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INTERCEPTING A SUPERIOR MISSILE: A REACHABILITY ANALYSIS OF AN APOLLONIUS CIRCLE-BASED MULTIPLAYER DIFFERENTIAL GAME

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Abstract. *This paper investigates the existence of a reachable set in which we can move backward in time to reach an interception set of lower-speed pursuers with a superior missile in a multi-player pursuit-evasion differential game. The solution of the game is derived as a viscosity solution of the time-dependent Hamilton-Jacobi-Isaacs (HJI) partial differential equations. The final value function of the HJI, uses the Apollonius circle to perform a perfectly encircled formation of the pursuers around the missile.*
Keywords: Differential game, Pursuit-evasion, Trajectory optimization, Reachability analysis

1. **Introduction.** The problem of intercepting a superior evader has been studied in many research works considering the simple motion case with a geometrical point of view and a variant of the Proportional Navigation Guidance Law. R. Isaacs initiated differential games when he studied the military pursuit-evasion type of problems in the Rand Corporation [1, 2]. The pursuit-evasion game that he studied is a two-player zero-sum game where the players have completely opposite interests. Isaacs used the method of "tenet of transition" to resolve the strategic dynamic decision-making in this adversarial environment. The theory of two-player zero-sum differential game has been generalized to the N-player noncooperative (nonzero-sum) game case [3]. The Nash-equilibrium is adopted as the solution concept when the roles of all players are symmetric. The concept of the Stackelberg solution that appeared in a hierarchical (static) optimization problem has also been extended on the differential games due to the works [4, 5]. Through the experiment, Su et al. [6] discovered that the performance of pursuit success is determined by the number of pursuers. In a general situation, the more the number of pursuers is the easier to pursuit the evader. In the field of missile guidance, many two-player differential games have been studied. In [7], a nonlinear differential game intercept guidance law for short-range missiles is derived. S. Kang and H. J. Kim [8] introduced an engagement scenario where the missile guidance was based on a new guidance strategy, which exploits the information from the eigenvalues in the framework of a pursuit-evasion differential game. The multi-player pursuit-evasion games, with simple motion dynamics (where the capability of pursuers and evaders is unequal) as a geometrical problem, are considered in many types of research [9, 10, 11].

In [9] and based on the formation control, the authors studied the pursuit strategy by exploiting the cooperation among pursuers in capturing an evader with higher speed; the

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resulted pursuit strategy provides pursuers future heading direction. Here, the formation means the angular distribution of the pursuers around the evader. In their study on pursuit-evasion robot [10], they concluded that even though the maximum velocity of pursuit may be lower than the evasion's; the pursuer should catch the evader when the evader locates in the convex polygon and all adjacent Apollonius circles (which are formed by each pursuer) and the evader should be tangent or intersect with each other. Recently, in [11] the researchers studied pursuit-evasion games of multiple pursuers and a single high-speed evader with the perfectly encircled formation as an initial condition; it is worth pointing out that their research considers the geometrical approach.

The problem of pursuit-evasion games could be investigated by Optimal Control techniques [12]. One approach is to find one optimal trajectory, for each pursuer, from an initial state to the state of interception (i.e., so-called the Controllability). In this approach, the dynamic game is formulated using Pontryagin Minimum Principle which converts the problem into two-points Boundary Value Problem then solved using the numerical analysis techniques.

Another approach is to solve the Pursuit-Evasion Differential Game using a well-known technique of Optimal Control, the Principle of Dynamic Programming. The Principle of Dynamic Programming considers the viscosity solution of time-dependent Hamilton-Jacobi-Isaacs (HJI) Partial Differential Equations (PDEs), then taking advantage of an implicit surface function of the continuous Backward Reachable Set (BRS) (i.e., so-called the Reachability) for the HJI viscosity solution.

Although a considerable amount of literature has been published on Reachability Analysis of continuous-system for safety verification or Obstacle-avoidance guarantee, we use it to investigate the interception of a superior evader/missile in a pursuit-evasion differential game. Assuming a constant speed ratio at engagement scenarios, a perfectly encircled formation of the pursuers around the missile is considered as the terminal path constraint (or the final value function of the HJI) which enforces the necessary condition to achieve the interception.

To put it briefly, the purpose of this study is to investigate the existence of a reachable set in which we can move backward in time to reach an interception set of the lower-speed pursuers with an evader. It is worth to mention that the pursuers use a non-anticipative strategy as an information pattern of the game.

This research paper is organized as follows. Section 2 presents the problem statement of the research, Subsection 3.1 gives a background about the Hamilton-Jacobi-Isaacs, Subsection 3.2 introduces the backward reachable set, Subsection 3.3 introduces the interception set using Apollonius circles formation, Section 4 shows a numerical solution of the mathematical model, and Section 5 presents the conclusion and future work.

2. Problem Statement and Preliminaries.

Assume a situation where a speed-superior missile m seeks a fixed target g and a group of n pursuers defends g by intercepting m before it reaches its destination. Mathematically, this scenario is known as a multiplayer pursuit-evasion differential game.

Consider one evader e (i.e., missile m) and a set of slower pursuers p_j, $j = 1, \ldots, n$. N denotes the number of all players (one attacker missile and many defending pursuers) and n denotes the number of pursuers only and $\mu^i \in U^i$ represents the control decision for the i^{th} player, U^i is the set of inputs, from which the i^{th} player can choose from, $\mathcal{X}$ is the state vector of the game $\mathcal{X} \coloneqq (e, p_1, \ldots, p_n) \in \mathbb{R}^{N\times m}$, the objective functional of such system is

$$J = \phi\left(\mathcal{X}, t_f\right) + \int_{t_0}^{t_f} L\left(\mathcal{X}, \mu_e, \mu_{p_1}, \mu_{p_2}, \ldots, \mu_{p_n}, t\right) dt \tag{1}$$

The optimal terminal cost is

$$\overset{*}{J} = \left(\overset{*}{\mu}_e, \overset{*}{\mu}_{p_1}, \overset{*}{\mu}_{p_2}, \ldots, \overset{*}{\mu}_{p_n}\right) = \max_{\mu_e \in U^e} \min_{\mu_{p_j} \in \gamma_{p_j}} J\left(\mu_e, \mu_{p_1}, \mu_{p_2}, \ldots, \mu_{p_n}\right) \tag{2}$$

The system dynamics are defined as

$$\dot{\mathcal{X}} = f\left(\mathcal{X}, \mu_e, \mu_{p_1}, \ldots, \mu_{p_n}, t\right) \quad a.e. \;\; t \in [t_0, t_f] \tag{3}$$

where f is the mapping

$$f: \; \mathbb{R}^{N\times m} \times U^e \times U^{p_1} \times \ldots \times U^n \times [t_0, t_f] \rightarrow \mathbb{R}^{N\times m} \tag{4}$$

With the following assumptions:

1) a dynamic game defined by Equation (1), governed by the dynamics given in Equation (3) and constrained by the velocity ratio α between the pursuers and the missile where $\alpha = \frac{v_p}{v_e} < 1$;
2) the interception set $\mathcal{I}_{t_0}$ as zero level set of an implicit, bounded, *Lipschitz* and continuous level set function ϕ_0

$$\mathcal{I}_{t_0} := \left\{\mathcal{X} \in \mathbb{R}^2 \times [0, 2\pi] : \phi_0\left(\mathcal{X}, t_f\right) \le 0\right\} \tag{5}$$

We need to investigate the Backward Reachable Set (BRS) of continuous states started from the interception set $\mathcal{I}_{t_0}$ (or target set), where the pursuers and the missile collide.

3. Reachability Analysis.

3.1. Hamilton-Jacobi-Isaacs (HJI) background.

As we stated before, the differential games are formulated and solved using optimal control techniques. So, the terminal cost value function of the objective functional, given in Equation (1), is described as:

$$V(\mathcal{X}) = \inf_{\mu_{p_j} \in \gamma_{p_j}} \sup_{\mu_e \in U^e} J\left(\mathcal{X}, \mu_e, \mu_{p_1}, \mu_{p_2}, \ldots, \mu_{p_n}, t\right) \tag{6}$$

Using dynamic programming and the principle of optimality, the solution of time-dependent Hamilton-Jacobi-Isaacs (HJI) partial differential equations of the previous problem is the viscosity solution, following reference [21] gives the following two equations:

$$\frac{\partial V(\mathcal{X}, t)}{\partial t} + H\left(\mathcal{X}, V_x, t\right) = 0 \tag{7}$$

subject to

$$V\left(\mathcal{X}, t_f\right) = \phi_0\left(\mathcal{X}, t_f\right) \tag{8}$$

$H(.)$ is the Hamiltonian and is given by

$$\begin{aligned} H\left(\mathcal{X}, V_{\mathcal{X}}, t\right) = \max_{\mu_e \in U^e} \min_{\mu_{p_j} \in U^{p_j}} & \frac{\partial V(\mathcal{X}, t)}{\partial \mathcal{X}} f\left(\mathcal{X}, \mu_e, \mu_{p_1}, \ldots, \mu_{p_n}, t\right) \\ & + L\left(\mathcal{X}, \mu_e, \mu_{p_1}, \ldots, \mu_{p_n}, t\right) \end{aligned} \tag{9}$$

The previous equations are solved backwards in time with terminal value $\phi(\mathcal{X}, 0) = \phi_0(\mathcal{X})$. The Lagrangian $L(.)$ equals zero as there is no running cost.

The optimal decision for the missile is

$$\overset{*}{\mu}_e = \arg\max_{\mu_e \in U^e} \min_{\mu_{p_j} \in \gamma_{p_j}} \frac{\partial V(\mathcal{X}, t)}{\partial \mathcal{X}} f\left(\mathcal{X}, \mu_e, \mu_{p_1}, \ldots, \mu_{p_n}, t\right) \tag{10}$$

The optimal decisions for the pursuers are

$$\overset{*}{\mu}_{p_j} = \arg\max_{\mu_e \in U^e} \min_{\mu_{p_j} \in \gamma_{p_j}} \frac{\partial V(\mathcal{X}, t)}{\partial \mathcal{X}} f\left(\mathcal{X}, \overset{*}{\mu}_e, \mu_{p_1}, \ldots, \mu_{p_n}, t\right) \tag{11}$$

3.2. **Backward Reachable Set (BRS) background.** The problem of pursuit game could be investigated by one optimal trajectory from an initial state to the intercepting state (i.e., Controllability). Another approach, which we consider in this work, is to adopt the Backward Reachable Set (BRS) (i.e., Reachability) as a solution for the pursuit-evasion game described in the previous section. A method for computing backward reachable sets of a solution to the Hamilton-Jacobi, based on level set techniques, is presented in Level Set Toolbox. For more information about the toolbox, we refer the readers to [13].

The reachable set is zero sub-level set of an appropriate function, and the boundary of this set propagated under a nonlinear flow field using a validated numerical approximation of a time-dependent Hamilton-Jacobi-Isaacs (HJI) Partial Differential Equation (PDE). The boundary of the backward reachable set is computed as the zero level set of the viscosity solution of the Hamilton-Jacobi partial differential equation on a grid representing a discretization of the state space [14-20].

Definition 3.1. *The backward reachable set Γ of a system, starting from the interception set $\mathcal{I}_{t_0}$, is the set of all trajectories $\mathcal{X}(t)$, for which there exists an admissible strategy of each pursuer μ_{p_j} for all admissible inputs of the evader, such that $\mathcal{X}(t_f) \in \mathcal{I}_{t_0}$ is reachable from $\mathcal{X}(t) \in \Gamma$ and $t \in [t_0, t_f]$.*

The previous definition might be expressed more formally as:

$$\Gamma = \left\{\mathcal{X} : \exists \mu_{p_j} \in \gamma_{p_j}, \forall \mu_e \in U^e, s \leq t, \exists \tau \in [s, 0], \phi_0\left(\tau, \mathcal{X}, t, \mu_e, \mu_{p_j}(\mu_e)\right) \in \mathcal{I}_{t_0}\right\} \tag{12}$$

The mapping γ_{p_j} is the set of non-anticipative strategies or the information pattern of the game. The information pattern of the game addresses what information the players know about each other's decisions, which directly affects their strategies and the outcome of the game. Consequently, we assume that the set U^e is a set of strategies for the missile. For the pursuers, we give more knowledge about missile's decisions. In other words, the input for each pursuer depends on the decisions of the missile, so-called non-anticipative strategies. For each pursuer, there is a mapping γ_{p_j}:

$$\begin{aligned}\gamma_{p_j} := \Big\{\alpha : U^e \to U^{p^j},\ t > 0,\ e_1,\ e_2 \in U^e, \\ e_1(\tau) = e_2(\tau) \quad \forall \tau \leq t \Rightarrow \alpha[e_1](\tau) = \alpha[e_2](\tau)\Big\}\end{aligned} \tag{13}$$

In backward reachable set, we need to specify the target set and to determine the set of states from which trajectories start, and terminate at that specific target set at exactly time t_0.

In our case as the target set is the interception set $\mathcal{I}_{t_0}$, then Γ_{t_f} is the set of initial configurations where, for any possible control input is chosen by the missile, the pursuers generate disturbance policies that lead to an interception.

3.3. **Interception set using Apollonius circles-based formation.** Following [2, 11], in a pursuit-evasion game with three pursuers $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ and $p_3(x_3, y_3)$, form three Apollonius circles centered at c_1, c_2 and c_3 with radii r_1, r_2 and r_3 respectively.

Pursuer p_j forms an Apollonius circle with a radius equaling to:

$$r_j = \left(\frac{\alpha}{1-\alpha^2}\right)\left(\sqrt{\left(x_{p_j} - x_e\right)^2 + \left(y_{p_j} - y_e\right)^2}\right) \tag{14}$$

and centered at:

$$c_j = \left(\frac{x_{p_j} - x_e\alpha^2}{1 - \alpha^2}, \frac{y_{p_j} - y_e\alpha^2}{1 - \alpha^2}\right) \tag{15}$$

where $\alpha = \frac{v_p}{v_e} < 1$.

A formation of n pursuers is achieved when each pursuer is with the angular separation with the evader where $\theta = \arcsin(\alpha)$, $\frac{v_p}{v_e} \geq \sin(180°/n)$. Figure 1 depicts a simple illustration of the geometry of pursuit-evasion game using Apollonius circles. Figure 2 shows the

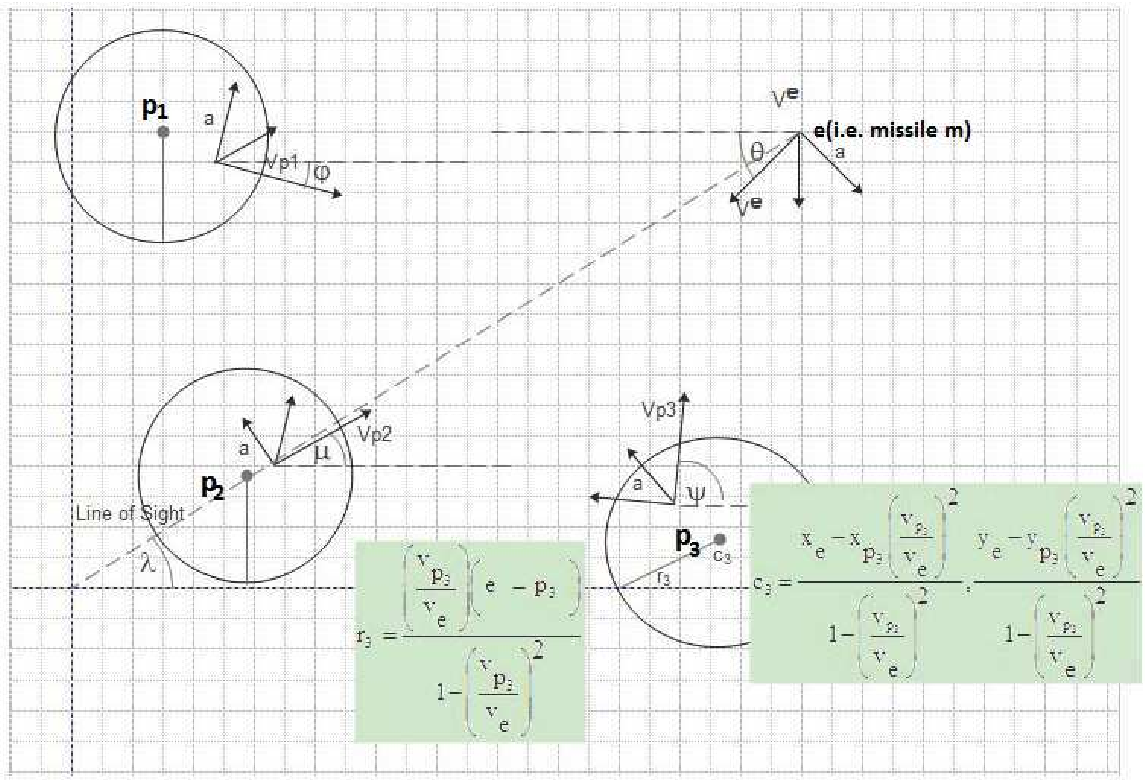


FIGURE 1. The geometry of an engagement scenario among three pursuers and one missile. Pursuer p_3 forms an Apollonius circle centered at c_3 with radius r_3, with velocity vector v_{p_3} and a is the lateral acceleration.

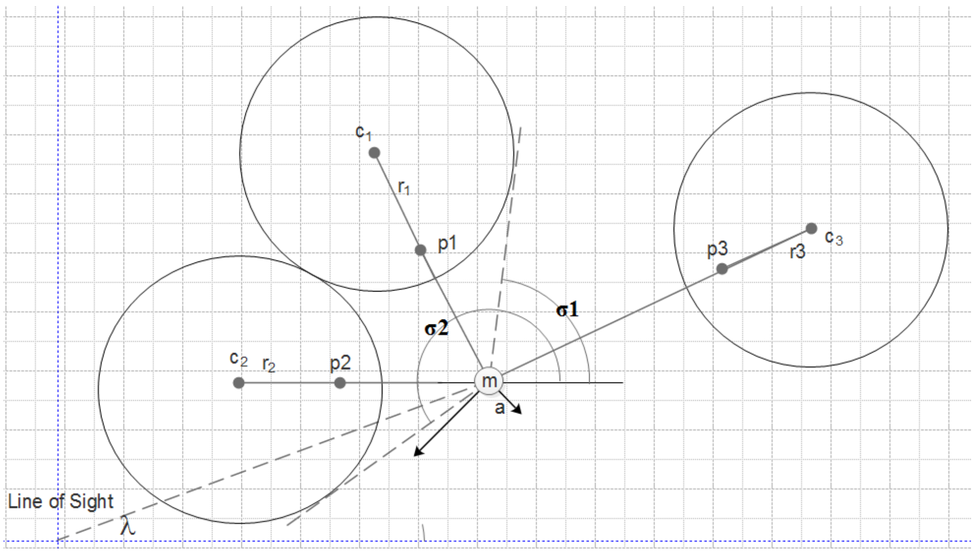


FIGURE 2. The geometry of two slow pursuers p_1 and p_2 forming two tangent Apollonius circles, if the heading angle of the missile lies between σ_1 and σ_2, one of the two pursuers might intercept the missile.

geometry of three slow pursuers and one fast missile, two slow pursuers p_1 and p_2 form two tangent Apollonius circles, if the heading angle of the missile lies between σ_1 and σ_2 then an interception might occur, otherwise, the missile could escape using the gap between p_2 and p_3 or using the gap between p_1 and p_3. Figure 3 shows a perfectly encircled formation of pursuers around the missile which is the initial condition or the terminal value function of the HJI PDE. On the contrary of other geometrical approaches, the solution starts from the assumed interception backward in time to determine the reachable set. In order to determine the BRS starting from the interception set $\mathcal{I}_{t_0}$ with the terminal value function ϕ_0 in Equation (8), we have to define the value function of the interception and the information pattern of the game.

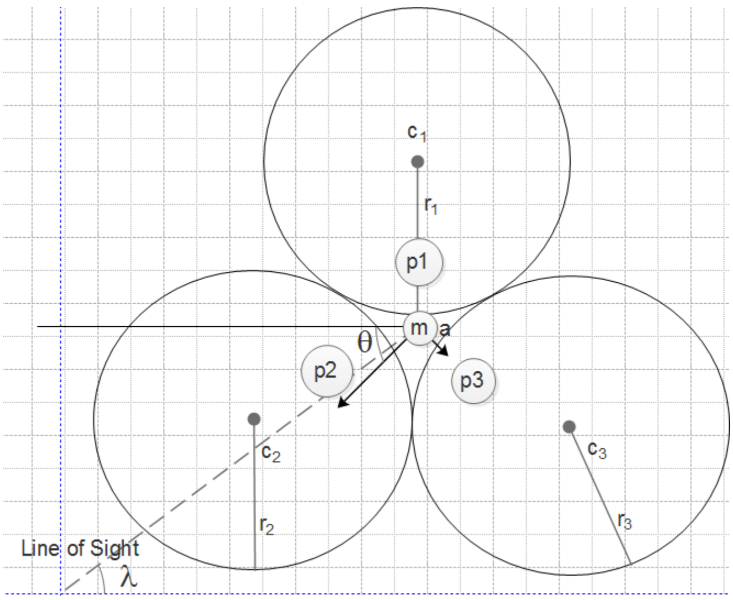


FIGURE 3. The three pursuers perfectly encircle the missile making a possible interception.

Interception set $\mathcal{I}_{t_0}$ is zero level set of the scalar signed distance function ϕ_0, where pursuers' Apollonius circles radii approach distance δ^j with the missile and the relative heading angles should guarantee a perfectly encircled formation constraint. This terminal condition, ensured the pursuers surround the evader with counterclockwise, equally distributed angles. Formally, we formulated the interception set $\mathcal{I}_{t_0}$ of our pursuit-evasion game as the following:

$$\mathcal{I}_{t_0} = \Bigg\{ \left(x_{r_j}, y_{r_j}, \theta_{r_j}\right) \in \mathbb{R}^2 \times [0, 2\pi] : \left(\frac{\alpha}{1-\alpha^2}\right) \left(\sqrt{\left(x_{r_j}\right)^2 + \left(y_{r_j}\right)^2}\right) \leq \delta^j \\ \cap \left(x_{r_j}, y_{r_j}, \theta_{r_j}\right) \in \mathbb{R}^2 \times [0, 2\pi] : \theta_{r_j} = \frac{j\pi}{n}, j \in \{1, 2, \ldots, n\} \Bigg\} \tag{16}$$

In this differential game, the evader is attempting to escape by maximizing the cost function represented by Equation (6), using optimal decision μ_e. The goal of the pursuer p_j is to drive the system into a capture set, as in Equation (16), by minimizing the cost function using optimal disturbance μ_{p_j} from non-anticipative strategy set $\gamma^t_{p_j}$.

4. **Simulation Results.** In this section, we show the results of implementing the solution of a pursuit-evasion differential game with one missile and three pursuers and perform the numerical computation using the Level Set Method Toolbox [22].

4.1. **Mathematical model.** Consider a system dynamics of a pursuit-evasion game, where each player's kinematic equations are as the following:

$$\dot{x} = v \sin\theta \tag{17a}$$

$$\dot{y} = v \cos\theta \tag{17b}$$

$$\dot{\theta} = \omega \tag{17c}$$

$\mathcal{X} = [x, y, \theta]$ is the state space where x, y are the coordinates of the location and θ is the heading angle (deg.), the control input $\mu := |\omega| \leq 1$ is the angular speed and v is the translational velocity (m/sec), $a_c^N = \omega$ is the command lateral acceleration. The evader centric reference frame is a common trick to reduce the dimensionality of the problem by rewriting the dynamics, given in Equation (17), in a new relative coordinate system by fixing the evader e at the origin; the system dynamics of each pursuer p_j becomes:

$$\begin{bmatrix} x_{r^j} \\ y_{r^j} \\ \theta_{r^j} \end{bmatrix} = \begin{bmatrix} (x^j - x^e)\cos\theta^e - (y^j - y^e)\sin\theta^e \\ (x^j - x^e)\sin\theta^e + (y^j - y^e)\cos\theta^e \\ \theta^j - \theta^e \end{bmatrix} \tag{18}$$

The kinematic equations become:

$$\begin{bmatrix} \dot{x}_{r^j} \\ \dot{y}_{r^j} \\ \dot{\theta}_{r^j} \end{bmatrix} = \begin{bmatrix} -\omega^j y_{r^j} + v^j \sin(\theta_r\ ) \\ -v^e + \omega^e x_{r^j} + v^j \cos(\theta_r) \\ \omega^j - \omega^e \end{bmatrix} \tag{19}$$

θ_{r^j} is the relative heading angle between pursuer j and the evader and computed in a counterclockwise direction from positive x-axis.

From Equations (9)-(11) we obtain:

$$\begin{aligned} & H(\mathcal{X}, V_{\mathcal{X}}, \mu_e, u_{p_1}, \mu_{p_2}, \mu_{p_3}, t) \\ &= \lambda_1 v_{p_1} \sin(\theta_{r^1}) + \lambda_2 v_{p_1} \cos(\theta_{r^1}) - \lambda_2 v_e + \mu_e(-y_{r^1}\lambda_1 + x_{r^1}\lambda_2 - \lambda_3) - \mu_{p_1}(\lambda_3) \\ &\quad + \lambda_3 v_{p_2} \sin(\theta_{r^2}) + \lambda_5 v_{p_2} \cos(\theta_{r^2}) - \lambda_5 v_e + \mu_e(-y_{r^2}\lambda_4 + x_{r^2}\lambda_5 - \lambda_6) - \mu_{p_2}(\lambda_6) \\ &\quad + \lambda_6 v_{p_3} \sin(\theta_{r^3}) + \lambda_8 v_{p_3} \cos(\theta_{r^3}) - \lambda_8 v_e + \mu_e(-y_{r^3}\lambda_7 + x_{r^3}\lambda_8 - \lambda_9) - \mu_{p_3}(\lambda_9) \end{aligned} \tag{20}$$

where $\lambda_1, \ldots, \lambda_9$ are the components of the gradient $V_{\mathcal{X}}$.

The optimal decisions for evader and pursuers are computed using:

$$\overset{*}{\mu}_e = sign(-y_{r_1}\lambda_1 + x_{r_1}\lambda_2 - \lambda_3 - y_{r_2}\lambda_4 + x_{r_2}\lambda_5 - \lambda_6 - y_{r_3}\lambda_7 + x_{r_3}\lambda_8 - \lambda_9) \tag{21}$$

$$\overset{*}{\mu}_{p_1} = sign(\lambda_3) \tag{22a}$$

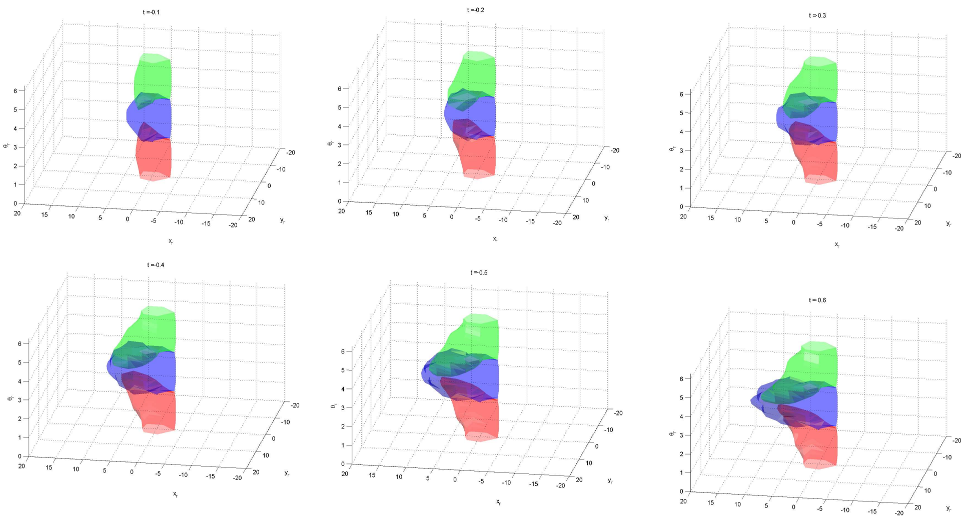
$$\overset{*}{\mu}_{p_2} = sign(\lambda_6) \tag{22b}$$

$$\overset{*}{\mu}_{p_3} = sign(\lambda_9) \tag{22c}$$

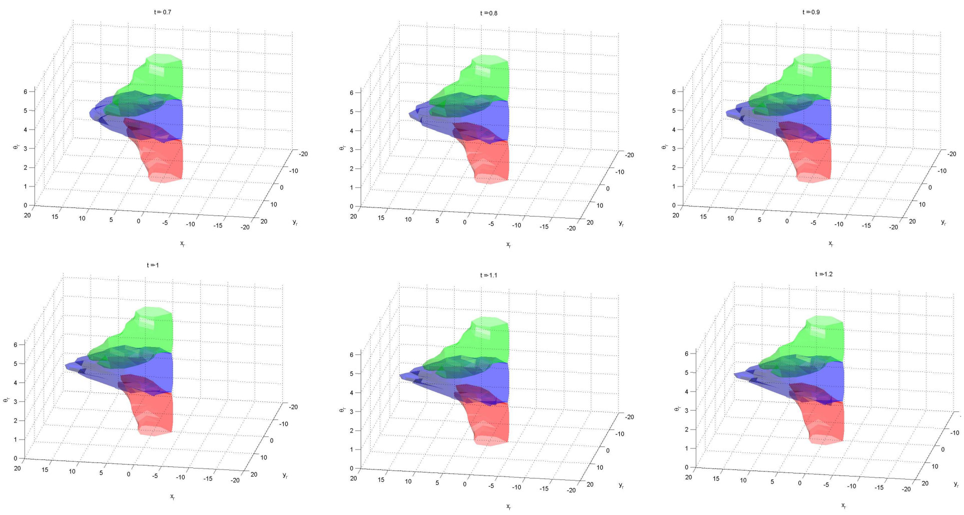
Finally, we define the terminal value function ϕ_0 as the intersection of two implicit surface functions (with max function), which results in an implicit surface function of the set $\mathcal{I}_{t_0}$ (16), as:

$$\phi_0 := \max\left\{ \frac{\alpha}{1-\alpha^2}\sqrt{(x_{r^j})^2 + (y_{r^j})^2}, \frac{j\pi}{n} \right\} \tag{23}$$

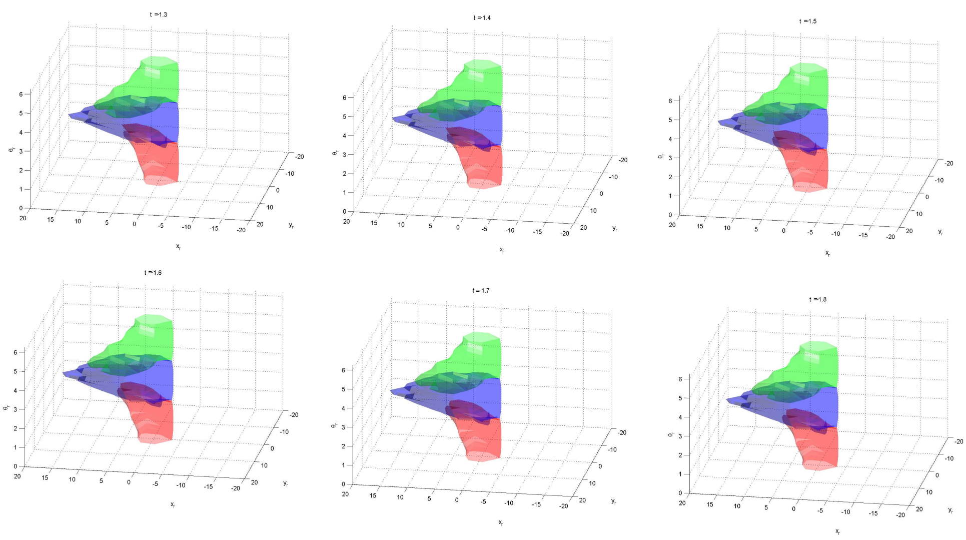
The solution ϕ_t is an implicit surface representation of the finite time backwards reachable set. While the aforementioned toolbox is designed for solving initial value and not terminal value PDEs [13], converting to the initial value PDE form used in the toolbox simply requires multiplying the output of $H(.)$ in Equation (9) by negative one.



(a) Growing of backward reachable sets from $t = 0$ to $t = -0.6$

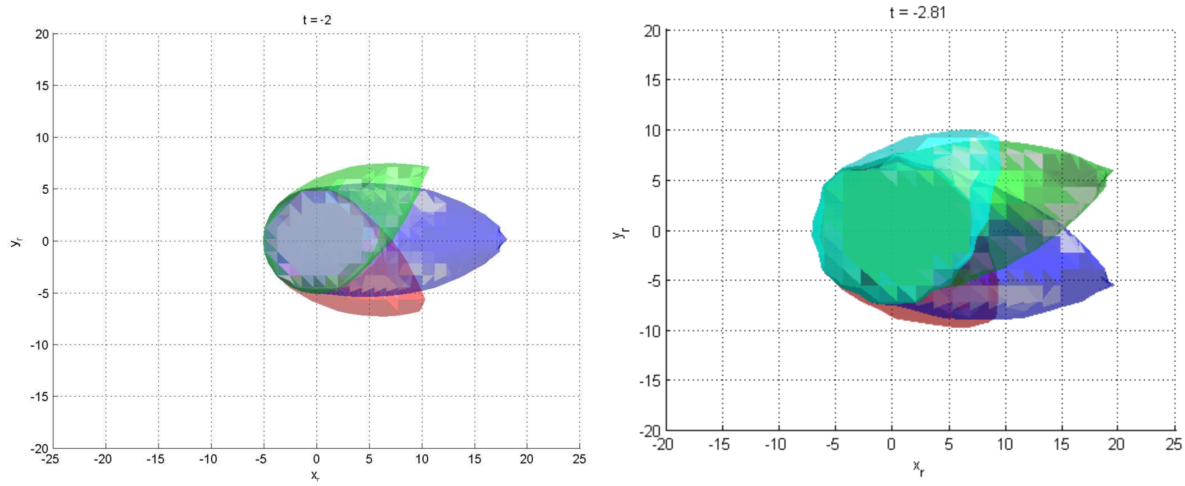


(b) Growing of backward reachable sets from $t = -0.7$ to $t = -1.2$



(c) Growing of backward reachable sets from $t = -1.3$ to $t = -1.8$

FIGURE 4. (color online) Illustration of the growing of the backward reachable set of one missile and three pursuers game



(a) Plan view of backward reachable sets of 3 pursuers

(b) Plan view of backward reachable sets of 4 pursuers

FIGURE 5. The first subplot(a) shows the cross-section of the backward reachable set of the three pursuers and one evader fixed position. Note that as the movement of the evader is parallel with the x axis the maximum reach is due to the opposite direction of one pursuer which has a relative angle equaling 180° with the evader. The second subplot(b) shows the plan view of the reachable set of four pursuers where the same concepts could be applied.

4.2. **An example of four-players game and its system decomposition.** As a proof of concept, a game of four players is considered where the number of pursuers $n = 3$. We have chosen the maximum speed and turn rate of the evader as $v_e = 10$, $\mu_e \in [-1, 1]$, respectively. Each pursuer is with turn rate $\mu_p \in [-1, 1]$ and velocity $v_p = 8.7$; hence, $\alpha = \frac{8.7}{10} \geq \sin(180°/n)$, and the interception set to be approximately an Apollonius circle of radius $\frac{\alpha}{1-\alpha^2}\sqrt{(x_{rj})^2 + (y_{rj})^2} \leq \delta^j$. The algorithms in the toolbox can be used in any number of dimensions, although computational cost and visualization difficulty make dimensions four and higher a challenge [13]. Due to the exponential growth of memory and computational cost, as system dimensions increase, we need to decrease the computations of the value function. The full-dimensional BRS is decomposable when the subsystems have no overlapping states, reduces computation complexity by decomposing the higher-dimensional full system into three lower-dimensional subsystems.

$$\Gamma = \bigcup_{j=1}^{n} (\Gamma_j) \tag{24}$$

The BRS Γ of the game is decomposed into three subsystems. Each Γ_j is computed over 18 time steps on a grid cell size of 60^m, where $m = 3$ is the state space dimension of one pursuer-evader relative measurement. The computation is done using the MATLAB Level Set Toolbox [22] on a PC with a 3 GHz Intel Core i7 processor and memory of 8 GB.

4.3. **Trajectory synthesis for *n* pursuers.** Next in Algorithm 1, we present an algorithm for evaluating each pursuer's trajectory in the differential game. We start by iterating over a specific period of time from $t_f = -4$, and for each pursuer, we calculate the optimal decisions and substitute it in the game dynamics. The trajectory terminates at $t_0 = 0$ or reaches the interception set $\mathcal{I}_{t_0}$ which is an initial set of the backward reachable set Γ produced by the Level Set Toolbox. As an implementation of the previous algorithm, Table 1 summarizes the numerical initial values of the game of three pursuers for three different runs. Figure 6 depicts the movement of each pursuer and the series of slices where the shape, of each slice of the reachable set, depends on the relative heading of the missile and the pursuer.

Algorithm 1 Trajectory generation for each pursuer in the game

Require: $\mathcal{I}_{t_0} \subset \Gamma$ ▷ Γ is the BRS calculated using the Level Set Toolbox [22]
for all $\tau = -t : 0$ **do** ▷ move forward from specific point in time
  for all $j = 1 : n$ **do** ▷ for each pursuer j
    $\mu_e \leftarrow \overset{*}{\mu}_e \in U^e$ ▷ missile's optimal decision using Equation (21)
    $\mu_{p_j} \leftarrow \overset{*}{\mu}_{p_j} \in U^{p^j}$ ▷ pursuer's optimal disturbance using Equation (22)
    $\mathcal{X} \leftarrow f(\mathcal{X}, \mu_e, \mu_{p_j}, \tau)$ ▷ update state using Equation (17)
    if $\mathcal{X} \in \mathcal{I}_{t_0}$ **then** ▷ pursuer's state is in $\mathcal{I}_{t_0}$
      break ▷ terminate current trajectory
    end if
  end for
end for

Table 1. The numerical initial values of the game of four players for three different runs

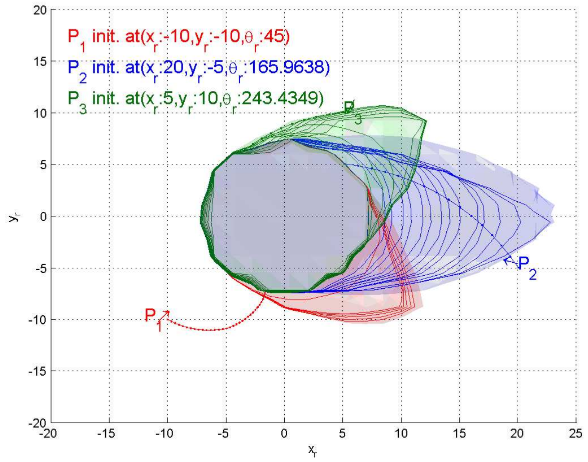
state / $player(i)$	v_i	x_r	y_r	θ_r	x_r	y_r	θ_r	x_r	y_r	θ_r
e	10	0	0							
p_1	8.7	−10	−10	45°	−15	−10	33.7°	20	−15	143.1°
p_2	8.7	20	−5	165.9°	20	−15	134.1°	15	15	255°
p_3	8.7	5	10	243.4°	−5	15	288°	−15	0	360°

4.4. **Discussion.** Our experimental results suggest that the reachability set analysis of the pursuit-evasion game, based on Apollonius circle, helps in determining all reachable sets in which a group of slower pursuers could be in an interception course with a superior missile. The pursuit-evasion game assumes perfect information, and the missile uses an open-loop trajectory while the pursuers use non-anticipative strategy. In a game of four players, where the missile is fixed at the center and surrounded by n pursuers, each pursuer starts at a distance of δ from the center. We set up the initial condition using Level Set Toolbox by constructing three cylinders centered at the original point with radii equal δ^j; each represents the initial interception set, with the missile, from a different set of angles. If one of the pursuers starts outside of the reachable set then the missile has a guaranteed escape strategy.

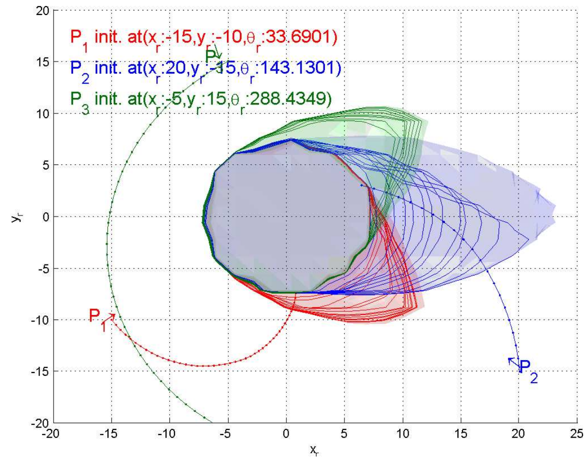
The pursuer p_1 is with relative heading $\theta_{r_1} \in [0, 2\pi/3]$ and colored in red, pursuer p_2 is with relative heading $\theta_{r_2} \in [2\pi/3, 4\pi/3]$ and colored in blue and pursuer p_3 is with relative heading $\theta_{r_3} \in [4\pi/3, 2\pi]$ and colored in green, see Figure 4(a) subplot at time $t = -0.1$.

Starting from interception states, we integrated forward in time until we found the backward reachable set stops growing when time $t \approx -2.0$.

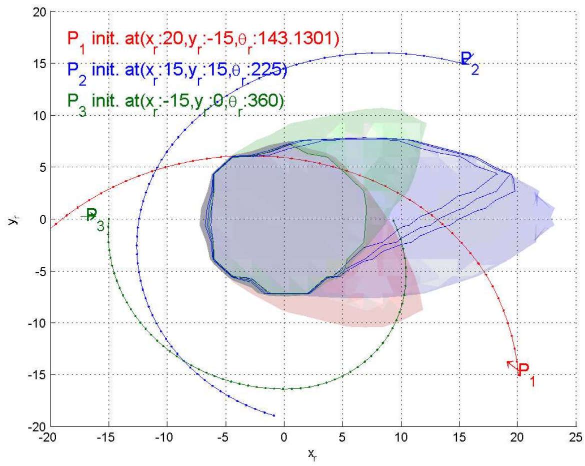
Figure 4, shows the growth of the backward reachable sets of one missile and three pursuers starting from time $t \approx -2$ to time $t = 0$. As the missile moves in parallel to the



(a) Growing of backward reachable sets from $t = -4$, player p_1 starts at $(-10, -10)$ with angle $45°$ and player p_2 starts at $(20, -5)$ with angle $165°$ and player p_3 starts at $(5, 10)$ with angle $243°$.



(b) Growing of backward reachable sets from $t = -4$, player p_1 starts at $(-15, -10)$ with angle $33°$ and player p_2 starts at $(20, -15)$ with angle $143°$ and player p_3 starts at $(-5, 15)$ with angle $288°$.



(c) Growing of backward reachable sets from $t = -4$, player p_1 starts at $(20, -15)$ with angle $143°$ and player p_2 starts at $(15, 15)$ with angle $225°$ and player p_3 starts at $(-15, 0)$ with angle $360°$.

FIGURE 6. Illustration of a set of pursuers which try to reach the initial reachable set making an interception

x-axis, the pursuer p_2 colored in blue, which makes a relative heading angle equals $180°$, moves in the opposite direction which leads to a maximum reach as shown in Figure 5. A plan view of a reachable set of a game of four players and another game of five players are also depicted in Figure 5.

After computing the backward reachable sets of the game, which constructs the control policies that ensure an interception, each pursuer determines its control actions at each instant of time using this BRS. We introduced a simple algorithm which takes a number of pursuers with arbitrary positions and using the resulted BRS to check if the pursuer generates a trajectory that intercepts the missile or not. We run different simulations with a different number of pursuers as well as an arbitrary distribution of their locations and orientations. The trajectory of pursuer $p_1(10, 10, 45°)$ starts from its arbitrary location and ends in the reachable set returned by the Level Set Toolbox, see Figure 6(a).

From the experiment depicted in Figure 6(b) for pursuer $p_3(-5, 15, 288°)$ and the experiment depicted in Figure 6(c) for pursuer $p_2(15, 15, 225°)$, we noticed from the results that the missile could avoid the interception if the pursuer starts outside the collision state.

The same concept could be applied to games of more than three pursuers using system decomposition techniques in order to avoid the curse of dimensionality problem. We believe that our results introduced the use of reachability analysis not limited to safety-critical scenarios for real-time dynamical systems, but as a defense analysis method. The approach could be applied on different scenarios involving a system that is governed by a set of differential equations, such as a set of slower torpedoes facing a faster enemy submarine or a group of slower UAV planned to collide with a faster one.

5. Conclusion and Future Work.

From the outcome of our investigation, it is possible to conclude that the backward reachability analysis could be used in presenting the solution of a multiplayer differential game where a number of pursuers intercept a faster missile. We converted the differential game into time-dependent Hamilton-Jacobi-Isaacs (HJI) PDE, considering the terminal value function to be the Apollonius circle-based path constraint on the pursuers formation.

As an example, we investigated the growing backward reachable sets of one missile and three pursuers and how to generate a trajectory for each pursuer in order to check if the final state of pursuer reaches the interception set. By making use of system decomposition, the same concepts could be extended to many pursuers and one missile in order to overcome the exponential complexity of solving such kind of games. As current and future work, a further experimental investigation of other dynamic games to find one optimal trajectory for each pursuer in a deterministic or a stochastic situation (uncertain environments) using Pontryagin Minimum Principle should be conducted. The dynamic game is supposed to use an open loop strategy which requires that all players decide their entire controls input without any knowledge of the other players' decisions.

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