*International Journal of Innovative Computing, Information and Control*
Volume **15**, Number **4**, August **2019**
 ISSN 1349-4198
pp. 1337–1350

# COMBINING PROJECTION ALGORITHM AND SCENARIO APPROACH FOR VIRTUAL REFERENCE FEEDBACK TUNING CONTROL

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Received December 2018; revised April 2019

Abstract. *The problem of designing two feedback controllers for an unknown plant based on measured data is discussed here in a linear setting. Virtual reference feedback tuning control is a direct method that aims at minimizing a cost function of the 2-norm type by using a set of data. When constructing the cost function, we consider two model-matching problems between closed loop transfer function and sensitivity function simultaneously. In these model-matching procedures, we analyze how to design the reference input signal and disturb signal respectively. After transforming the problem of identifying unknown parameter vectors in virtual reference feedback tuning control into two linear regression models, the projection algorithm is introduced to achieve our first on-line parameter estimation scheme. And we also see that the projection algorithm is equivalent to one optimization problem. Further in this paper, scenario optimization is used to determine the number of scenarios that determine the solution. The maximum number of data points that determine the so-called support points equals the number of optimization variables.*
**Keywords:** Virtual reference feedback tuning control, Scenario program, Parameter identification, Projection algorithm

1. **Introduction.** The common control system structure includes the open loop system and closed loop system, and gradually in engineering the open loop system is replaced by the closed loop system, due to its simple structure or poor performance. The characteristic of the open loop system is that no feedback effect exists between measured output and input, so the open loop system is used only in case of little internal disturbance or external disturbance. One feedback part is added in the closed loop system, and then the function of this feedback part is to send the measured output back to the input and compare them to get one error signal. This error signal can be used to generate one controller, such as PID controller, which makes the output tend to one given value, so the essence of the closed loop system is to reduce this error signal by using the negative feedback part. Due to this function of automatically correcting the derivation or error from the given value, the closed loop system can not only alleviate the error coming from internal or external disturbance, but also achieve the purpose of automatic control. Generally for many industrial production processed, safety and production restrictions are strong reasons for not allowing control experiments in the open loop system, and then the closed loop system is most needed in all of our engineering.

In designing the controller for every control system structure, we always assume that the mathematical model corresponding to the plant is known in prior, i.e., the process

DOI: 10.24507/ijicic.15.04.1337

of designing the controller is based on the priori knowledge of the plant. However, this assumption does not hold in reality, and then the mathematical model of the plant may be identified by lots of system identification methods. Generally in order to design the controller in the closed loop system, there exist two kinds of approaches based on the priori knowledge of the plant. These two approaches are divided into the model based control and direct data driven control. The main difference between these two approaches is that whether the plant is needed to be identified in prior. The model based control approach needs to construct the mathematical model of the plant by using mechanical modeling technique or system identification idea, and then this identified mathematical model is used to design the controller. So the controller performance of this model based control approach depends on the accuracy of the identified plant closely. In order to alleviate this dependency, some scholars propose to apply the input-output measured data to designing the controller directly and avoid the tedious process of identifying the plant. From a system identification idea point of view, the above second approach is called the direct data driven control approach. As the direct data driven control approach avoids the identification process with respect to the plant and simplifies the whole designing process about the controller, in theory or engineering field it is more widely considered to achieve the control performance than the classical model based control method. Many common direct data driven control approaches include virtual reference feedback tuning control, subspace predictive control and iterative feedback tuning control, etc.

Virtual reference feedback tuning control considered in this paper is a direct data driven method, as it designs the closed loop controller based on the input-output measured data directly without identifying the plant. The main difficulty of the controller design in the closed loop system is that the researcher would consider the correlation between the input and external disturbance induced by the feedback loop. When considering the problem of designing controller in the closed loop system, the controller is designed based on a known plant. However, the mathematical model of the plant cannot be easily determined in the industrial process, as the cost of developing the mathematical model of the plant is very high. In order to avoid the identification process corresponding to the plant, one of direct data driven method-virtual reference feedback tuning control was proposed to design the controller in 1940s [3]. The main essence of virtual reference feedback tuning control is that the controller can be designed directly by measured data without any priori knowledge about the plant model. In [8], the problem of designing the controller was transformed to identify one unknown parameter vector of the controller, when the structure of the controller was parameterized by an unknown parameter vector. Then the idea of virtual reference feedback tuning control was applied in a benchmark problem [9], where a control cost of the 2-norm type was minimized by using a batch of data collected from the plant. Virtual reference feedback tuning control for controller tuning in a nonlinear setup was introduced in [10], and it was based on a global model reference optimization procedure. In [8], the extension of virtual reference feedback tuning control to the design of two degrees of freedom controller was presented to shape the input-output transfer function of the closed loop system. As the direct data driven control approach often consists of iterative adjustment of the controller's parameters towards the parameter values, the H2 performance criterion is analyzed in order to characterize and enlarge the set of initial parameter values from which a gradient descent algorithm can converge to its global minimization [1]. The contribution of [2] introduces an invalidation test step based on the available data to check if the flexibility of the controller parameter is suitable for the design objectives. In recent years, we publish many papers on virtual reference feedback tuning control, for example, in [11], the model-matching problems between the closed loop transfer function and sensitivity function were considered simultaneously, and the iterative sequence generated by the iterative least squares identification algorithm is analyzed for identifying the unknown parameter vector. Virtual reference feedback tuning control for constrained closed loop system was studied in [14], where the ellipsoid optimization iterative algorithm was adopted to generate a sequence of ellipsoids with decreasing volume. To compromise the gap between the linear and nonlinear controller, the problem of designing a linear time varying controller was solved by using virtual reference feedback tuning control [14], and a new regularization term about the two unknown time varying parameter vectors was added to restrict the jump property. When the unknown controller is a nonlinear function form, virtual reference feedback tuning control is applied to constructing a linear affine function of the output. Then through minimizing one approximation error, the adjustable weights in the linear affine function are given in [12]. The most interesting contribution of our published paper [13] is to combine virtual reference feedback tuning control into internal model control strategy, i.e., to avoid the process of identifying the plant in internal model control, the idea of virtual reference feedback tuning control is applied into the internal model control in order to identify the identified plant and internal model controller simultaneously.

Above mentioned references are only concentrated on parameter identification for virtual reference feedback tuning control, and to the best of our knowledge that there is no any other subject on this special control strategy. Here in this paper, based on above mentioned references and our published contributions, we treat the projection identification and scenario program in virtual reference feedback tuning control for closed loop system with two degrees of freedom controllers. In the process of designing the two unknown controllers by virtual reference feedback tuning control, the model-matching problems with respect to the expected closed loop transfer function and expected sensitivity function are taken into account simultaneously. Then one additional term about the sensitivity function would be added to the former objective function, so before constructing the new objective function, virtual input and virtual disturbance must be generated respectively. When the two unknown controllers are all parameterized by their own parameter vectors respectively, one important step in virtual reference feedback tuning control is to identify these two unknown parameter vectors. After transforming the problem of identifying unknown parameter vectors in virtual reference feedback tuning control into two linear regression models, the projection algorithm is introduced to achieve our first on-line parameter estimation scheme. And we also see that the projection algorithm is equivalent to one optimization problem. Further in this paper, scenario optimization is used to determine the number of scenarios that determine the solution. The maximum number of data points that determine the so-called support points equals the number of optimization variables.

The paper is organized as follows. In Section 2, the problem formulation is addressed, and the structure of our considered closed loop system with two degrees of freedom controllers is presented. To compare virtual reference feedback tuning control and classical model reference control, a short introduction of model reference control is given to design the two unknown parametrized controllers in Section 3, and the deficiency of model reference control is also pointed out. In Section 4, virtual reference feedback tuning control is proposed to design these two parametrized controllers; furthermore, virtual input and virtual disturbance are constructed to get an optimization problem. The projection algorithm is applied to identifying the unknown parameter vector in Section 5. In Section 6, scenario program is presented to determine the number of data points. A simulation example is given in Section 7. Section 8 ends the paper with final conclusion and points out our future work.

2. **Problem Description.** Assume the plant is a linear time invariant discrete time single input and single output process. The plant is denoted by a rational transfer function form $P(z)$, and $P(z)$ is unknown. Throughout the closed loop experimental process, only a sequence of input-output measured data corresponding to the plant $P(z)$ are collected. The input-output relation is described as follows

$$y(t) = P(z)u(t) + d(t) \tag{1}$$

where $z$ is a time shift operator, i.e., $(zu(t) = u(t-1))$, $P(z)$ is one transfer function of the plant, $u(t)$ is the measured input, $y(t)$ is the measured output corresponding to the plant $P(z)$, and $d(t)$ is the external noise. When $d(t)$ in (1) is unknown but has known bounds, we regard to the uncertainty associated with $d(t)$ as additive noise because of the way it enters the input-output relation in (1).

Consider the following simple closed loop system with two degrees of freedom controllers in Figure 1, the input-output relations in the whole closed loop system are written as follows

$$\begin{cases} y(t) = P(z)u(t) + d(t) \\ u(t) = C_1(z,\theta)\varepsilon(t) = C_1(z,\theta)[r(t) - C_2(z,\eta)y(t)] \end{cases} \tag{2}$$

where $r(t)$ is the excited signal, $C_1(z,\theta)$ and $C_2(z,\eta)$ are two degrees of freedom controllers which are parametrized by unknown parameter vectors $\theta$ and $\eta$ respectively, i.e., these two controllers $C_1(z,\theta)$ and $C_2(z,\eta)$ are independently parametrized as the following linear affine forms

$$\begin{aligned} &C_1(z,\theta) = \alpha^T(z)\theta,\ C_2(z,\eta) = \beta^T(z)\eta \\ &\alpha(z) = [\alpha_1(z), \alpha_2(z), \ldots, \alpha_n(z)]^T, \\ &\beta(z) = [\beta_1(z), \beta_2(z), \ldots, \beta_n(z)]^T \\ &\theta = [\theta_1, \theta_2, \ldots, \theta_n]^T,\ \eta = [\eta_1, \eta_2, \ldots, \eta_n]^T \end{aligned} \tag{3}$$

where $\alpha(z)$ and $\beta(z)$ denote two known basic function vectors, and $\theta$ and $\eta$ are two unknown parameter vectors with dimension $n$.

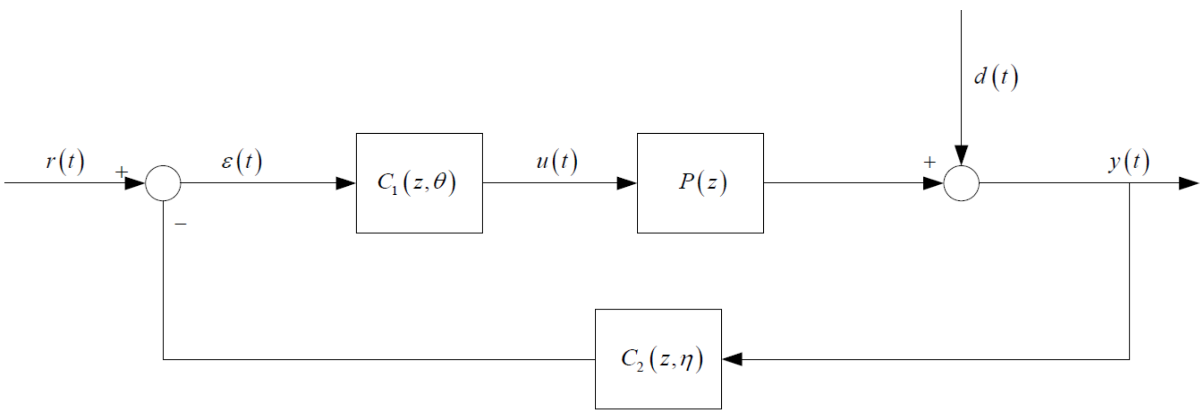



FIGURE 1. The closed loop system with two degrees of freedom controllers

Formulating Equation (2) again, the new input-output relation in terms of excited input $r(t)$ and external noise $d(t)$ are given as

$$y(t) = \frac{P(z)C_1(z,\theta)}{1 + P(z)C_1(z,\theta)C_2(z,\eta)} r(t) + \frac{1}{1 + P(z)C_1(z,\theta)C_2(z,\eta)} d(t) \tag{4}$$

3. **Classical Model Reference Control.** In the closed loop system with two degrees of freedom controllers, the first transfer function from the excited signal $r(t)$ to measured output $y(t)$ is closed loop transfer function, and the second transfer function from the external signal $d(t)$ to measured output $y(t)$ is sensitivity function. From Equation (4), the control task of classical model reference control is to tune the unknown parameter vectors $\theta$ and $\eta$ corresponding to two degrees of freedom controllers $(C_1(z,\theta), C_2(z,\eta))$ in order to achieve the expected closed loop transfer function and expected sensitivity function. Given the closed loop transfer function $M(z)$ and expected sensitivity function $S(z)$, we want to guarantee that the closed loop transfer function approximates to its expected function $M(z)$, and the sensitivity function tends to its expected function $S(z)$, too. The problem of tuning these two unknown parameter vectors $\theta$ and $\eta$ is formulated as the following classical model reference control optimization problem

$$\begin{aligned}\min_{\theta,\eta} J_{MR}(\theta,\eta) = &\left\| \frac{P(z)C_1(z,\theta)}{1+P(z)C_1(z,\theta)C_2(z,\eta)} - M(z) \right\|_2^2 \\ &+ \left\| \frac{1}{1+P(z)C_1(z,\theta)C_2(z,\eta)} - S(z) \right\|_2^2\end{aligned} \tag{5}$$

where $\|.\|_2^2$ is the common Euclidean norm. In Equation (5), before solving this optimization problem with respect to unknown parameter vectors $\theta$ and $\eta$, the priori knowledge about the plant $P(z)$ may be needed. As the plant $P(z)$ is unknown in this model reference control design method, the identification strategy is used to identify $P(z)$. To avoid the identification process of the plant $P(z)$, virtual reference feedback tuning control is proposed to directly identify the unknown parameter vectors $\theta$ and $\eta$ in two controllers $(C_1(z,\theta), C_2(z,\eta))$ from the measured input-output data set $Z^N = \{u(t), y(t)\}_{t=1}^N$, where $N$ is the number of data points.

4. **Virtual Reference Feedback Tuning Control.** Given two controllers $\{(C_1(z,\theta), C_2(z,\eta))\}$, as the closed loop transfer function from $r(t)$ to $y(t)$ is $M(z)$, then we apply one arbitrary signal $r(t)$ to exciting the closed loop system (2), and the output of the closed loop system is described as

$$y(t) = M(z)r(t)$$

Consider one special excited input $\bar{r}(t)$, the necessary condition about that the closed loop transfer function be $M(z)$ is that the two closed loop systems have the same output $y(t)$ under a given input. During classical model reference control, this necessary condition holds in case of choosing suitable controller and excited signal. However, above description does not hold, due to unknown plant. The idea of virtual reference feedback tuning control means that virtual input $\bar{r}(t)$ and virtual disturbance $\bar{d}(t)$ need to be constructed firstly, so we give the detailed process of constructing virtual input $\bar{r}(t)$ and virtual disturbance $\bar{d}(t)$.

4.1. **Virtual input.** Collecting input-output sequence $\{u(t), y(t)\}_{t=1}^N$ of the plant $P(z)$, then for every measured output $y(t)$, define one virtual input $\bar{r}(t)$ such that

$$y(t) = M(z)\bar{r}(t) \tag{6}$$

This virtual input $\bar{r}(t)$ does not exist in reality and it cannot be used to generate actual measured output $y(t)$. However, virtual input $\bar{r}(t)$ can be obtained by equation $y(t) = M(z)\bar{r}(t)$, i.e., $y(t)$ is the measured output of the closed loop system, when the excited signal $\bar{r}(t)$ is applied with no disturbance $d(t) = 0$.

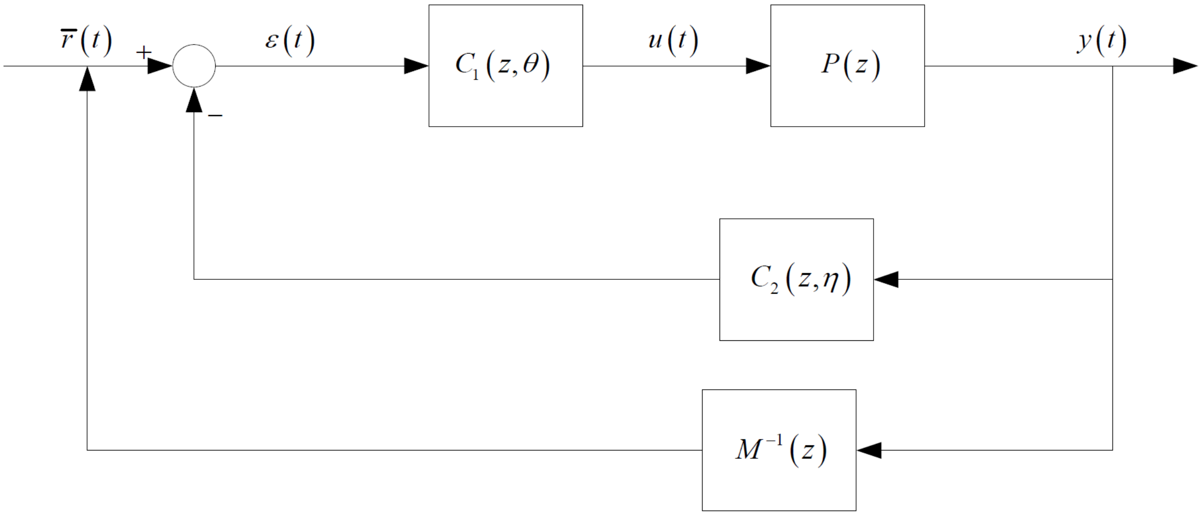



FIGURE 2. Construction of virtual input

Due to the unknown plant $P(z)$, and when $P(z)$ is excited by $u(t)$, its output is $y(t)$, so we choose two suitable controllers $\{(C_1(z,\theta), C_2(z,\eta))\}$ to obtain one expected signal $u(t)$, if the closed loop system is excited by virtual input $\bar{r}(t)$ and $y(t)$ simultaneously. The construction of virtual input $\bar{r}(t)$ can be seen in Figure 2, where the tracking error $\varepsilon(t)$ is defined as

$$\varepsilon(t) = \bar{r}(t) - C_2(z,\eta)y(t) = \left(M^{-1}(z) - C_2(z,\eta)\right) y(t) \tag{7}$$

From Figure 2 we see that when the closed loop system is excited by $(\bar{r}(t), y(t), d(t) = 0)$, the expression of $u(t)$ is got

$$u(t) = C_1(z,\theta)\varepsilon(t) = C_1(z,\theta)\left(M^{-1}(z) - C_2(z,\eta)\right) y(t) \tag{8}$$

4.2. **Virtual disturbance.** Given the measured output $y(t)$, define one virtual disturbance $\bar{d}(t)$ to guarantee that when the closed loop system is excited by virtual disturbance $\bar{d}(t)$, the obtained output $\bar{y}(t)$ is defined as

$$\bar{y}(t) = y(t) + \bar{d}(t) \tag{9}$$

The construction of virtual disturbance $\bar{d}(t)$ is shown in Figure 3, where this virtual disturbance $\bar{d}(t)$ satisfied that

$$y(t) + \bar{d}(t) = S(z)\bar{d}(t) \tag{10}$$

Equation (10) is used to generate virtual disturbance $\bar{d}(t)$, and it means that when the closed loop system is excited by the following signals $(r(t) = 0, \bar{d}(t), \bar{y}(t) = y(t) + \bar{d}(t))$, the obtained signal $u(t)$ is got

$$u(t) = C_1(z,\theta)\varepsilon(t) = -C_1(z,\theta)C_2(z,\eta)\bar{y}(t) \tag{11}$$

As $\bar{y}(t)$ in Equation (11) is not the true output, and instead the true output is $y(t)$, $\bar{y}(t)$ in Equation (11) must be changed into $y(t)$.

Combining two Equations (9) and (11), we get that

$$\begin{aligned} y(t) &= (S(z) - 1)\bar{d}(t) = (S(z) - 1)S^{-1}(z)\bar{y}(t) \\ \bar{y}(t) &= \frac{S(z)}{S(z) - 1} y(t) \end{aligned} \tag{12}$$

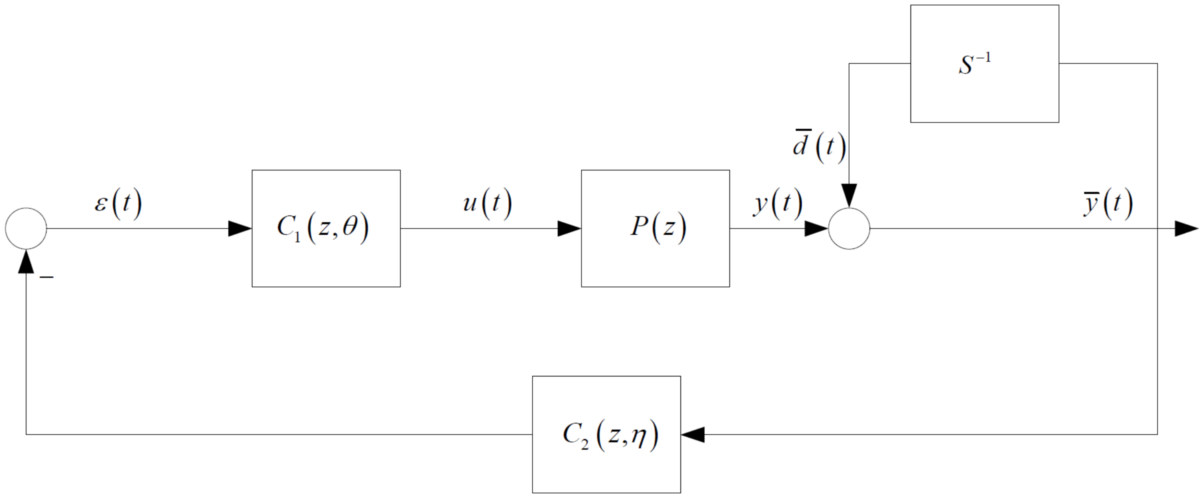



FIGURE 3. Construction of virtual disturbance

Substituting (12) into (11), we obtain that

$$u(t) = -C_1(z,\theta)C_2(z,\eta)\frac{S(z)}{S(z)-1}y(t) \tag{13}$$

Using Equations (8) and (13), two unknown parameter vectors $\theta$ and $\eta$ in two controllers $\{(C_1(z,\theta), C_2(z,\eta))\}$ can be identified by solving the following optimization problem

$$\begin{aligned}\min_{\theta,\eta} J_{VR}^N(\theta,\eta) &= \frac{1}{N}\sum_{t=1}^{N}\left[u(t) - C_1(z,\theta)\left(M^{-1}(z) - C_2(z,\eta)\right)y(t)\right]^2 \\ &+ \frac{1}{N}\sum_{t=1}^{N}\left[u(t) + C_1(z,\theta)C_2(z,\eta)\frac{S(z)}{S(z)-1}y(t)\right]^2\end{aligned} \tag{14}$$

Observing optimization problem (14), all variables are known except for two parametrized controllers $\{(C_1(z,\theta), C_2(z,\eta))\}$. More specifically input-output data $\{u(t), y(t)\}_{t=1}^N$ can be collected by sensors, and these two expected transfer functions $M(z)$ and $S(z)$ are priori known. Roughly speaking, the plant $P(z)$ is not in optimization problem (14), which is the main contribution in virtual reference feedback tuning control. Furthermore, optimization problem (14) embodied that the problem of designing two controllers can be transformed to identify two unknown parameter vectors. This transformation will simplify the latter computational complexity for designing controllers.

5. **Projection Algorithm.** From above descriptions on virtual reference feedback tuning, the main step is to identify two unknown parameter vectors $(\theta, \eta)$. However, now lots of identification algorithms proposed in references are based on probability distribution on external disturbance. Generally to validate the applied model in engineering, some uncertainties can be added to the mathematical model. Frequently disturbance inflecting the real system has to be taken into account in the mathematical model in order to ensure a similar behavior of the real system and the mathematical model. In [4], there are two ways to represent uncertainties: the statistical approach and the deterministic approach. In the statistical approach, the uncertainty or disturbance is modeled by a random process with a known statistical property, when estimates of the probability distributing of the uncertainty or disturbance are available. Here we introduce one practical algorithm to achieve the identification goal.

5.1. **Preliminaries.** From the theoretical perspective, firstly we transform the problem of identifying unknown parameter vectors in virtual reference feedback tuning control into two linear regression models, which are suitable for our proposed projection algorithm.

Observing the optimization problem (14) in virtual reference feedback tuning control, our goal is to identify two optimal parameter vectors such that the following ideal forms hold

$$\begin{cases} u(t) = C_1(z,\theta)\big(M^{-1}(z) - C_2(z,\eta)\big)y(t) \\ u(t) = -C_1(z,\theta)C_2(z,\eta)\dfrac{S(z)}{S(z)-1}y(t) \end{cases} \tag{15}$$

Let

$$C_1(z,\theta)\left(M^{-1}(z) - C_2(z,\eta)\right)y(t) = -C_1(z,\theta)C_2(z,\eta)\frac{S(z)}{S(z)-1}y(t) \tag{16}$$

Then

$$\left(M^{-1}(z) - C_2(z,\eta) + C_2(z,\eta)\frac{S(z)}{S(z)-1}\right)y(t) = 0 \tag{17}$$

It means that

$$\begin{cases} \big((S(z)-1)M^{-1}(z) + C_2(z,\eta)\big)y(t) = 0 \\ C_2(z,\eta)y(t) - (S(z)-1)M^{-1}(z)y(t) = 0 \end{cases} \tag{18}$$

Substituting the linear affine form (3) of parametrized controller $C_2(z,\eta)$ into above Equation (18), then we have

$$\begin{bmatrix} 1 & z & z^2 & \dots & z^n \end{bmatrix}\begin{bmatrix} 1 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} y(t) = (S(z)-1)M^{-1}(z)y(t) \tag{19}$$

where one special basic function vector is that

$$\alpha(z) = \beta(z) = \begin{bmatrix} 1 & z & z^2 & \dots & z^n \end{bmatrix}^T$$

Rewrite Equation (19) as one linear regression model

$$y(t) = y(t-1)\eta_1 + y(t-2)\eta_2 + \cdots + y(t-n)\eta_n + (S(z)-1)M^{-1}(z)y(t)$$

Hence

$$y(t) = \frac{M(z)}{M(z)-S(z)+1} \times \begin{bmatrix} y(t-1) & y(t-2) & \dots & y(t-n) \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} \tag{20}$$

Introduce the regression vector $\phi(t)$ as

$$\phi(t) = \frac{M(z)}{M(z)-S(z)+1}\begin{bmatrix} y(t-1) & y(t-2) & \dots & y(t-n) \end{bmatrix}$$

Then Equation (20) will be a linear regression model

$$y(t) = \phi(t)\eta \tag{21}$$

After unknown parameter vector $\eta$ is identified and substituting its estimator $\hat{\eta}$ into the parametrized controller, then controller $C_2(z, \hat{\eta})$ is obtained. Applying this obtained controller $C_2(z, \hat{\eta})$ into Equation (15), we have

$$u(t) = -C_1(z, \theta)C_2(z, \hat{\eta})\frac{S(z)}{S(z) - 1}y(t) \tag{22}$$

Similarly based on that special basic function vector $\alpha(z)$, Equation (22) can be written as another linear regression model

$$u(t) = \varphi(t)\theta \tag{23}$$

where the regressor vector $\varphi(t)$ is defined as

$$\varphi(t) = -\frac{S(z)}{S(z) - 1}C_2(z, \hat{\eta})y(t)$$

From these two linear regression models (21) and (23), we see that firstly unknown parameter vector $\eta$ can be identified on the basis of linear regression model (21), and then after substituting its estimator $\hat{\eta}$ into regressor vector $\alpha(z)$, another unknown parameter vector $\theta$ is obtained from linear regression model (23). So here we only give a detain zonotope identification algorithm to identify unknown parameter vector $\eta$ in linear regression model (21), and the identification of $\theta$ is similar to it.

5.2. **Projection algorithm.** As linear regression model (21) is an ideal form, actually disturbance inflecting the real system must be taken into account to ensure a similar behavior of the real system, i.e., the real system is that

$$y(t) = \phi(t)\eta + d(t) \tag{24}$$

where $d(t)$ represents the considered disturbance.

Based on the linear regression model (24), which is linear in the unknown parameter vector $\eta$, now here our first on line parameter estimation scheme is introduced to identify this unknown parameter vector $\eta$. This on line parameter estimation scheme is named as the projection algorithm. Then the detailed projection algorithm is given as follows

$$\hat{\eta}(t) = \hat{\eta}(t-1) + \frac{\phi(t)}{\phi^T(t)\phi(t)}\left[y(t) - \phi^T(t)\hat{\eta}(t-1)\right] \tag{25}$$

with the initial estimate $\hat{\eta}(0)$ and $\hat{\eta}(t)$ being the current parameter estimate at time instant $t$, $\hat{\eta}(t-1)$ is the previous estimate. In the projection algorithm (25), the current parameter estimate $\hat{\eta}(t)$ is computed in terms of the previous estimate $\hat{\eta}(t-1)$.

The projection algorithm above can be formulated as that. Given $y(t)$ and $\phi(t)$, and possible values of initial estimate $\hat{\eta}(0)$ satisfying that linear regression model lie on the hyper-surface.

$$\{\eta : y(t) = \phi(t)\eta + d(t)\} \tag{26}$$

To further analyze the above projection algorithm (25), we obtain the following Theorem 5.1.

**Theorem 5.1.** *The projection algorithm is equivalent to the following optimization problem. Given $\hat{\eta}(t-1)$ and $y(t)$, determine $\hat{\eta}(t)$ such that*

$$J_1 = \frac{1}{2}\left\|\hat{\eta}(t) - \hat{\eta}(t-1)\right\| \tag{27}$$

*is minimized subject to*

$$y(t) = \phi(t)\hat{\eta}(t) + d(t)$$

**Proof:** Setting a Lagrange multiplier $\lambda$ for the constraint $y(t) = \phi(t)\hat{\eta}(t)$, we have

$$J_2 = \frac{1}{2}\|\hat{\eta}(t) - \hat{\eta}(t-1)\| + \lambda[y(t) - \phi(t)\hat{\eta}(t)] \tag{28}$$

Using the necessary conditions for a minimum is that

$$\begin{cases} \dfrac{\partial J_2}{\partial \hat{\eta}(t)} = 0 \\ \dfrac{\partial J_2}{\partial \lambda} = 0 \end{cases} \tag{29}$$

Substituting Equation (28) into Equation (29), we have

$$\hat{\eta}(t) - \hat{\eta}(t-1) - \lambda\phi(t) = 0 \tag{30}$$

$$y(t) - \phi(t)\hat{\eta}(t) = 0 \tag{31}$$

Substituting Equation (30) into (31) gives

$$y(t) - \phi^T(t)\left[\hat{\eta}(t-1) + \lambda\phi(t)\right] = 0 \tag{32}$$

Then that Lagrange multiplier $\lambda$ is obtained as

$$\lambda = \frac{y(t) - \phi^T(t)\hat{\eta}(t-1)}{\phi^T(t)\phi(t)} \tag{33}$$

Substituting Equation (33) into (30), it holds that

$$\hat{\eta}(t) = \hat{\eta}(t-1) + \frac{\phi(t)}{\phi^T(t)\phi(t)}\left[y(t) - \phi^T(t)\hat{\eta}(t-1)\right]$$

which corresponds to the projection algorithm (25).

6. **Scenario Approach for Virtual Reference Feedback Tuning Control.** Scenario optimization is a well established approach to data driven optimization, whose solution comes accompanied by precise generalization guarantees. Suppose that $\Delta$ is a probability space, endowed with a $\sigma$ algebra $D$ and a probability $P$. Let $\left(\Delta^N, D^N, P^N\right)$ be the $N$ fold Cartesian product of $\Delta$ equipped with the product $\sigma$ algebra $D^N$ and the probability $P^N = P \times \cdots \times P$. A point in $\left(\Delta^N, D^N, P^N\right)$ is a sample $\left(\delta^{(1)} \ldots \delta^{(N)}\right)$ of $N$ components extracted independently from $\Delta$ according to the same probability $P$. Each $\delta^{(i)}$ is called a scenario, and a scenario is a pair of input-output, i.e., $\delta^{(i)} = (u(i), y(i))$. Let $X$ be a subset of $R^{2n}$, and $2n$ denotes the dimension of the unknown parameter vector $(\theta, \eta)$. For each $\delta \in \Delta$, set $X_\delta$ to be a subset of $R^{2n}$, then for any sample $\left(\delta^{(1)} \ldots \delta^{(N)}\right)$, the corresponding sets $X_{\delta^{(1)}} \ldots X_{\delta^{(N)}}$ are considered.

Here we assume that an algorithm $A$ is available, which maps $\left(\delta^{(1)} \ldots \delta^{(N)}\right)$ to a possibly sub-optimal solution $\left(\hat{\theta}, \hat{\eta}\right)$, before extending our deep analysis, we need the following assumption and definition.

**Assumption 6.1.** *$P^N$ almost surely for all $N$, the solution $\left(\hat{\theta}, \hat{\eta}\right) = A\left(\delta^{(1)} \cdots \delta^{(N)}\right)$ exists, it is unique, and it satisfies all constraints, that is $\left(\hat{\theta}, \hat{\eta}\right) \in X_{\delta^{(i)}}$, $i = 1, 2, \ldots, N$. Moreover, $\left(\hat{\theta}, \hat{\eta}\right)$ is invariant with respect to any permutation of the sample $\left(\delta^{(1)} \ldots \delta^{(N)}\right)$.*

**Definition 6.1.** *Let $(\theta, \eta)$ be a given point in $X$, the violation probability of $(\theta, \eta)$ is defined as*

$$V(\theta, \eta) = P\{\delta \in \Delta : (\theta, \eta) \notin X_\delta\} \tag{34}$$

*when consider a fixed reliability parameter* $\epsilon \in (0,1)$, $(\theta,\eta) \in X$ *is* $\epsilon$ *feasible if* $V(\theta,\eta) \leq \epsilon$. *And* $V(\theta,\eta)$ *is a number for any fixed value of* $(\theta,\eta)$, *if* $V(\theta,\eta)$ *is computed, then* $V\left(\hat{\theta},\hat{\eta}\right)$ *is a random variable over* $\Delta^N$.

We want to provide a guarantee in the following form

$$P^N\left\{V\left(\hat{\theta},\hat{\eta}\right) > \epsilon\right\} \leq \beta \tag{35}$$

which states that

$$P^N\left\{\left(\hat{\theta},\hat{\eta}\right) \text{ is } \epsilon\text{-feasible}\right\} \geq 1-\beta \tag{36}$$

It means that the reliability guarantee will depend on the assessed cardinality of the set. Another definition formulates the idea of subset of constraints sufficient to support the solution.

**Definition 6.2.** *Consider a sample* $\left(\delta^{(1)}\dots\delta^{(N)}\right) \in \Delta^N$, *and let* $\left(\hat{\theta},\hat{\eta}\right)$ *be the corresponding solution. A support set is a subset of elements* $S = \left\{\delta^{(i_1)}\dots\delta^{(i_K)}\right\} \subset \left\{\delta^{(1)}\dots\delta^{(N)}\right\}$ *such that the problem obtained by removing all the constraints except* $(\theta,\eta) \in X_{\delta_{i_1}}\dots X_{\delta_{i_k}}$ *has the same solution* $\left(\hat{\theta},\hat{\eta}\right)$.

A support set can be found by a simple greedy algorithm, further if $\left(\delta^{(1)}\dots\delta^{(N)}\right)$ is a random sample, then its cardinality

$$s_N^* = \left\|B\left(\delta^{(1)}\dots\delta^{(N)}\right)\right\| \tag{37}$$

is a random variable over $\Delta^N$. Here a function $B : \left(\delta^{(1)}\dots\delta^{(N)}\right) \to \{i_1\dots i_k\}$ such that $\left\{\delta^{(i_1)}\dots\delta^{(i_k)}\right\}$ is a support set.

Then from above assumption and definitions, we state the following result.

**Theorem 6.1.** *Suppose that Assumption 6.1 holds, and let* $\beta \in (0,1)$. *Define the function* $\epsilon : \{0\dots N\} \to [0,1]$ *as follows*

$$\epsilon(k) = \begin{cases} 1 & \text{if } k = N \\ 1 - \sqrt[N-k]{\dfrac{\beta}{NC_N^k}} & \text{otherwise} \end{cases} \tag{38}$$

*Set that some algorithm* $B : \Delta^N \to 2^{\{1\dots N\}}$ *designed to select a support set is provided and let* $s_N^* = \left\|B\left(\delta^{(1)}\dots\delta^{(N)}\right)\right\|$, *then we have*

$$P^N\left\{V\left(\hat{\theta},\hat{\eta}\right) > \epsilon(s_N^*)\right\} \leq \beta \tag{39}$$

*And it is sufficient to select a sample size*

$$N \geq \frac{2}{\epsilon}\left(2n - 1 + \ln\left(\frac{1}{\beta}\right)\right) \tag{40}$$

7. **Simulation Example.** Here in this section, one example is given to prove the efficiency of this finite sample properties corresponding to virtual reference feedback tuning control. Consider one discrete time linear system, its transfer function form is described as follows

$$P(z) = \frac{(z-1.2)(z-0.4)}{z(z-0.3)(z-0.8)}$$

Two classical PID controllers are used here in the simulation

$$C_1(z,\theta) = \alpha^T(z)\theta = \begin{bmatrix} \frac{z^2}{z^2-z} & \frac{z}{z^2-z} & \frac{1}{z^2-z} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$C_2(z,\eta) = \beta^T(z)\eta = \begin{bmatrix} \frac{z^4}{z^4-z} & \frac{z^3}{z^4-z} & \frac{z^2}{z^4-z} & \frac{z}{z^4-z} & \frac{1}{z^4-z} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix}$$

Two true PID controllers are given as

$$C_1(z,\theta) = \alpha^T(z)\theta = \begin{bmatrix} \frac{z^2}{z^2-z} & \frac{z}{z^2-z} & \frac{1}{z^2-z} \end{bmatrix} \begin{bmatrix} 0.86 \\ 0.2 \\ 0.1 \end{bmatrix}$$

$$C_2(z,\eta) = \beta^T(z)\eta = \begin{bmatrix} \frac{z^4}{z^4-z} & \frac{z^3}{z^4-z} & \frac{z^2}{z^4-z} & \frac{z}{z^4-z} & \frac{1}{z^4-z} \end{bmatrix} \begin{bmatrix} 0.28 \\ 0.12 \\ 0.05 \\ -0.02 \\ -0.06 \end{bmatrix}$$

The expected closed loop transfer function is chosen as

$$M(z) = \frac{z(z-1)\left(0.86z^4 - 1.1z^3 + 3.9z^2 + 0.8z + 0.48\right)}{z^7 - 3z^6 - 0.96z^5 - 0.72z^4 - 0.93z^3 + 3.9z^2 + 0.8z + 0.48}$$

The expected sensitivity function is given as

$$S(z) = \frac{0.04z^5 + 0.01z^4 - 0.006z^3 - 0.035z^2 - 0.04z + 0.01}{-0.5z^6 + z^5 - z^4 + 0.24z^3 + 0.12z^2 + 0.1z + 1}$$

The input-output measured data $\{u(t), y(t)\}_{t=1,2,\ldots,1000}$ are collected in the closed loop environment, and the number of data points is set 2000. To use the idea of virtual reference feedback tuning control in designing parameter vectors, the plant model $P(z)$ is excited by zero mean Gaussian white noise, which is plotted in Figure 4 with the number of data points being 1000. The measured output data is seen in Figure 5. The nonlinear least squares algorithm is used to solve the optimization problem (14). Before starting this iteration algorithm, the initial values of the unknown parameter vector are selected as

$$\theta = \begin{bmatrix} 0.75 \\ 0.25 \\ 0.15 \end{bmatrix}, \quad \eta = \begin{bmatrix} 0.2612 \\ 0.1277 \\ 0.0854 \\ -0.0125 \\ -0.0784 \end{bmatrix}$$

In Figure 6, the asymptotic curves of three PID parameters in the unknown parameter vector $\hat{\theta}_N^T$ are plotted. The dotted lines denote the sample variances which are computed by sample data; the solid lines denote the asymptotic variances which are computed by using the above asymptotic analysis formula, i.e., these three unknown parameters are estimated with finite data and infinite data. The variance corresponding to finite data is the sample variance, and the other case is the asymptotic variance. Due to the fact that the infinite data is an ideal case, the asymptotic variance is not useful in practice. Here the goal of Figure 6 is to verify whether the sample variance can approximate the asymptotic variance. According to Figure 6, the solid lines approximate to the dotted

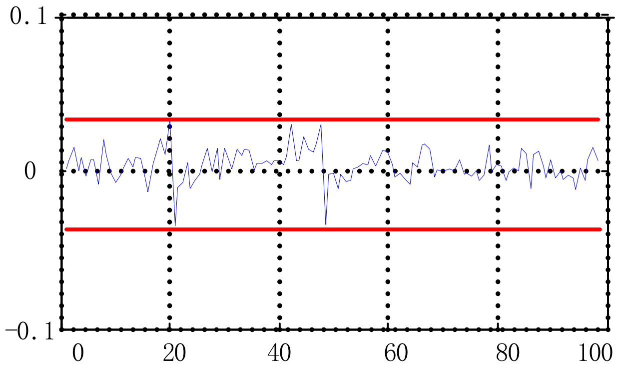



FIGURE 4. The input signal as a white noise

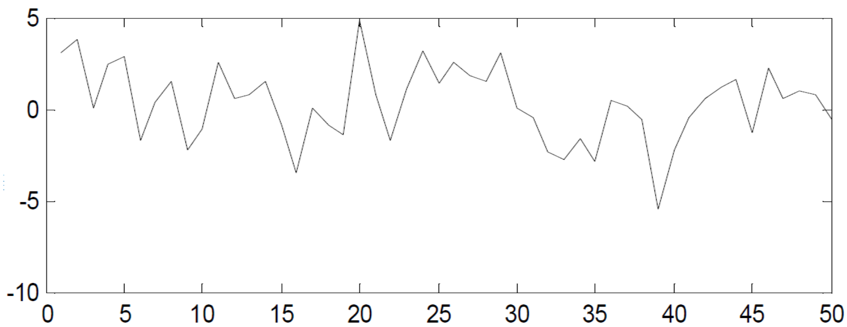
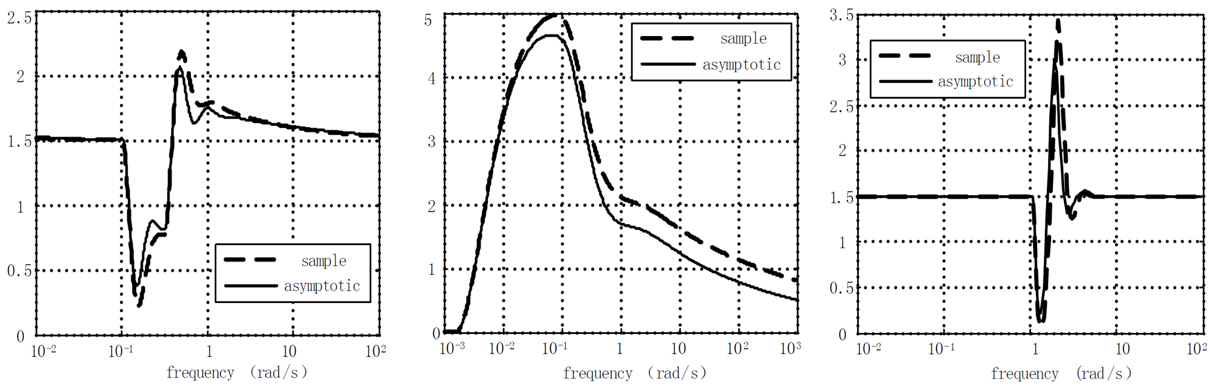



FIGURE 5. The measured output signal




(a) Compare the variance of the first P parameter.

(b) Compare the variance of the second I parameter.

(c) Compare the variance of the third D parameter.

FIGURE 6. The approximated curves of three PID parameters

lines very closely, so it means that the asymptotic value can be more close to the real sample value.

8. **Conclusion.** In this paper, virtual reference feedback tuning control is applied to designing controllers for closed loop system with two degrees of freedom controllers. The model-matching problems corresponding to the expected closed loop transfer function and expected sensitivity function are considered simultaneously. The projection algorithm is introduced to achieve our first on-line parameter estimation scheme. Furthermore, scenario optimization is used to determine the number of scenarios that determine the

solution. However, in this paper, no any simulations in practice are considered, so that how to use projection algorithm and scenario approach for virtual reference feedback tuning control in practice or engineering is our further work.

**Acknowledgment.** This work is partially supported by the Grants from the National Natural Science Foundation of China (No. 61364014) and (No. jxxjb18020).

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