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FURTHER STUDIES ON ROBUST H_∞ STATE FEEDBACK PLUS STATE-DERIVATIVE FEEDBACK CONTROLLER FOR UNCERTAIN FUZZY DYNAMIC SYSTEMS

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Abstract. *This paper examines the problem of designing a robust H_∞ state feedback plus state-derivative feedback control for a class of uncertain nonlinear systems that is described by a Takagi-Sugeno (T-S) fuzzy model. Linear matrix inequalities (LMIs) approach is employed to obtain the robust controller for such a system. Simultaneously, the illustrative example is given to show the effectiveness of the proposed methodology. The results show that the proposed approach guarantees the fulfillment of both the asymptotic stability and the performance index.*
Keywords: Robust H_∞ control, State feedback, State-derivative feedback, Linear matrix inequalities (LMIs), Takagi-Sugeno (T-S) fuzzy model

1. Introduction.

H_∞ theories for nonlinear problems have been extensively studied and developed in the last 2 decades [1,2]. The aim of H_∞ methods is to achieve stabilization with the prescribed performance index [32,33]. However, the higher-order nonlinear estimation of real-life dynamical system is an important issue in both the analysis and the design of nonlinear control systems [35,36]. Presently, the T-S fuzzy model has been attracted by most researchers due to the fact that the T-S fuzzy model is appropriated for simplifying the dynamics of complex nonlinear systems [37,38] and has been widely used in many different areas [3-6]. The global behavior of a nonlinear system can be explained by the T-S fuzzy model construction procedures. The T-S fuzzy control design is derived by utilizing the concept of parallel distributed compensation (PDC); i.e., a fuzzy system is represented by each plant rule model [7,34]. In addition, the T-S fuzzy model based on the LMIs techniques can be used to solve the stability analysis and the control design problems [8-12]. LMIs based T-S fuzzy model techniques ensure not only stabilization but also important issue of control performance, namely, robustness in fuzzy control system designs [7]. Thus, unquestionably, during the past two decades, various robust H_∞ design approaches based on T-S fuzzy model techniques for uncertain nonlinear systems have been developed in several works [13-17].

The obtained measurable signals, which is one of problems occurring in real mechanical control systems, are the state feedback and state-derivative feedback signals such as the control of suppression systems, where the accelerometers serve as principal sensors of vibration [18]. As previous research works have been shown in [19], it has been found that the state has been greatly limited by the necessity for accurate information about

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parameters that may be difficult to estimate with high precision, while the state derivative is easily obtained. Furthermore, according to [20], the results have shown that the state-derivative signals are easier to obtain than the state signals are. In the case of the controlled vibration suppression of mechanical systems, the main vibration sensors are accelerometers [21]. Thus, for actual accelerations, it is possible to reconstruct velocities with reasonable accuracy, but not displacements [22]. In the case of temperature measurement inside a bauxite smelter, the state-derivative feedback approach can be used in mechanical control systems since the state cannot be measured [23]. The simulation results of controlled design for the rejection of sinusoidal disturbances and tracking sinusoidal reference signals based on state-derivative feedback have shown that the controlled system can reject the disturbances and track the reference signal [24]. In addition, the state-derivative feedback approach provides results with better performance when used as the estimator [25,26]. Moreover, in most cases, the state-derivative feedback design normally provides smaller gains than those for the conventional state feedback design approach [27]. Recently, [28,29] acquired novel results by designing the H_∞ fuzzy state-derivative feedback control using the LMIs technique. Unfortunately, those results have not been applied to a nonlinear system that includes uncertainties. As reported in several studies, these designed approaches have not yet been adequately researched, and these design problems are still challenging.

According to computing perspectives, the design of robust H_∞ fuzzy state feedback plus state-derivative feedback controllers for uncertain nonlinear systems has been aggregated to examine a set of LMIs in conjunction with the T-S fuzzy model approach. The LMIs are quickly solved by employing the convex optimization algorithm. The approach proposed in this paper can significantly mitigate computational difficulties. As T-S fuzzy controller gains are acquired, one is able to directly apply the controller to such a system. The technique reduces design costs associated with the practical use of theoretical outcomes. Therefore, research on robust H_∞ fuzzy state feedback plus state-derivative feedback control design for a class of uncertain nonlinear systems can be conducted from a theoretical or practical point of view, further motivating us to conduct the present study. Consequently, from these motivations, we examine the problem of designing a robust H_∞ fuzzy state feedback plus state-derivative feedback controller for a class of uncertain nonlinear systems.

Therefore, the main contributions and novelty of this paper are threefold. First, the definitions of the H_∞ control problem and asymptotic stability are introduced for the system. Second, the T-S fuzzy model is applied to approximate uncertain nonlinear systems. Third, the LMIs approach is used to develop a means of designing a robust H_∞ fuzzy state feedback plus state-derivative feedback controller that adheres to performance and robustness specifications. This paper is organized as follows. Preliminaries are explained in Section 2. In Section 3, the proposed control strategy is illustrated as a means of designing a robust H_∞ fuzzy state feedback plus state-derivative controller such that the L_2 gain derived from mapping from exogenous input noise to the regulated output is less than a prescribed value for the uncertain nonlinear system as described in Section 2. The results of this approach are demonstrated through an example presented in Section 4. Finally, the conclusions are summarized in Section 5.

2. **Preliminaries.** The T-S fuzzy model is explained by IF-THEN rules that can be used to approximate the nonlinear system by combining the linear models via nonlinear membership functions. A T-S fuzzy model is examined by the i-th rule as follows:

Plant Rule i: IF $\upsilon_1(t)$ is $M_{i1}(t)$ and ... and $\upsilon_\vartheta(t)$ is $M_{i\vartheta}(t)$, THEN

$$\dot{x}(t) = A_i x(t) + B_i u(t) + B_w w(t), \tag{1}$$

$$z(t) = C_i x(t), \tag{2}$$

where $i = 1, 2, \ldots, r$, M_{ij} $(j = 1, 2, \ldots, \vartheta)$ are fuzzy sets, r is the number of IF-THEN rules, $\upsilon(t)$ represents the premise variables, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input, $w(t) \in \Re^p$ is the input disturbance belonging to $L_2[0, \infty)$, $z(t) \in \Re^s$ is the controlled output, and matrices A_i, B_i, B_w and C_i are suitable matrices of the system. In this paper, it is assumed that $\upsilon(t)$ is the vector containing all individual elements $\upsilon_1(t), \ldots, \upsilon_\vartheta(t)$. For any specified state vector and control input, the T-S fuzzy model is inferred as follows.

Let

$$\varpi_i(\upsilon(t)) = \prod_{j=1}^{\vartheta} M_{ij}(\upsilon_j(t))$$

and

$$\mu_i(\upsilon(t)) = \frac{\varpi_i(\upsilon(t))}{\sum_{i=1}^{r} \varpi_i(\upsilon(t))},$$

where $M_{ij}(\upsilon_j(t))$ is the grade of membership of $\upsilon_j(t)$ in M_{ij}. It is assumed in this paper that

$$\varpi_i(\upsilon(t)) \geq 0, \quad \sum_{i=1}^{r} \varpi_i(\upsilon(t)) > 0, \quad i = 1, 2, \ldots, r, \tag{3}$$

for all t. Therefore,

$$\mu_i(\upsilon(t)) \geq 0, \quad \sum_{i=1}^{r} \mu_i(\upsilon(t)) = 1, \quad i = 1, 2, \ldots, r, \tag{4}$$

for all t. To keep our notations simple, we use $\varpi_i = \varpi_i(\upsilon(t))$ and $\mu_i = \mu_i(\upsilon(t))$. Thus, we can generalize that the T-S fuzzy models represent the weighted average of the following forms:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i[A_i x(t) + B_i u(t) + B_w w(t)], \tag{5}$$

$$z(t) = \sum_{i=1}^{r} \mu_i(C_i x(t)), \quad i = 1, 2, 3, \ldots, r. \tag{6}$$

In most real physical systems, depending on the nature of the information of states that is available to the controller, uncertain parameters and disturbances are found within the complexities of the design problem. Robust control methods are designed to achieve robust performance and stability in the presence of bounded modeling errors. Thus, the T-S fuzzy system can be considered with parametric uncertainties as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i[(A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + (B_{w_i} + \Delta B_{w_i})w(t)], \tag{7}$$

$$z(t) = \sum_{i=1}^{r} \mu_i[(C_i + \Delta C_i)x(t)] \quad i = 1, 2, 3, \ldots, r. \tag{8}$$

With the identical controller shown in Figure 1, the robust H_∞ state feedback plus state-derivative controller is written as follows:

$$u(t) = \sum_{j=1}^{r} \mu_j\big(K_{s_j} x(t) - K_{d_j} \dot{x}(t)\big), \quad \forall j = 1, 2, 3, \ldots, r, \tag{9}$$

where matrices A_i, B_i, B_{w_i} and C_i are defined as in the previous section and matrices ΔA_i, ΔB_i, ΔB_{w_i} and ΔC_i represent uncertainties of the system and satisfy the following assumption.

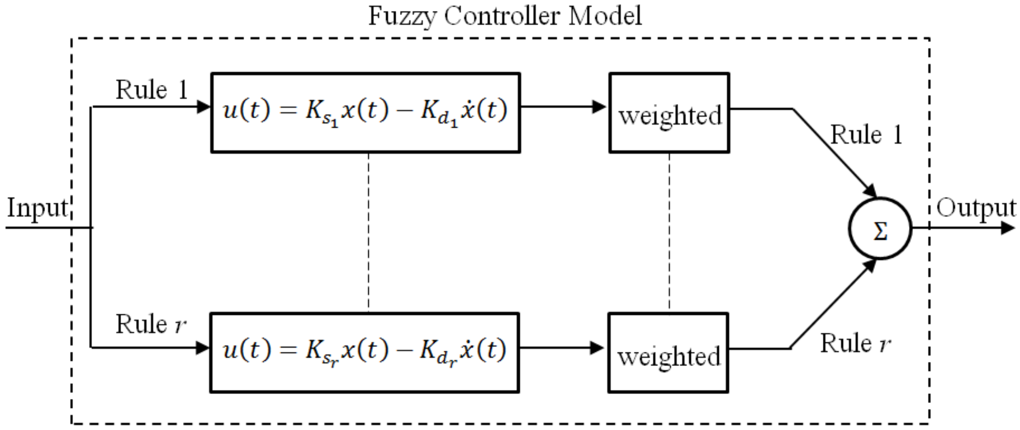

FIGURE 1. The weighted average of the fuzzy controller model

Assumption 2.1. [39]

$$\Delta A_i = F(x(t),t)H_{1_i}, \quad \Delta B_{w_i} = F(x(t),t)H_{2_i},$$
$$\Delta B_i = F(x(t),t)H_{3_i}, \quad \Delta C_i = F(x(t),t)H_{4_i},$$

where H_{j_i}, $j = 1,2,3,4$ are known matrix functions that characterize the structure of uncertainties. Furthermore, the following inequality holds:

$$\|F(x(t),t)\| \le \rho$$

for any known positive constant ρ.

Note that according to [39], for simplicity, in this paper we assume that the uncertainties of the system satisfy Assumption 2.1 due to the fact that it is possible to apply for the real physical system. In addition, in the computation point of view, we can easily obtain the results since it has less computational complexity and less computational time.

Next, let us recall the following definitions.

Definition 2.1. *Suppose γ is a given positive real number. A system of form (7) is said to have an L_2 gain less than or equal to γ if*

$$\int_0^{T_f} z^T(t)z(t)dt \le \gamma^2 \left[\int_0^{T_f} w^T(t)w(t)dt\right] \tag{10}$$

for all $T_f \ge 0$ and $w(t) \in L_2[0,T_f]$.

Definition 2.2. (Asymptotic stability [30,31]) *Let $x_e = 0$ be an equilibrium for $\dot{x} = f(x)$. Let $V : R^n \longrightarrow R$ be a continuously differentiable function such that*

- *$V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.*
- *$\dot{V}(x) < 0$ for all $x \neq 0$, $\dot{V}(0) = 0$.*

Then, x_e is asymptotically stable and is the unique equilibrium point.

Note that for the symmetric block matrices, we use (*) as an ellipsis for terms induced by symmetry. Thus, the following results address system (7) and (8).

3. **Main Results.** This section opens by considering the problem of designing an H_∞ state feedback plus state-derivative feedback controller that guarantees L_2 gains from exogenous input noise to a regulated output of less than or equal to a prescribed value and ensures that the closed-loop system is asymptotically stable. An LMI approach is used to derive a fuzzy controller that stabilizes the system (7) and (8). Suppose that there is a fuzzy state feedback plus state-derivative controller of the following terms:

Controller Rule j: IF $x_{k_1}(t)$ is $M_{1i}(t)$ and … and $x_{k_j}(t)$ is $M_{ji}(t)$, THEN

$$u(t) = K_{s_j}x(t) - K_{d_j}\dot{x}(t), \quad \forall j = 1, 2, 3, \ldots, r, \tag{11}$$

where $x(t)$ is a state vector and K_{s_j} and K_{d_j} are the controller gains of an H_∞ state feedback controller and of a state-derivative feedback controller, respectively. Finally, the fuzzy controller shown in Figure 1 can be inferred as

$$u(t) = \sum_{j=1}^{r} \mu_j\big(K_{s_j}x(t) - K_{d_j}\dot{x}(t)\big), \quad \forall j = 1, 2, 3, \ldots, r. \tag{12}$$

Before presenting the next results, the following lemma is recalled.

Lemma 3.1. [28] *Given the system (5) and (6), a scalar* $\gamma > 0$ *and the inequality (10) holds if there exists a positive definite symmetric matrix* $P > 0$ *and matrices* Y_{s_j} *and* Y_{d_j}*,* $j = 1, 2, \ldots, r$*, satisfying the following linear matrix inequalities:*

$$P > 0, \tag{13}$$

$$\begin{pmatrix} \Pi_{ij} & (*)^T & (*)^T & (*)^T \\ B_{w_i}^T & -\gamma^2 I & (*)^T & (*)^T \\ C_iP + C_iY_{d_j}^TB_i^T & 0 & -I & (*)^T \\ (Y_{s_j} + Y_{d_j})^TB_i^T & 0 & 0 & -P \end{pmatrix} < 0, \quad \forall i, j = 1, 2, \ldots, r, \tag{14}$$

where

$$\Pi_{ij} = PA_i^T + A_iP + Y_{s_j}^TB_i^T + B_iY_{s_j} + B_iY_{d_j}A_i^T + A_iY_{d_j}^TB_i^T.$$

The suitable choice of the fuzzy controller is

$$u(t) = \sum_{j=1}^{r} \mu_j\big(K_{s_j}x(t) - K_{d_j}\dot{x}(t)\big), \quad \forall j = 1, 2, 3, \ldots, r, \tag{15}$$

where $K_{s_j} = Y_{s_j}P^{-1}$ *and* $K_{d_j} = Y_{d_j}P^{-1}$*.*

Regarding [28] and Lemma 3.1, the controllers using the fuzzy state feedback plus state-derivative feedback based on LMIs technique to achieve a prescribed performance and stability are developed. Unfortunately, that approach has not been applied to an uncertainty nonlinear system. Especially, the phenomena of uncertain parameters and disturbances are frequently encountered in most real dynamical systems. These problems are found within the complexity of designing the problems. Thus, by motivated from [28], this research work then proposes the robust control methods for a class of uncertain nonlinear system with aiming to achieve the robust performance and the stability in the presence of bounded modeling errors. From Assumption 2.1, the closed-loop fuzzy system (7) and (8) and the controller (12) shown in Figure 2 can be expressed as follows:

$$\left[I + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j B_iK_{d_j}\right]\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \left[A_ix(t) + B_iK_{s_j}x(t) + \tilde{B}_{w_i}\tilde{w}(t)\right], \tag{16}$$

where $\tilde{B}_{w_i} = [\delta I \;\; I \;\; \delta I \;\; B_{w_i}]$ and the disturbance is

$$\tilde{w}(t) = \begin{bmatrix} \frac{1}{\delta} F(x(t), t) H_{1_i} E_{ij} x(t) \\ F(x(t), t) H_{2_i} w(t) \\ 0 \\ w(t) \end{bmatrix}.$$

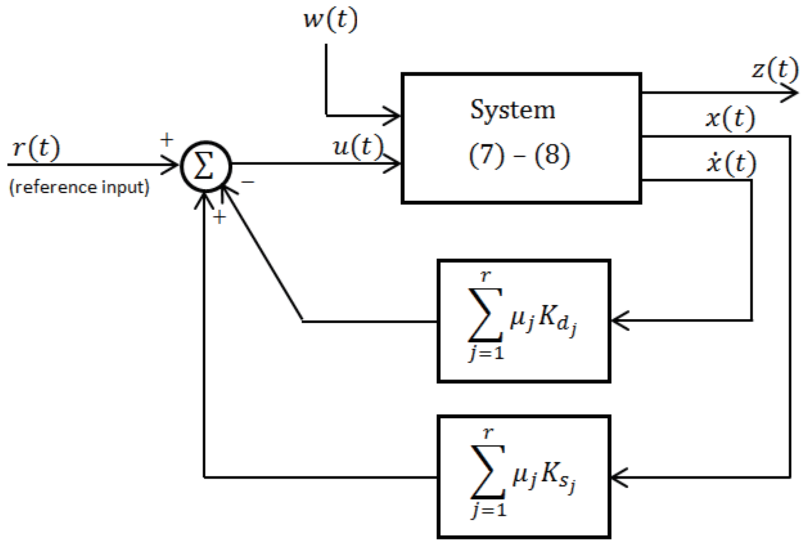

FIGURE 2. The closed-loop fuzzy system

Remark 3.1. *The goal is to obtain state feedback gains and state-derivative feedback gains K_{s_j} and K_{d_j} $(j = 1, 2, \ldots, r)$, respectively, such that the following conditions hold.*

1) Matrices $(I + B_i K_{d_j})$, $\forall i, j = 1, 2, 3, \ldots, r$ have full rank.

2) The system (7) and (8) with the fuzzy controller (12) is asymptotically stable, and the H_∞ performance is satisfied for all admissible values based on the sufficient condition for a prescribed scalar$y > 0$.

To establish the proposed results and without sacrificing generality, we apply the following assumption: rank $[I \,|\, B_i] = n$ exists. Thus, it is easy to conclude that if rank $[I \,|\, B_i] = n$ holds, then K_{d_j} exists such that rank $[I + B_i K_{d_j}] = n$ (i.e., matrices $(I + B_i K_{d_j})$, $\forall i, j = 1, 2, 3, \ldots, r$ have full rank).

From Remark 3.1 and Assumption 2.1, we define

$$E_{ij} = (I + B_i K_{d_j})^{-1}, \tag{17}$$

and thus, (16) can be written as

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \left[E_{ij} \left(A_i + B_i K_{s_j} \right) x(t) + E_{ij} \tilde{B}_{w_i} \tilde{w}(t) \right]. \tag{18}$$

An LMI approach is applied to deriving a fuzzy controller that stabilizes the system (18) and that guarantees the disturbance rejection of level $\gamma > 0$ immediately. First, to design the state feedback plus state-derivative feedback controller, the following design objectives must be satisfied.

(a) The closed-loop system is asymptotically stable when $w(t) = 0$.

(b) Under zero initial conditions, the system (18) satisfies $\|z\|_2 \le \gamma \|w\|_2$ for any nonzero $w(t) \in L_2 [0, +\infty)$, where $\gamma > 0$ is a prescribed constant.

The following theorem provides sufficient conditions for the existence of a robust H_∞ fuzzy state feedback plus state-derivative feedback. These sufficient conditions can be derived by the Lyapunov approach.

Theorem 3.1. *Consider the system (7) and (8). Given a prescribed H_∞ performance $\gamma > 0$ and a positive constant δ, there are symmetric matrices $P > 0$ and matrices Y_{s_j} and Y_{d_j}, $j = 1, 2, \ldots, r$, satisfying the following linear matrix inequalities:*

$$\Xi_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{19}$$

$$\Xi_{ij} + \Xi_{ji} < 0, \quad i < j \le r, \tag{20}$$

where

$$\Xi_{ij} = \begin{pmatrix} \Phi_{ij} & (*)^T & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T & (*)^T \\ \tilde{C}_i P + \tilde{C}_i Y_{d_j}^T B_i^T & 0 & -I & (*)^T \\ (Y_{s_j} + Y_{d_j})^T B_i^T & 0 & 0 & -P \end{pmatrix}, \tag{21}$$

with

$$\Phi_{ij} = PA_i^T + A_i P + Y_{s_j}^T B_i^T + B_i Y_{s_j} + B_i Y_{d_j} A_i^T + A_i Y_{d_j}^T B_i^T,$$

$$\tilde{C}_i = \left[\begin{array}{cccc} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \sqrt{2}\lambda\rho H_{3_i}^T & \sqrt{2}\lambda C_i^T \end{array} \right]^T,$$

$$\lambda = \left(1 + \rho^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \left[\| H_{2_i}^T H_{2_j} \| \right] \right)^{\frac{1}{2}}.$$

Furthermore, the suitable fuzzy controller is written as

$$u(t) = \sum_{j=1}^{r} \mu_j \big(K_{s_j} x(t) - K_{d_j} \dot{x}(t) \big), \quad \forall j = 1, 2, 3, \ldots, r, \tag{22}$$

where

$$K_{s_j} = Y_{s_j} P^{-1},$$

and

$$K_{d_j} = Y_{d_j} P^{-1}.$$

Proof: Refer to Appendix 1 for the proof. □

4. Numerical Example.

Example 4.1. *This example presents the model of tunnel diode circuit which is one of the well-known benchmarks in uncertain nonlinear problems. A common issue of a control system is the voltage and current control in a tunnel diode circuit. Let us consider the following characterized equation, a nonlinear tunnel diode circuit system with an uncertainty parameter and disturbance is investigated in this example* [16,30]*:*

$$\begin{aligned} C\dot{x}_1(t) &= -0.2x_1(t) + 0.01x_1^3(t) + x_2(t) + 0.01w(t), \\ L\dot{x}_2(t) &= -x_1(t) - (R \pm \Delta R)x_2(t) + u(t), \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \end{aligned} \tag{23}$$

where $x_1(t) = v_C(t)$ and $x_2(t) = i_L(t)$ are the state variables, $u(t)$ is the control input, $w(t)$ is the disturbance input noise, and $z(t)$ is the controlled output. The parameters in *the circuit are given as follows:* $R = 1\ \Omega$, $C = 100\ mF$, $L = 1000\ mH$, *and* $\Delta R = 0.3\%$ *is an uncertain term. Substituting the parameters into (23), we obtain*

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + 0.1x_1^3(t) + 10x_2(t) + 0.1w(t), \\ \dot{x}_2(t) &= -x_1(t) - (1 \pm 0.3\%)x_2(t) + u(t), \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{24}$$

For the sake of simplicity, we will use as few rules as possible. Assuming that $|x_1(t)| \le 3$, *and the nonlinear system plant can be approximated using two T-S fuzzy rules. Let us choose the membership functions of the fuzzy sets as follows:*

$$N_1(x_1(t)) = 1 - \frac{|\ x_1(t)\ |}{3}, \text{ and } N_2(x_1(t)) = \frac{|\ x_1(t)\ |}{3}. \tag{25}$$

Note that $N_1(x_1(t))$ *and* $N_2(x_1(t))$ *can be interpreted as membership functions of the fuzzy sets shown in Figure 3. Using these two fuzzy sets, the uncertain nonlinear system can be represented by the following T-S fuzzy model:*

Plant Rule 1: IF $x_1(t)$ *is* $N_1(x_1(t))$, *THEN*

$$\begin{aligned} \dot{x}(t) &= [A_1 + \Delta A_1]x(t) + B_w w(t) + B_1 u(t), \\ z(t) &= C_1 x(t), \end{aligned}$$

Plant Rule 2: IF $x_1(t)$ *is* $N_2(x_1(t))$, *THEN*

$$\begin{aligned} \dot{x}(t) &= [A_2 + \Delta A_2]x(t) + B_w w(t) + B_2 u(t), \\ z(t) &= C_2 x(t) \end{aligned}$$

where

$$A_1 = \begin{bmatrix} 2 & 10 \\ -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.9 & 10 \\ -1 & -1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

With $\Delta A_1 = F(x(t), t)H_{1_1}$, $\Delta A_2 = F(x(t), t)H_{1_2}$ *and assuming that* $\|F(x(t), t)\| \le \rho = 1$, *we have*

$$H_{1_1} = H_{1_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

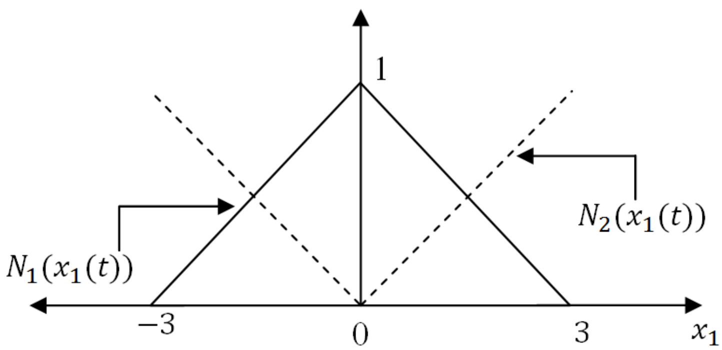

FIGURE 3. Membership functions for the two fuzzy sets used in Example 4.1

From the LMI optimization algorithm and Theorem 3.1 with $\gamma = 1$*, we have*

$$K_{s_1} = \begin{bmatrix} 8.7892 & 19.8802 \end{bmatrix}, \quad K_{s_2} = \begin{bmatrix} 8.9069 & 20.1551 \end{bmatrix},$$
$$K_{d_1} = \begin{bmatrix} -9.3789 & -12.8072 \end{bmatrix} \text{ and } K_{d_2} = \begin{bmatrix} -9.3694 & -12.8004 \end{bmatrix}.$$

The resulting fuzzy controller is

$$u(t) = \sum_{j=1}^{2} \mu_j \left(K_{s_j} x(t) - K_{d_j} \dot{x}(t) \right), \tag{26}$$

where $\mu_1 = N_1(x_1(t))$ *and* $\mu_2 = N_2(x_1(t))$.

Remark 4.1. *The fuzzy controller (26) ensures that the inequality (10) holds. Figures 4 and 5 present the state variables ($x_1(t)$ and $x_2(t)$) of Theorem 3.1 and the disturbance input signal, $w(t)$, used during the simulation, respectively. As shown in Figure 6, after 0.55 seconds, the ratio of the regulated output energy to the disturbance input noise energy approaches a constant value of less than the prescribed value of 1.*

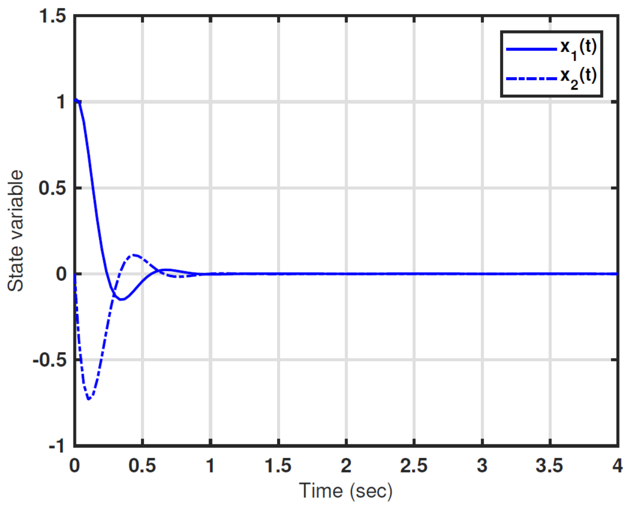

FIGURE 4. State variables of Example 4.1, $x_1(t)$, and $x_2(t)$

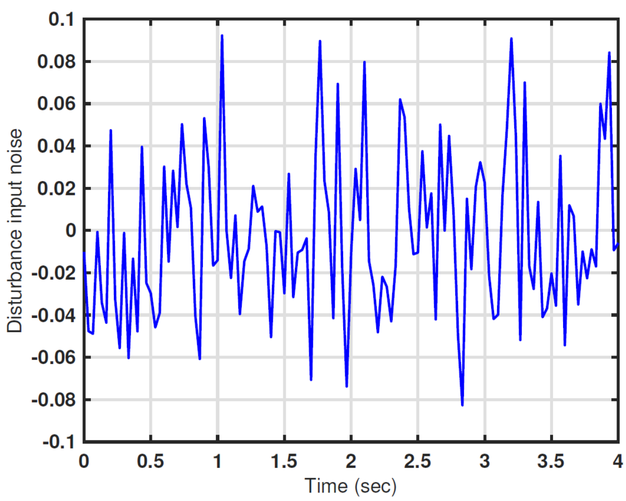

FIGURE 5. Disturbance input noise used in Example 4.1, $w(t)$

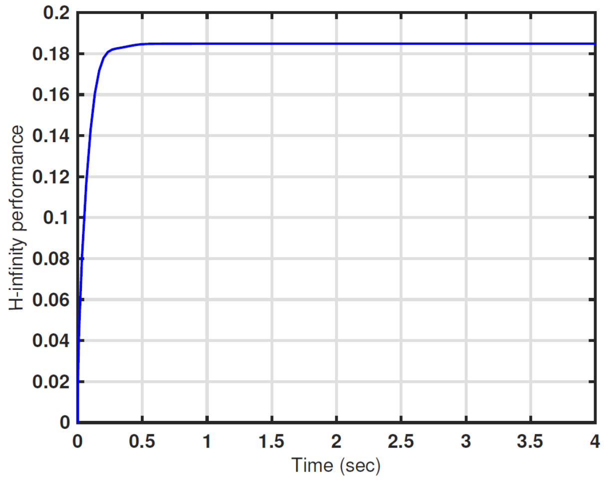

FIGURE 6. H_∞ performance of Example 4.1, $\left(\sqrt{\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}}\right)$

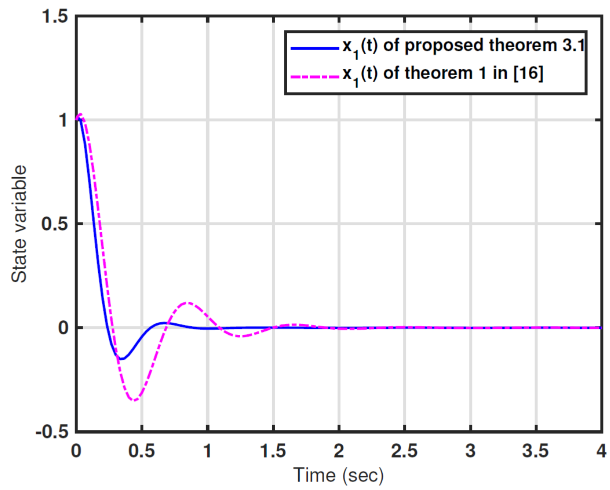

FIGURE 7. Comparison of state variable of Example 4.1, $x_1(t)$

Remark 4.2. *According to Theorem 1 used in* [16] *and Theorem 3.1 used in this paper, Figure 7 presents the comparative results for the state variable* $x_1(t)$ *at the same* $\gamma = 1$, *and* $\Delta R = 0.3$. *Figure 7 shows that Theorem 3.1 used in this study generates a response faster than that shown in* [16]. *This result shows that the uncertain nonlinear system is effectively controlled using the proposed fuzzy controller (26).*

Example 4.2. *Let us consider the uncertain nonlinear problem of balancing an inverted pendulum on a cart. The movement equations are* [4]*:*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= \frac{f(x(t)) - a\cos(x_1(t))u(t)}{4(l+\Delta l)/3 + am(l+\Delta l)\cos^2(x_1(t))} + 0.01w(t),\\ z(t) &= \begin{bmatrix} 0.01x_1(t) \\ 0.01u(t) \end{bmatrix},\end{aligned} \tag{27}$$

where $f(x(t)) = g\sin(x_1(t)) - am(l+\Delta l)x_2^2(t)\sin(2x_1(t))/2$, $x_1(t)$ *represents the angle from the vertical axis (in radians),* $x_2(t)$ *is the angular velocity of the pendulum,* $u(t)$ *is*

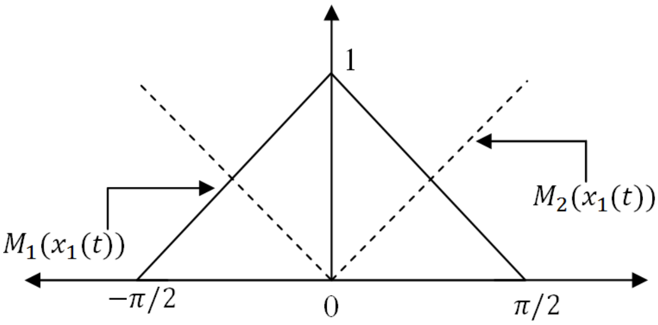

FIGURE 8. Membership functions for the two fuzzy sets used in Example 4.2

the control force applied to the cart (in Newtons), $z(t)$ is the regulated output, $w(t)$ is the disturbance, $g = 9.8$ m/s^2 is the gravity constant, M is the cart mass, $2l$ is the pendulum length, m is the pendulum mass and $x_1(t) \in [-\pi/2,\ \pi/2]$. Define $M = 8$ kg, $m = 2$ kg, $2l = 1$ m, $a = 1/(m+M)$ and Δl as an uncertain term that is bounded in $[0\ \ 0.10]$. Note that the system is uncontrollable when $x_1(t) = \pm\pi/2$; therefore, we linearize the system around $0°$ and $88°$ instead. Therefore, it is assumed that $x_1(t) \in [-88°,\ 88°]$. The nonlinear system plant can be approximated using two T-S fuzzy rules. Let us choose the membership functions of the fuzzy sets as follows:

$$M_1(x_1(t)) = 1 - \frac{2}{\pi}|x_1(t)| \quad \text{and} \quad M_2(x_1(t)) = \frac{2}{\pi}|x_1(t)|. \tag{28}$$

Note that $M_1(x_1(t))$ and $M_2(x_1(t))$ can be interpreted as the membership functions of the fuzzy sets shown in Figure 8. Using these two fuzzy sets, the uncertain nonlinear system can be represented by the following T-S fuzzy model:

Plant Rule 1: IF $x_1(t)$ is $M_1(x_1(t))$, THEN

$$\begin{aligned}\dot{x}(t) &= [A_1 + \Delta A_1]x(t) + B_w w(t) + B_1 u(t),\\ z(t) &= C_1 x(t),\end{aligned}$$

Plant Rule 2: IF $x_1(t)$ is $M_2(x_1(t))$, THEN

$$\begin{aligned}\dot{x}(t) &= [A_2 + \Delta A_2]x(t) + B_w w(t) + B_2 u(t),\\ z(t) &= C_2 x(t),\end{aligned}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{4l/3 - aml} & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix},$$

$$B_{2_1} = \begin{bmatrix} 0 \\ -\dfrac{a}{4l/3 - aml} \end{bmatrix},\ B_{2_2} = \begin{bmatrix} 0 \\ -\dfrac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},\ C_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix},\ \beta = \cos(88°),$$

$$\Delta A_1 = F(x(t),t)H_{1_1} \text{ and } \Delta A_2 = F(x(t),t)H_{1_2},$$

and assuming that $\|F(x(t),t)\| \le \rho = 1$, we have

$$H_{1_1} = \begin{bmatrix} 0 & 0 \\ 4.32 & 0 \end{bmatrix} \text{ and } H_{1_2} = \begin{bmatrix} 0 & 0 \\ 2.75 & 0 \end{bmatrix}.$$

From the LMI optimization algorithm and Theorem 3.1 with $\gamma = 1$*, we have*

$$K_{s_1} = \begin{bmatrix} 37.5502 & 43.6603 \end{bmatrix}, \quad K_{s_2} = \begin{bmatrix} 37.3845 & 43.6047 \end{bmatrix},$$

$$K_{d_1} = \begin{bmatrix} -43.7071 & -29.4621 \end{bmatrix} \text{ and } K_{d_2} = \begin{bmatrix} -43.7065 & -29.4600 \end{bmatrix}.$$

The resulting fuzzy controller is

$$u(t) = \sum_{j=1}^{2} \mu_j \left(K_{s_j} x(t) - K_{d_j} \dot{x}(t) \right) \tag{29}$$

where $\mu_1 = M_1(x_1(t))$ *and* $\mu_2 = M_2(x_1(t))$*.*

Remark 4.3. *The fuzzy controller (29) guarantees that the inequality (10) holds. The histories of state variables of Theorem 3.1, ($x_1(t)$ and $x_2(t)$) are given in Figure 9. Figure 10 presents the disturbance input signal, $w(t)$, used during the simulation. The ratio of the regulated output energy to the disturbance input noise energy obtained from the robust H_∞ fuzzy state feedback plus state-derivative controller (29) is illustrated in Figure 11. After 1.5 seconds, the ratio of the regulated output energy to the disturbance input noise*

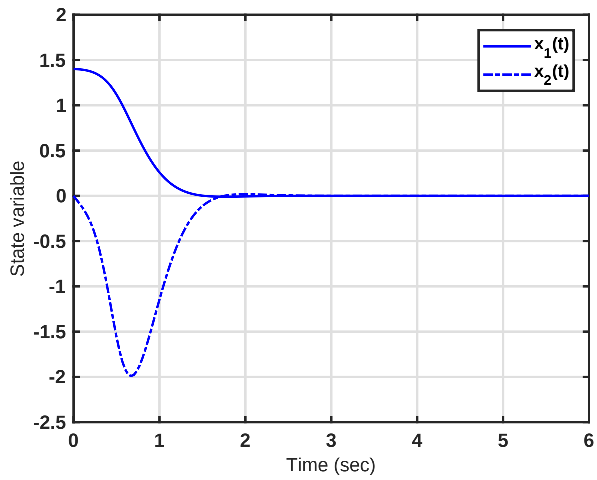

FIGURE 9. State variables of Example 4.2, $x_1(t)$, and $x_2(t)$

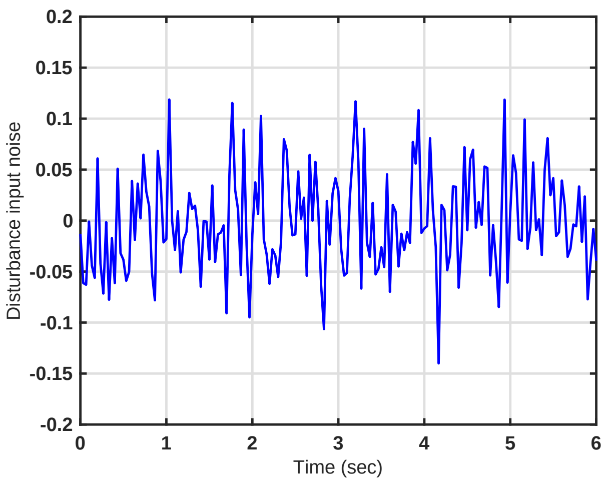

FIGURE 10. Disturbance input noise used in Example 4.2, $w(t)$

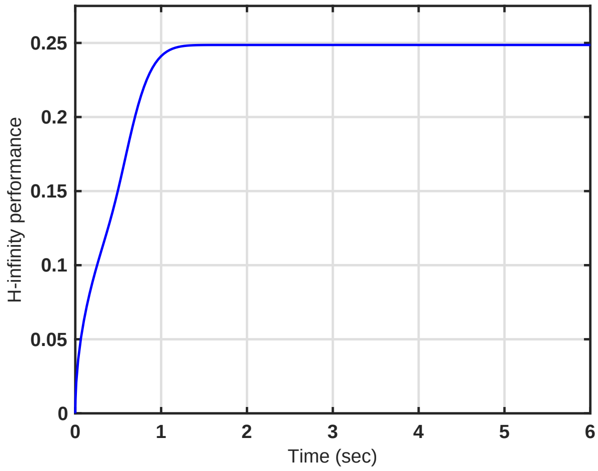

FIGURE 11. H_∞ performance of Example 4.2, $\left(\sqrt{\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}}\right)$

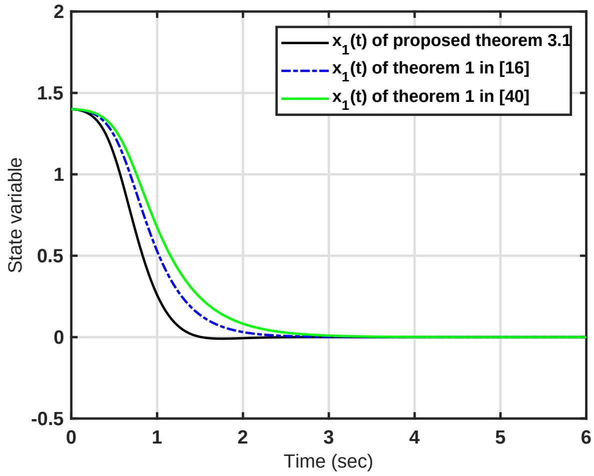

FIGURE 12. Comparison of state variable of Example 4.2, $x_1(t)$

energy tends toward a constant value of less than the prescribed value of 1. These results guarantee the asymptotical stability and the H_∞ performance index of the system.

Remark 4.4. *Based on Theorem 1 used in* [16,40], *and Theorem 3.1 used in this paper, Figure 12 presents the comparative results for the state variable $x_1(t)$ at the same $\gamma = 1$, and $\Delta l = 0.10$. Figure 12 shows that Theorem 3.1 used in this study generates a response faster than Theorem 1 shown in* [16,40]. *This result shows that the uncertain nonlinear system is effectively controlled using the proposed fuzzy controller (29).*

Remark 4.5. *According to the results shown in this section, the proposed controller for the uncertain nonlinear system is guaranteed to meet design requirements (e.g., the asymptotic stability and H_∞ performance index of the system). Practically, the failure of components can be easily found in many real physical control systems. The characteristics of dynamical systems do not easily achieve the desired objectives (e.g., the rise time, the settling time, and transient oscillations due to poor transient responses). However, this research is valid only when a closed-loop system with the proposed controller must not satisfy many* *transient response requirements at the same time. Thus, motivated by a lack of control over transient behaviors, the robust H_∞ fuzzy state feedback plus state-derivative feedback controller with D stability constraints for an uncertain nonlinear system can be considered in future work. We note that extensions of the proposed approach to the analysis and synthesis of fuzzy-affine dynamic systems in piecewise-Lyapunov-function frameworks may be another interesting avenue for future research. In addition, applications of the proposed theoretical approach to uncertain physical systems, such as wind energy systems and cascaded DC-DC converter-based hybrid battery energy storage systems, will be explored in the future work.*

5. **Conclusions.** This paper has investigated a robust H_∞ fuzzy state feedback plus state-derivative feedback controller design procedure for a class of uncertain nonlinear systems that guarantees the L_2-gain from an exogenous input to a regulated output to be less than or equal to a prescribed value. Based on LMIs approach, LMIs based sufficient conditions for the uncertain Takagi-Sugeno (T-S) fuzzy model to have an H_∞ performance are established. The effectiveness of the proposed design methodology is demonstrated through the illustrative examples. However, the failure of components can be easily found in many real physical control problems. Thus, a robust H_∞ fuzzy state-derivative feedback controller with D-stability constraints for an uncertain nonlinear system can be investigated in future research work.

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Appendix 1. Proof of Theorem 3.1.

Proof: Let us consider a Lyapunov function

$$V(x(t)) = x^T(t)Qx(t), \tag{30}$$

where $Q = P^{-1} > 0$. Taking the derivative of $V(x(t))$ along the closed-loop system (18), we have

$$\begin{aligned}\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big[&x^T(t)\Big((A_i + B_iK_{s_j})^T E_{ij}^T Q + QE_{ij}(A_i + B_iK_{s_j})\Big)x(t)\\ &+ \tilde{w}^T(t)\tilde{B}_{w_i}^T E_{ij}^T Qx(t) + x^T(t)QE_{ij}\tilde{B}_{w_i}\tilde{w}(t)\Big].\end{aligned} \tag{31}$$

Adding and subtracting the following

$$-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n\left[\ \tilde{w}^T(t)\tilde{w}(t)\ \right] \tag{32}$$

to and from (31), we acquire

$$\begin{aligned}\dot{V}(x(t)) = &\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n\left[\ x^T(t) \quad \tilde{w}^T(t)\ \right]\\ &\times\left(\begin{array}{cc}\left(\begin{array}{c}(A_i + B_iK_{s_j})^T E_{ij}^T Q\\ + QE_{ij}(A_i + B_iK_{s_j}) + \tilde{C}_i^T\tilde{C}_i\end{array}\right) & (*)^T\\ \tilde{B}_{w_i}^T E_{ij}^T Q & -\gamma^2 I\end{array}\right)\left[\begin{array}{c}x(t)\\ \tilde{w}(t)\end{array}\right]\\ &-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n\left[\ \tilde{w}^T(t)\tilde{w}(t)\ \right],\end{aligned} \tag{33}$$

where

$$\tilde{z}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\tilde{C}_i x(t). \tag{34}$$

Next, let us consider Theorem 3.1; we have

$$\left(\begin{array}{cccc}\Phi_{ij} & (*)^T & (*)^T & (*)^T\\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T & (*)^T\\ \tilde{C}_iP + \tilde{C}_iY_{d_j}^T B_i^T & 0 & -I & (*)^T\\ (Y_{s_j} + Y_{d_j})^T B_i^T & 0 & 0 & -P\end{array}\right) < 0, \tag{35}$$

where $\Phi_{ij} = PA_i^T + A_iP + Y_{s_j}^T B_i^T + B_iY_{s_j} + B_iY_{d_j}A_i^T + A_iY_{d_j}^T B_i^T$. By applying the Schur complement, we obtain

$$\begin{pmatrix} \Phi_{ij} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T \\ \tilde{C}_iP + \tilde{C}_iY_{d_j}^T B_i^T & 0 & -I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} B_i\left(Y_{s_j} + Y_{d_j}\right)P^{-1} \\ \left(Y_{s_j} + Y_{d_j}\right)^T B_i^T \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < 0. \quad (36)$$

By applying the algebraic inequality

$$aXb + b^TXa^T \leq (a+b)X(a+b)^T, \quad (37)$$

then (36) yields

$$\begin{pmatrix} \Phi_{ij} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T \\ \tilde{C}_iP + \tilde{C}_iY_{d_j}^T B_i^T & 0 & -I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} B_iY_{s_j}P^{-1}Y_{d_j}^T B_i^T \\ +B_iY_{d_j}P^{-1}Y_{s_j}^T B_i^T \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < 0, \quad (38)$$

and by substituting Φ_{ij} into (38), we have

$$\begin{pmatrix} \begin{pmatrix} PA_i^T + A_iP + Y_{s_j}^T B_i^T + B_iY_{s_j} + B_iY_{d_j}A_i^T + A_iY_{d_j}^T B_i^T \\ +B_iY_{s_j}P^{-1}Y_{d_j}^T B_i^T + B_iY_{d_j}P^{-1}Y_{s_j}^T B_i^T \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T \\ \tilde{C}_iP + \tilde{C}_iY_{d_j}^T B_i^T & 0 & -I \end{pmatrix} < 0, \quad (39)$$

or

$$\begin{pmatrix} \begin{pmatrix} P\left(A_i + B_iY_{s_j}P^{-1}\right)^T + \left(A_i + B_iY_{s_j}P^{-1}\right)P \\ +B_iY_{d_j}P^{-1}P(A_i + B_iY_{s_j}P^{-1})^T \\ +\left(A_i + B_iY_{s_j}P^{-1}\right)PP^{-1}Y_{d_j}^T B_i^T \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T \\ \tilde{C_i}P + \tilde{C}_iPP^{-1}Y_{d_j}^T B_i^T & 0 & -I \end{pmatrix} < 0, \quad (40)$$

with $K_{d_j} = Y_{d_j}P^{-1}$ and $K_{s_j} = Y_{s_j}P^{-1}$. Then, (40) yields

$$\begin{pmatrix} \begin{pmatrix} (I + B_iK_{d_j})P(A_i + B_iKs_j)^T \\ +(A_i + B_iKs_j)P(I + B_iK_{d_j})^T \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T & -\gamma^2 I & (*)^T \\ \tilde{C}_iP(I + B_iK_{d_j})^T & 0 & -I \end{pmatrix} < 0. \quad (41)$$

Pre- and post-multiplying by $\begin{pmatrix} \left(I + B_iK_{d_j}\right)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ and $\begin{pmatrix} \left(I + B_iK_{d_j}\right)^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, respectively, on both sides of (41), we obtain

$$\begin{pmatrix} \begin{pmatrix} P(A_i + B_iK_{s_j})^T(I + B_iK_{d_j})^{-T} \\ +(I + B_iK_{d_j})^{-1}(A_i + B_iK_{s_j})P \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T(I + B_iK_{d_j})^{-T} & -\gamma^2 I & (*)^T \\ \tilde{C}_iP & 0 & -I \end{pmatrix} < 0, \quad (42)$$

or, in a more compact form,

$$\begin{pmatrix} \left(P\big(A_i + B_iK_{s_j}\big)^T E_{ij}^T + E_{ij}\big(A_i + B_iK_{s_j}\big)P \right) & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T E_{ij}^T & -\gamma^2 I & (*)^T \\ \tilde{C}_i P & 0 & -I \end{pmatrix} < 0. \tag{43}$$

By multiplying both sides of (43) by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, we obtain

$$\begin{pmatrix} \left((A_i + B_iK_{s_i})^T E_{ii}^T Q + QE_{ii}(A_i + B_iK_{s_i}) \right) & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T E_{ii}^T Q & -\gamma^2 I & (*)^T \\ \tilde{C}_i & 0 & -I \end{pmatrix} < 0, \tag{44}$$

$i = 1, 2, 3, \ldots, r$, and

$$\begin{aligned} &\begin{pmatrix} \left((A_i + B_iK_{s_j})^T E_{ij}^T Q + QE_{ij}(A_i + B_iK_{s_j}) \right) & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T E_{ij}^T Q & -\gamma^2 I & (*)^T \\ \tilde{C}_i & 0 & -I \end{pmatrix} \\ &+ \begin{pmatrix} \left((A_j + B_jK_{s_i})^T E_{ji}^T Q + QE_{ji}(A_j + B_jK_{s_i}) \right) & (*)^T & (*)^T \\ \tilde{B}_{w_i}^T E_{ji}^T Q & -\gamma^2 I & (*)^T \\ \tilde{C}_j & 0 & -I \end{pmatrix} < 0, \end{aligned} \tag{45}$$

$i < j \leq r$. Applying the Schur complement to (44) and (45) and rearranging them, we then have

$$\begin{pmatrix} \left((A_i + B_iK_{s_i})^T E_{ii}^T Q + QE_{ii}\left(A_i + B_iK_{s_i}\right) + \tilde{C}_i^T \tilde{C}_i \right) & (*)^T \\ \tilde{B}_{w_i}^T E_{ii}^T Q & -\gamma^2 I \end{pmatrix} < 0, \tag{46}$$

$i = 1, 2, 3, \ldots, r$, and

$$\begin{aligned} &\begin{pmatrix} \left(\left(A_i + B_iK_{s_j}\right)^T E_{ij}^T Q + QE_{ij}\left(A_i + B_iK_{s_j}\right) + \tilde{C}_i^T \tilde{C}_i \right) & (*)^T \\ \tilde{B}_{w_i}^T E_{ij}^T Q & -\gamma^2 I \end{pmatrix} \\ &+ \begin{pmatrix} \left((A_j + B_jK_{s_i})^T E_{ji}^T Q + QE_{ji}\left(A_j + B_jK_{s_i}\right) + \tilde{C}_j^T \tilde{C}_j \right) & (*)^T \\ \tilde{B}_{w_i}^T E_{ji}^T Q & -\gamma^2 I \end{pmatrix} < 0, \end{aligned} \tag{47}$$

$i < j \leq r$. Using (46) and (47) and the fact that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{m=1}^{r} \sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n M_{ij}^T N_{mn} \leq \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i\mu_j \left[M_{ij}^T M_{ij} + N_{ij} N_{ij}^T \right], \tag{48}$$

it is clear that

$$\begin{pmatrix} \left(\left(A_i + B_iK_{s_j}\right)^T E_{ij}^T Q + QE_{ij}\left(A_i + B_iK_{s_j}\right) + \tilde{C}_i^T \tilde{C}_i \right) & (*)^T \\ \tilde{B}_{w_i}^T E_{ij}^T Q & -\gamma^2 I \end{pmatrix} < 0, \tag{49}$$

where $i, j = 1, 2, \ldots, r$. Since (49) is less than zero and because $\mu_n \geq 0$ and $\sum_{n=1}^{r} \mu_n = 1$, then (33) becomes

$$\dot{V}(x(t)) \leq -\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n \left[\, \tilde{w}^T(t)\tilde{w}(t) \,\right]. \tag{50}$$

Integrating both sides of (50) yields

$$\int_0^{T_f} \dot{V}(x(t))dt \leq \int_0^{T_f} \left[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n \left[\, \tilde{w}^T(t)\tilde{w}(t) \,\right]\right] dt, \tag{51}$$

$$\begin{aligned} &V\left(x(T_f)\right) - V\left(x(0)\right) \\ &\leq \int_0^{T_f} \left[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n \left[\, \tilde{w}^T(t)\tilde{w}(t) \,\right]\right] dt. \end{aligned} \tag{52}$$

Because $V(x(0)) = 0$ and $V\left(x(T_f)\right) \geq 0$ for all $T_f \neq 0$, we obtain

$$\int_0^{T_f} \tilde{z}^T(t)\tilde{z}(t)dt \leq \gamma^2 \left[\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n \left[\, \tilde{w}^T(t)\tilde{w}(t) \,\right] dt\right]. \tag{53}$$

Inserting $\tilde{z}(t)$ and $\tilde{w}(t)$, respectively, given in (34) and (16) into (53), and using the fact that $\|F(x(t), t)\| \leq \rho$, and (50), we have

$$\begin{aligned} &\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \left(2\lambda^2 x^T(t)C_i^T C_i x(t) + 2\lambda^2\rho^2 x^T(t)H_{3_i}^T H_{3_i} x(t)\right) dt \\ &\leq \gamma^2 \left[\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \left[w^T(t)w(t)\right] dt + \rho^2 \int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \left[w^T(t)H_{2_i}^T H_{2_i} w(t)\right] dt\right], \end{aligned} \tag{54}$$

and using $\lambda^2 = 1 + \rho^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\left[\| \, H_{2_i}^T H_{2_j} \, \|\right]$, we obtain

$$\begin{aligned} &\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \left(2\lambda^2 x^T(t)C_i^T C_i x(t) + 2\lambda^2\rho^2 x^T(t)H_{3_i}^T H_{3_i} x(t)\right) dt \\ &\leq \gamma^2\lambda^2 \left[\int_0^{T_f} \left[w^T(t)w(t)\right] dt\right]. \end{aligned} \tag{55}$$

Adding and subtracting

$$\lambda^2 z^T(t)z(t) = \lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \Big(x^T(t)\big(C_i + F\big(x(t), t\big)H_{3_i}\big)^T \big(C_i + F\big(x(t), t\big)H_{3_i}\big)x(t)\Big) \tag{56}$$

to and from (55), one obtains

$$\begin{aligned} &\int_0^{T_f} \Bigg[\lambda^2 z^T(t)z(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \Big[\Big(2\lambda^2 x^T(t)C_i^T C_i x(t) + 2\lambda^2\rho^2 x^T(t)H_{3_i}^T H_{3_i} x(t)\Big) \\ &\quad - \Big(\lambda^2\Big(x^T(t)(C_i + F(x(t), t)H_{3_i})^T \times (C_i + F(x(t), t)H_{3_i})x(t)\Big)\Big)\Big]\Bigg] dt \\ &\leq \gamma^2\lambda^2 \left[\int_0^{T_f} \left[w^T(t)w(t)\right] dt\right]. \end{aligned} \tag{57}$$

Using the triangular inequality and the fact that $\|F(x(t),t)\| \leq \rho$, we have

$$\begin{aligned}
&\lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \left[\Big(x^T(t)\big(C_i + F\big(x(t),t\big)H_{3_i}\big)^T \times \big(C_i + F(x(t),t)H_{3_i}\big)x(t)\Big)\right] \\
&\leq \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \big[2\lambda^2 x^T(t)C_i^T C_i x(t) + 2\lambda^2\rho^2 x^T(t)H_{3_i}^T H_{3_i} x(t)\big].
\end{aligned} \tag{58}$$

Substituting (58) into (57), we obtain

$$\int_0^{T_f} z^T(t)z(t)dt \leq \gamma^2 \left[\int_0^{T_f} w^T(t)w(t)dt\right]. \tag{59}$$

Hence, the inequality (10) holds. When $w(t) = 0$, (50) becomes $\dot{V}(t) \leq -z^T(t)z(t) \leq 0$. Therefore, the system (18) is asymptotically stable, and (b) is achieved. This completes the proof. $\square$