using Kalman filter.**

At this time, the option value is defined as the state variable $S_t$ by multiplying a sufficiently large positive constant $K[0,1]$ ($K \in R$). Here, assuming that filtration by a measurement process is $\left\{F_t^{\xi}\right\}_{t=\mathscr{R}}$,

$$F_t \supset F_t^{\xi}, \quad F_t \neq F_t^{\xi} \tag{20}$$

By this, the system becomes a non-complete model [16].

In Figure 8, let a stochastic model of a production process be a stochastic model of an observation equation set as follows.

**Definition 5.1.** *Observables expected value and variance*

$$\mu_t = E\left[S_t | F_t^{\xi}\right] \tag{21}$$

$$v_t = Variance\left[S_t | F_t^{\xi}\right] \tag{22}$$

**Definition 5.2.** *Observation system*

$$d\xi_t = S_t dt + \sigma_\alpha dW_t, \quad t \in \mathscr{R}_t,\ F_t^{\xi} \subset F_t,\ F_t \neq F_t^{\xi},\ \xi_t \in F_t^{\xi} \tag{23}$$

where $\sigma_\alpha$ indicates variance of $\xi_t$, and $W_t$ a standard Brownian motion.

From Kalman filter, we obtain again as follows [2, 17].

$$d\mu_t = m\mu_t dt + \frac{v_t}{\sigma_\alpha}\left[\sigma_\alpha(d\xi_t - \mu_t dt)\right] \tag{24}$$

$$dv_t = \left[-2mv_t + \sigma_\sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right]dt \tag{25}$$

Equation (25) is a Riccati type equation.

From Kalman filter, an expected value and variance value about throughput of all processes are estimated.

$$G(t, \mu_t) = \begin{cases} 0 & \text{Operation stop} \\ \mu_t e^{\theta t} & \text{In the course of determination} \end{cases} \tag{26}$$

Here, Ito's lemma is applied to Equation (26), and then we obtain as follows [18].

$$\begin{aligned} dG(t,\mu_t) &= \theta e^{\theta t}\mu_t dt + e^{\theta t} d\mu_t \\ &= \theta e^{\theta t}\mu_t dt + e^{\theta t}\left[\theta\left\{\left(1 + \frac{v_t}{a\sigma_\sigma^2}\right)\bar{S} - \mu_t\right\}dt + \frac{v_t}{\sigma_\sigma^2}d\xi_t\right] \\ &= \theta e^{\theta t}\mu_t dt + e^{\theta t}\left[\theta\left(1 + \frac{v_t}{a\sigma_\sigma^2}\right)\bar{S}dt + \frac{v_t}{\sigma_\sigma^2}d\xi_t\right] \end{aligned} \tag{27}$$

where let $\theta = \eta/(1+\eta\sigma_\sigma^2)$ and $v_t$ is constrained by the following equation.

$$dv_t = \left[-2av_t + \frac{1}{\sigma_\beta^2} - \frac{v_t^2}{\sigma_\sigma^2}\right]dt \tag{28}$$

Equation (28) is referred to as a Riccati type equation, and can derive an analytical solution.

We obtain as follows by integrating both sides of Equation (27). Please refer Appendix A.

$$\mu_t e^{\theta t} = g_0\mu_0 + \int_0^t e^{\theta t} \cdot \theta\left(1 + \frac{\upsilon_t}{a\sigma_\sigma^2}\right)\bar{S}dt + g_0\frac{\upsilon_t}{\sigma_\sigma^2}\int_0^t e^{\theta\tau}d\xi(\tau) \tag{29}$$

Then,

$$\mu_t = g_0\mu_0 e^{-\theta t} + \int_0^t \theta\left(1 + \frac{\upsilon_t}{a\sigma_\sigma^2}\right)\bar{S}d\tau + g_0\frac{\upsilon_t}{\sigma_\sigma^2}\int_0^t e^{\theta(\tau-t)}d\xi(\tau) \tag{30}$$

Here, the O-U process is depicted as follows.

$$\left.\begin{aligned} dS_t &= \eta\left(\bar{S} - S_t\right)dt + \sigma_\sigma dZ_t \\ d\xi_t &= gS_t dt + \sigma_\sigma dZ_t \end{aligned}\right\} \tag{31}$$

Then, we apply Ito's lemma to Equation (31) as follows.

$$\begin{aligned} \mu_t = g_0\mu_0 \exp&\left(-\frac{\eta}{(1+\eta\sigma_\sigma^2)}\right)t + g_0\int_0^t \frac{\eta}{(1+\eta\sigma_\sigma^2)}\left(1 + \frac{\upsilon_t}{\sigma_\sigma^2}\right)\bar{S}dt \\ &+ g_0\int_0^t \exp\left(\frac{\eta}{(1+\eta\sigma_\sigma^2)}(\tau - t)\right)d\xi(\tau) \end{aligned} \tag{32}$$

Therefore, we obtain as follows.

$$\begin{aligned} \mu_t &= g_0\mu_0 e^{-\theta t} + g_0\frac{\upsilon_t}{\sigma_\sigma^2}\int_0^t e^{\theta(t-s)}d\xi(s) + g_0\int_0^t \theta\left(1 + \frac{\upsilon_t}{\sigma_\sigma^2}\right)dt \\ &= g_0 e^{-\theta t}\left\{\mu_0 + \frac{\upsilon_t}{\sigma_\sigma^2}\int_0^t e^{\theta s}d\xi(s)\right\} + \int_0^t \theta\left(1 + \frac{\upsilon_t}{\sigma_\sigma^2}\right)\bar{S}dt \\ &= g_0 e^{-\theta t}\left\{\mu_0 + \frac{\upsilon_t}{\sigma_\sigma^2}\left(-\frac{1}{\theta}e^{-\theta t}\cdot\xi_t\right)\right\} + g_0\theta\left(1 + \frac{\upsilon_t}{\sigma_\sigma^2}\right)\bar{S}t \end{aligned} \tag{33}$$

According to above analysis, we discuss the following theoretical development.

$$dS_t = \eta\left(\bar{S} - S_t\right)dt + \sigma dZ_t \tag{34}$$

According to Appendix A the solution of Equation (34) is derived as follows.

$$S_t = S_0 e^{-\eta t} + \bar{S}\left(1 - e^{-\eta t}\right) + \int_0^t \sigma e^{-\eta(\tau-t)}dZ(\tau) \tag{35}$$

Therefore, let $S_0 \equiv \mathit{Constant}$ and then the average regression process of $S_t$ is as follows.

$$E[S_t] = S_0 e^{-\eta t} + \bar{S}\left(1 - e^{-\eta t}\right) \tag{36}$$

$$E[S_t] = \bar{S}, \quad t \to \infty \tag{37}$$

If there is no initial condition, Equation (35) is derived as follows.

$$S_t = \bar{S} + \frac{\sigma}{\sqrt{2\eta}}\left(Z\left(e^{2\eta t}\right)\right)e^{-\eta t} \tag{38}$$

If the initial condition is given, Equation (35) is derived as follows.

$$S_t = S_0 e^{-\eta t} + \bar{S}\left(1 - e^{-\eta t}\right) + \frac{\sigma}{\sqrt{2\eta}}Z\left(e^{2\eta t} - 1\right)e^{-\eta t} \tag{39}$$

5.2. **Risk premium value of production business by an estimated data.** We represent the risk premium value of production business. The value calculates the average production value using the estimated value. The observation system is derived again as follows.

$$d\xi_t = S_t dt + \sigma_\alpha dW_t \tag{40}$$

From Kalman filter, we obtain again as follows [2, 17].

$$d\mu_t = m\mu_t dt + \frac{v_t}{\sigma_\alpha}\left[\frac{1}{\sigma_\alpha}(d\xi_t - \mu_t dt)\right] \tag{41}$$

$$dv_t = \left[-2mv_t + \sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right]dt \tag{42}$$

Equation (42) is a Riccati type equation, and an analytical solution can be obtained; however, $v_t$ converges to a constant as $t \to \infty$. Therefore, we calculate the production value $G_t$.

**Definition 5.3.** *Estimated production value $\hat{G}(\mu_t, v_t)$*

$$G_t \equiv \hat{G}(\mu_t, v_t) \tag{43}$$

From Ito's lemma,

$$\begin{aligned}\hat{G}(\mu_t, v_t) &= \hat{G}_\mu\left[m\mu_t + \frac{v_t}{\sigma_\alpha}\left\{\frac{1}{\sigma_\alpha}(d\xi_t - \mu_t)\right\}\right] \\ &\quad + \hat{G}_v\left[-2mv_t + \sigma_\sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right]dt + \frac{1}{2}\hat{G}_{\mu\mu}\left[\left(\frac{v_t}{\sigma_\alpha}\right)^2 dt\right]\end{aligned} \tag{44}$$

Then, the production value evaluation is derived as follows.

$$\begin{aligned}E\left[\hat{G}_t\right] &= E\left[(1 - rdt)\left(\hat{G}_t + d\hat{G}_t\right) + d\xi_t | F_t^\xi\right] \\ &= E\left[\hat{G}_t\right] + E\left[d\hat{G}_t\right] - rE\left[\left(\hat{G}_t + d\hat{G}_t\right)dt\right] + E[\mu_t]\end{aligned} \tag{45}$$

From Equation (45),

$$\begin{aligned}E\left[d\hat{G}_t\right] &= rE\left[\left(\hat{G}_t + d\hat{G}_t\right)dt\right] - E[\mu_t] = r\hat{G}_t - \mu = E\left[m\hat{G}_\mu S_t dt + \frac{v}{\sigma_\alpha}dW_t\right] \\ &= E\left[\hat{G}_v\left(-2mv_t + \sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right)dt\right] + E\left[\frac{1}{2}\hat{G}_{\mu\mu}\left(\frac{v_t}{\sigma_\alpha}\right)^2\right]\end{aligned} \tag{46}$$

We obtain as follows by transforming Equation (46).

$$E\left[d\hat{G}_t\right] = E\left[m\hat{G}_\mu S_t dt\right] + E\left[\hat{G}_v\left(-2mv_t + \sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right)dt\right] + \frac{1}{2}E\left[\hat{G}_{\mu\mu}\left(\frac{v_t}{\sigma_\alpha}\right)^2 dt\right] \tag{47}$$

Therefore, we can obtain as follows.

$$mS_t\hat{G}_\mu dt + \left(-2mv_t + \sigma^2 - \frac{v_t^2}{\sigma_\alpha^2}\right)\hat{G}_v + \frac{1}{2}\hat{G}_{\mu\mu}\left(\frac{v_t}{\sigma_\alpha}\right)^2 = r\hat{G}_t - \mu_t \tag{48}$$

Here, assuming that $v_t$ is a constant data, that is $G_v = 0$, then we can obtain as follows.

$$\frac{1}{2}\left(\frac{v^*}{\sigma_\alpha}\right)^2\frac{\partial^2\hat{G}_\mu}{\partial\mu^2} + mS_t\frac{\partial\hat{G}_\mu}{\partial\mu} + \mu_t - r\hat{G}_\mu = 0 \tag{49}$$

where $v^*$ is a constant.

Then, we can obtain the risk premium value as the empirical equation as follows.

$$P(S_t) = \frac{\kappa_p}{\xi_p} \{(R + C) + \xi_f \cdot S_t\} \tag{50}$$

where $\kappa_p$, $\xi_p$, $R$, $C$ and $\xi_f$ are parameters respectively. $S_t$ denotes the OU process.

6. **Numerical Simulation.** With respect to Figure 9 (case 1) through Figure 12 (case 4), the line with rhombus is shown based on Equation (18) and the line with square is shown based on Equation (39). Figure 9 (case 1) through Figure 12 (case 4) are the comparison figures between normal process and OU process. Parameter values shown in Table 1 are set to appropriate values. The normal process and the OU process do not depend on the parameters. These graphs are presented for reference.

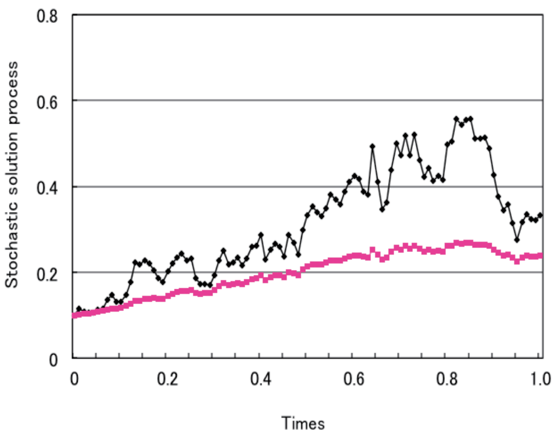


FIGURE 9. Comparison between normal process and O-U process (case 1)

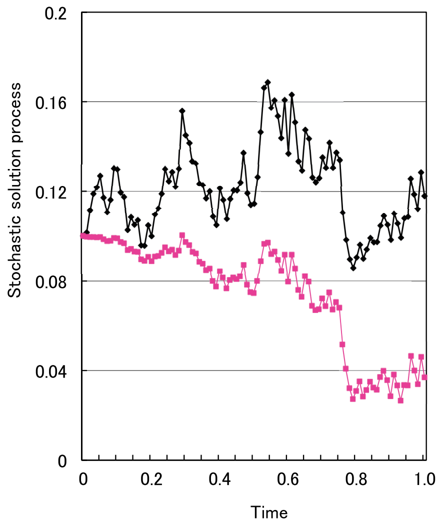


FIGURE 10. Comparison between normal process and O-U process (case 2)

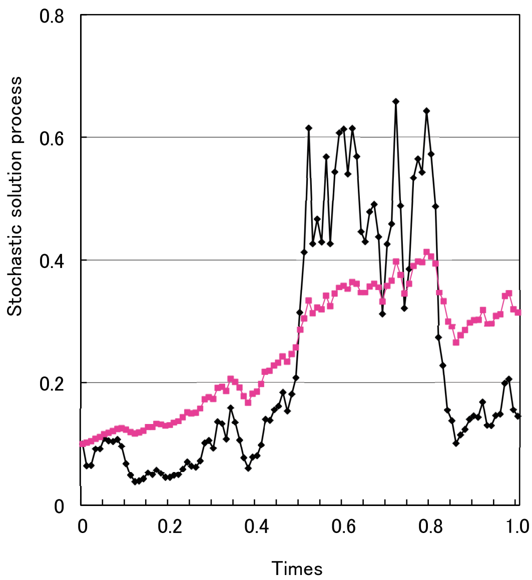


FIGURE 11. Comparison between normal process and O-U process (case 3)

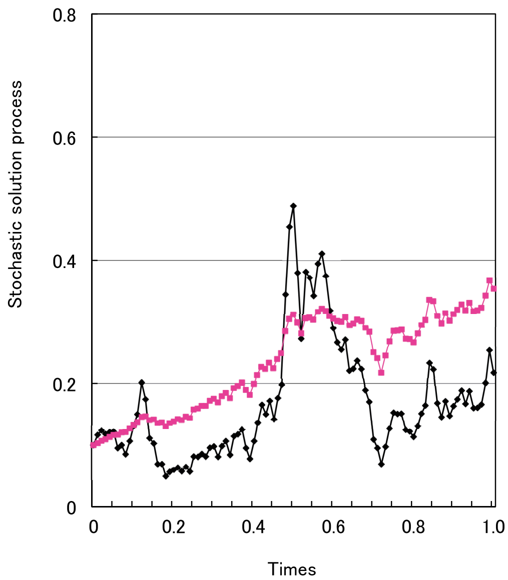


FIGURE 12. Comparison between normal process and O-U process (case 4)

Table 1. Numerical data for simulation

| | Figure 9 (case 1) | Figure 10 (case 2) | Figure 11 (case 3) | Figure 12 (case 4) |
|---|---|---|---|---|
| Normal process ($\mu_n$) | 0.9 | 0.5 | 0.7 | 0.7 |
| Normal process ($\sigma_n$) | 0.1 | 0.1 | 0.2 | 0.2 |
| O-U process ($\eta$) | 0.5 | 0.5 | 0.5 | 0.8 |
| O-U process ($S$) | 0.01 | 0.01 | 0.7 | 0.7 |

With respect to Figure 13 (case 1) through Figure 16 (case 4), the line with rhombus is shown based on Equation (18) and the line with square is shown based on Equation (33). Figure 13 (case 1) through Figure 16 (case 4) are the comparison figures between normal process and estimation process by Kalman filter. Parameter values shown in Table 2 are also set to appropriate values. These graphs are the normal process and its estimation process with several parameters. As far as these graphs are concerned, we estimate the normal process well.

With respect to Figure 17 (case 1) through Figure 20 (case 4), the line with rhombus is shown based on Equation (39) and the line with square is shown based on Equation (33). Figure 17 (case 1) through Figure 20 (case 4) are the comparison figures between OU process and estimation process by Kalman filter. Parameter values shown in Table 3 are also set to appropriate values. These graphs are OU process and its estimation process with several parameters. As you can see from these graphs, we often estimate the trend of the OU process. It seems that estimation accuracy is not good, but the precision fluctuates due to the winner process. It represents the limit of static numerical simulation.

Figure 21 shows the risk premium value, which is proportional to OU process based on Equation (50). The parameter settings are $\kappa_p = 0.8$, $\xi_p = 1$, $R + C = 0.6$ and $\xi_f = 0.6$ in Figure 21. Equation (50) expresses the product value at the exit of the production process, that is, just before shipment. What you can see from Figure 21 is that the value increases on average over time.

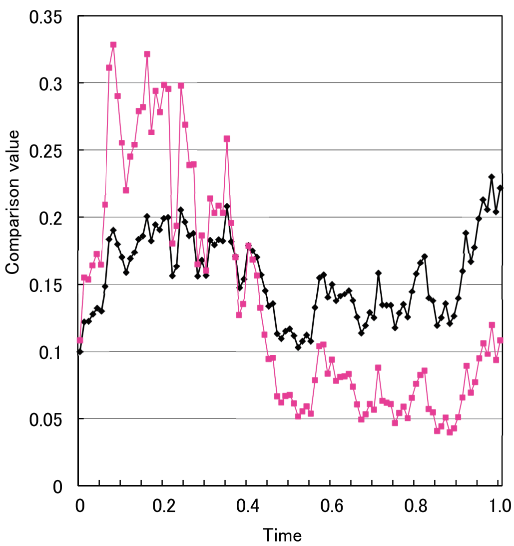


Figure 13. Comparison between normal process and estimation process (case 1)

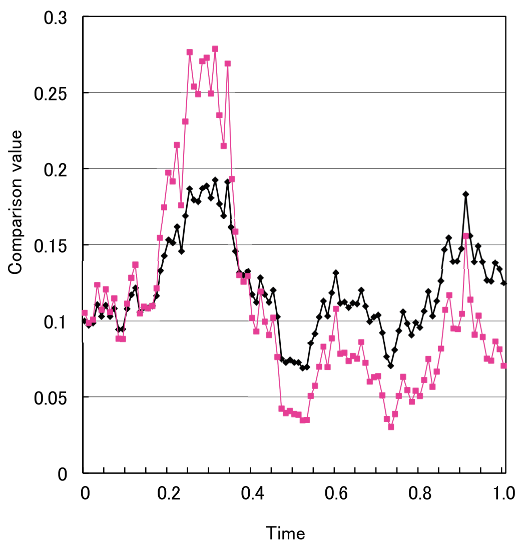


Figure 14. Comparison between normal process and estimation process (case 2)

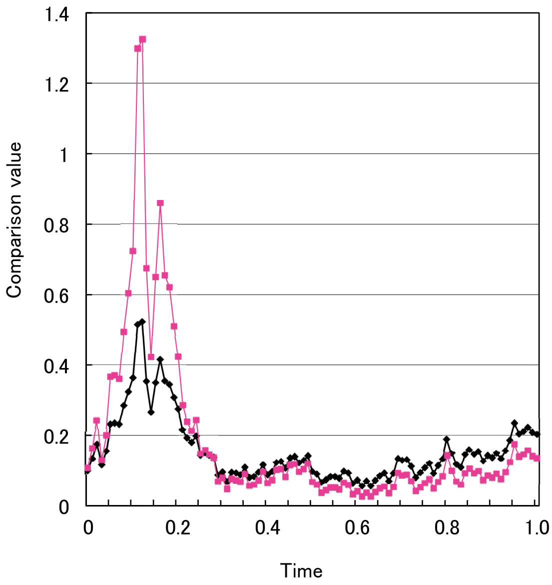


FIGURE 15. Comparison between normal process and estimation process (case 3)

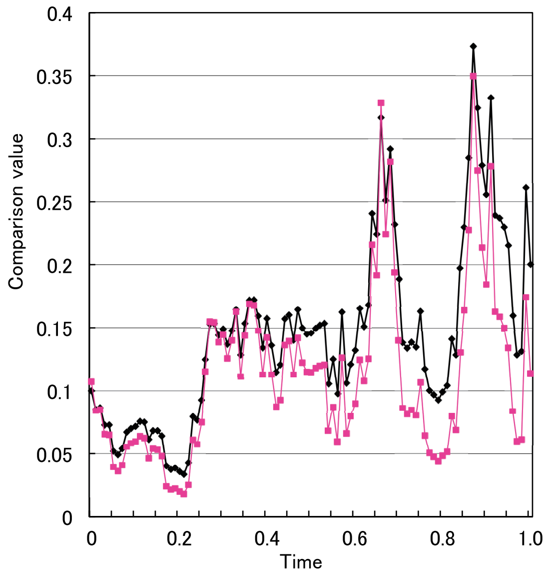


FIGURE 16. Comparison between normal process and estimation process (case 4)

TABLE 2. Numerical data for simulation

| | Figure 13 (case 1) | Figure 14 (case 2) | Figure 15 (case 3) | Figure 16 (case 4) |
|---|---|---|---|---|
| Normal process ($\mu_n$) | 0.9 | 0.5 | 0.7 | 0.7 |
| Normal process ($\sigma_n$) | 0.1 | 0.1 | 0.2 | 0.2 |
| Estimation process (Solution $S_t$) | 0.8 | 0.5 | 0.7 | 0.7 |
| Estimation process ($\sigma_\alpha$) | 0.18 | 0.18 | 0.28 | 0.28 |

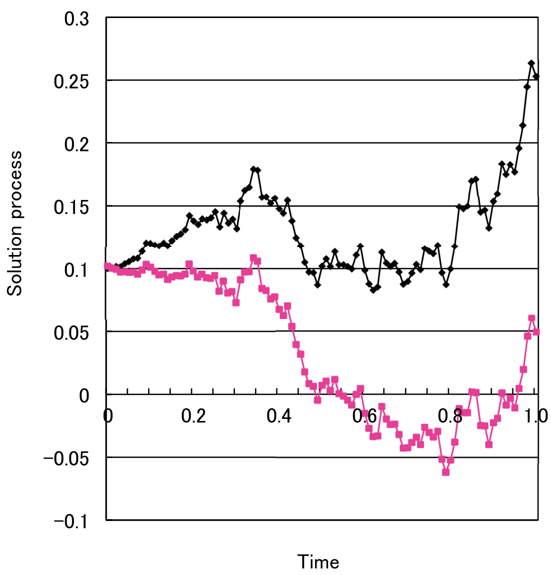


FIGURE 17. O-U process and estimation process (case 1)

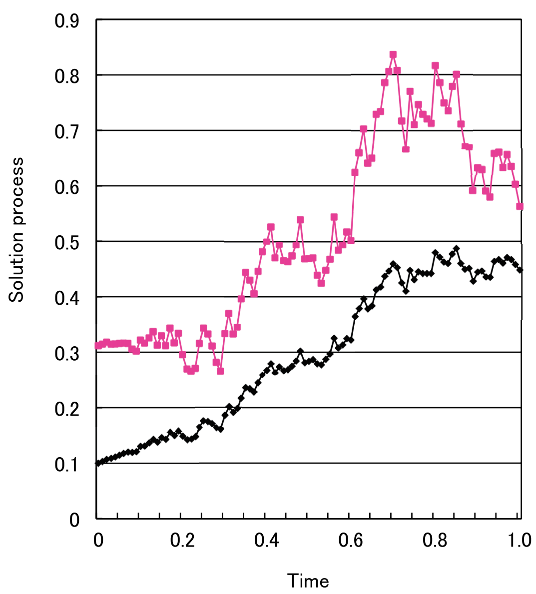


FIGURE 18. O-U process and estimation process (case 2)

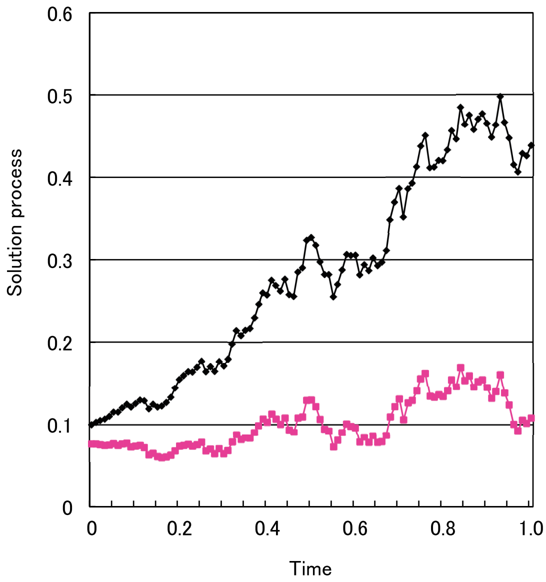


Figure 19. O-U process and estimation process (case 3)

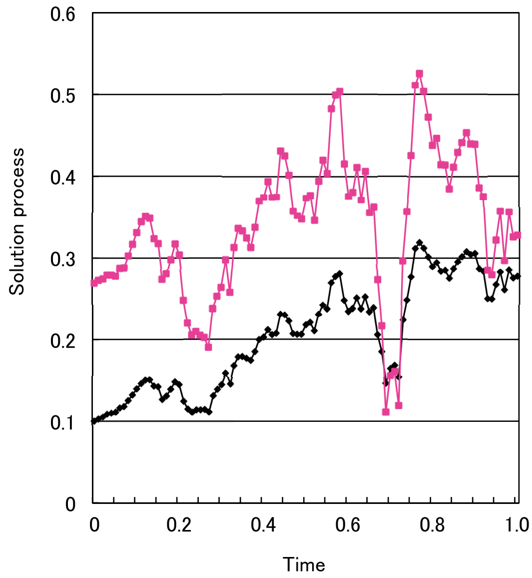


Figure 20. O-U process and estimation process (case 4)

Table 3. Numerical data for simulation

| | Figure 17 (case 1) | Figure 18 (case 2) | Figure 19 (case 3) | Figure 20 (case 4) |
|---|---|---|---|---|
| O-U process (Regression rate $\eta$) | 0.5 | 0.6 | 0.7 | 0.7 |
| O-U process (Initial value $S_0$) | 0.7 | 0.7 | 0.5 | 0.5 |
| Estimation process (Solution $S_t$) | 0.6 | 0.7 | 0.5 | 0.3 |
| Estimation process ($\sigma_\alpha$) | 0.2 | 0.1 | 0.2 | 0.1 |

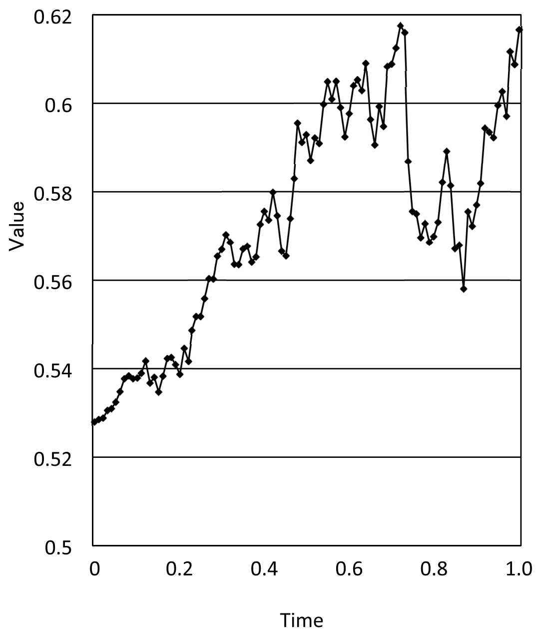


Figure 21. Premium value for a production process

7. **Conclusion.** In this study, we modeled the production process with delay utilizing the Ornstein-Uhlenbeck (OU) process in mathematical finance. We also proposed a system that estimates the OU process by utilizing the Kalman filter. Moreover, we proposed an empirical equation to express a risk premium value. It is a valuable equation to quantify risk premium. Our proposal is worthwhile. A production delay leads to cost increases, and a small-to-midsize firm will have a significant impact on revenues. With respect to the propagation between stages, we would like to propose a mathematical model that considers constraints on propagation on the upstream and downstream sides as a delay.

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**Appendix** A. **Ornstein-Uhlenbeck Process.**

$$r_t = -\theta(r_t - \mu)dt + \sigma dW_t \tag{51}$$

where $\theta$, $\mu$ and $\sigma$ are parameters respectively. $W_t$ is a Wiener process.

Equation (51) can be solved by a constant change method.

$$df(r_t, t) = \theta r_t e^{\theta t} dt + e^{\theta t} dr_t = e^{\theta t}\theta\mu dt + \sigma e^{\theta t} dW_t \tag{52}$$

We can obtain as follows by integration from 0 to $t$.

$$r_t e^{\theta t} = r_0 + \int_0^t e^{\theta s}\theta\mu ds + \int_0^t \sigma e^{\theta s} dW(s) \tag{53}$$

From Equation (53), we obtain as follows.

$$r_t = r_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \int_0^t \sigma e^{\theta(s-t)} dW(s) \tag{54}$$

The Ornstein-Uhlenbeck process can also be expressed as a Wiener process which has changed the scale and shifted time.

Assuming that $r_0$ is zero, $r_t$ is derived as follows.

$$E[r_t] = \mu + \frac{\sigma}{\sqrt{2\theta}} W\left(e^{2\theta t}\right) e^{-\theta t} \tag{55}$$

Assuming that $r_0$ is not zero, $r_t$ is derived as follows.

$$r_t = r_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \frac{\sigma}{\sqrt{2\theta}} W\left(e^{2\theta t} - 1\right) e^{-\theta t} \tag{56}$$