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STATIC OUTPUT FEEDBACK CONTROLLER FOR CONTINUOUS-TIME FUZZY SYSTEMS

Redouane Chaibi[1], El Houssaine Tissir[1], Abdelaziz Hmamed[1]
El Mostafa El Adel[2] and Mustapha Ouladsine[2]

[1]Laboratoire d'Electronique Signaux Systèmes et Informatique (LESSI)
Department of Physics
Faculty of Sciences
University Sidi Mohamed Ben Abdellah
Dhar El Mehraz B.P. 1796 Fes-Atlas, Morocco
c.redouane.chaibi@gmail.com; elhoussaine.tissir@usmba.ac.ma; hmamed_abdelaziz@yahoo.fr

[2]LIS – Laboratoire d'Informatique et Systèmes – UMR 7020
Aix Marseille Univ, University of Toulon
CNRS, LIS, Marseille, France
{ el-mostafa.el-adel; mustapha.ouladsine }@lis-lab.fr

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Abstract. *This paper will focus on static output feedback (SOF) control for continuous-time Takagi-Sugeno (T-S) fuzzy systems. Based on the use of the concept of decay rate in the quadratic Lyapunov function, sufficient conditions for designing fuzzy SOF control are given in terms of linear matrix inequalities (LMIs). Moreover, the proposed method overcomes the drawback of the iterative algorithms based on LMI approach, and does not need any equality constraints; in addition, it can provide less conservative results. In the end, numerical and practical examples are given to demonstrate the effectiveness of the proposed method.*

Keywords: Linear matrix inequalities, Takagi-Sugeno fuzzy models, Static output feedback

1. **Introduction.** In recent years, a large number of researchers are studying Takagi-Sugeno (T-S) systems, because although most physical systems are nonlinear, they can be adequately represented by this class of fuzzy systems. Takagi-Sugeno (T-S) fuzzy model approach has played an important role in designing stabilizing controllers for nonlinear systems, by interpolation of numerous linear models in terms of IF-THEN fuzzy rules [1]. Based on the T-S fuzzy model, some investigations have been interested in analyzing approaches of the stability under parallel distributed compensation (PDC) control. In [2, 3] sufficient conditions for discrete-time T-S fuzzy systems, subject to actuator saturation were presented, with the associated stabilization conditions solved using LMIs. The non-PDC control law is suggested in [4] to stabilize continuous-time T-S systems. In [5, 6], the problem of stability property has been studied for T-S fuzzy systems with time delay based on the Lyapunov-Krasovskii function. While [7, 8] have studied the exponential stability and stabilization of T-S fuzzy time-varying delay systems with parameter uncertainties.

Moreover, several relaxed stabilization conditions were also proposed to derive less conservative results in [9, 10, 11]. Most of the references cited previously have focused on state feedback control problems. However, in most industrial applications, state variables are not always completely measurable. Hence, researchers have been paying remarkable

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attention to the problems of output feedback control and particularly for dynamic output feedback. The major drawback of the dynamic output feedback approach is the increasing dimension of the closed-loop system.

Alternatively, static output feedback (SOF) control is proposed in a number of works such as [12, 13, 14]. The SOF is one of the most important open problems in control theory and applications, due to its simplicity to be implemented in practice, compared with dynamic output feedback control, see [15] and the references therein. Thus, many works have been done to relax LMI based stability conditions [16, 17, 18, 19, 20, 21]. In particular, sufficient conditions in terms of a set of strict LMIs for the SOF control for fractional-order T-S fuzzy systems were given in [19]. SOF controller for discrete-time T-S systems with time-delays was proposed in [20] using a multiple Lyapunov-Krasovskii functional when values of the disturbance attenuation and decay rate are imposed. Based on a quadratic Lyapunov function, using LMIs and some matrix transformations, a procedure to calculate SOF controller was given in [22]; a numerical procedure based on the cone complementary algorithm was given for the design of SOF stabilizing controllers of T-S systems in [18]; to solve the SOF for continuous-time T-S fuzzy models, an iterative algorithm is provided in [24] by using the common quadratic Lyapunov functions. In these addressed works, the conditions are bi-linear in the decision variables, so iterative algorithms based on LMI decomposition that depend on the initial values, have been developed to numerically solve the stabilization problem, which represents a weakness of the approach. In [13, 14, 16, 23], the bi-linear matrix inequalities (BMIs) problem for SOF control, has been solved by inserting an equality constraint condition for the Lyapunov matrix. In addition, a sufficient condition is described in [17, 29] for the SOF controller, which does not need any equality constraint. Motivated by this fact, this paper deals with the problem of SOF control design in terms of LMIs, for continuous-time T-S fuzzy systems. Based on the use of the concept of decay rate in the quadratic Lyapunov function, sufficient conditions for the existence of SOF are proposed. In comparison with the above-mentioned LMI design methods, the SOF control presented in this paper does not need any transformation matrices, and equality constraint and it is proved that the proposed method can give less conservative results. The remainder of this paper is organized as follows. The T-S system description and preliminary result are stated in Section 2. The suggested approach is provided in Section 3 and the principal results are proposed in LMIs formulation, while in Section 4, numerical and practical examples are given to illustrate the effectiveness of the proposed method, and the conclusion will be in Section 5.

Notations: Through this paper, $P > 0$ (< 0) means that P is positive (negative) definite matrices; the superscript T stands for matrix transpose; M^T denotes the transpose of M; $sym(M)$ represents $M + M^T$. I denotes the identity matrix with appropriate dimension. The symbol $*$ represents the symmetric term in a block matrix.

2. **Problem Formulation.** We consider the continuous T-S fuzzy mode which is described by the following fuzzy model.

Plant Rule i: If $z_1(t)$ is M_{i1} AND, …, AND $z_s(t)$ is M_{is}

$$\text{Then} \left\{ \begin{array}{l} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{array} \right. \tag{1}$$

where $i = 1, 2, \ldots, r$, $z(t) = [z_1(t)z_2(t)\ldots z_s(t)]$ are known premise variables, M_{ij} is the fuzzy sets and r is the number of rules. $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^l$ is the measurement output with full-row rank matrices C_i. A_i, B_i and C_i are constant matrices of compatible dimensions.

By using the commonly used center-average defuzzifier, product interference and singleton fuzzifier, the T-S fuzzy model (1) can be inferred as (2).

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + B_i u(t)] \\ y(t) = \sum_{i=1}^{r} h_i(z(t)) C_i x(t) \end{cases} \tag{2}$$

where

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \quad w_i(z(t)) = \prod_{j=1}^{s} M_{ij}(z_j(t))$$

$M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij} and $w_i(z(t))$ represents the weight of the i^{th} rule. In this paper, we assume that $w_i(z(t)) \geq 0$, for $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} w_i(z(t)) > 0$ for all t. Therefore, we get $h_i(z(t)) \geq 0$, for $i = 1, 2, \ldots, r$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$, for all t.

We consider the concept of PDC law [16] to design the static output feedback controller for T-S fuzzy systems (2).

Controller Rule i: If $z_1(t)$ is M_{i1} AND, ..., AND $z_s(t)$ is M_{is}

$$\text{Then } u(t) = F_i y(t) \tag{3}$$

where $i = 1, 2, \ldots, r$ and F_i are the feedback gain matrices.

The overall static feedback control is inferred as

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i y(t) \tag{4}$$

From (2) and (4), the closed-loop fuzzy system can be expressed as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} h_i h_j h_l [A_i + B_i F_j C_l] x(t) \tag{5}$$

where $h_i = h_i(z(t))$, $h_j = h_j(z(t))$, $h_l = h_l(z(t))$.

Lemma 2.1. [26] *For matrices T, Q, U, and W with appropriate dimensions and scalar ξ. The inequality*

$$T + W^T Q^T + QW < 0 \tag{6}$$

is fulfilled if the following condition holds:

$$\begin{bmatrix} T & * \\ \xi Q^T + UW & -\xi U - \xi U^T \end{bmatrix} < 0 \tag{7}$$

Lemma 2.2. [28] *Given a symmetric matrix $\Sigma \in \mathbb{R}^{p \times p}$ and two matrices X, Z of column dimension p, there exists a matrix Y such that the LMI*

$$\Sigma + sym\left\{X^T Y Z\right\} < 0 \tag{8}$$

holds if and only if the following two projection inequalities with respect to Y are satisfied:

$$X^{\perp T} \Sigma X^{\perp} < 0, \quad Z^{\perp T} \Sigma Z^{\perp} < 0 \tag{9}$$

where $X^{\perp}$ and $Z^{\perp}$ are arbitrary matrices whose columns form a basis of the null spaces of X and Z, respectively.

Remark 2.1. *There exist many convex methods for designing SOF controllers in the literature, for example, sufficient conditions with equality constraint were used in* [13, 16] *for the T-S fuzzy systems. Moreover, some methods without the need to impose any constraints on system matrices are also proposed for robust SOF H_∞ control, in* [25, 26] *for linear systems with polytopic and for discrete-time T-S fuzzy systems respectively. In the present contribution, the SOF control of continuous-time T-S fuzzy model is explicitly considered. Based on projection's lemma, new design conditions for SOF control have been proposed in the LMI framework, with additional slack variables. These extra variables offer more flexibility to reduce the design conservatism of previous works. In addition, with the aid of some special derivations, all complex couplings between Lyapunov matrices and feedback gain matrices are avoided. Hence, our results may offer a wider range of the feasible region than some existing results* [16, 17, 24]. *This fact will be further illustrated in the next section.*

Now, we shall present a numerically efficient technique to find the SOF gains of the form (4), such that the closed-loop system (5) is asymptotically stable.

3. **Main Results.** In this section, we shall present sufficient conditions for the existence of an SOF for the system (5), by using the concept of decay rate. The advantage of using the concept of decay rate in the quadratic Lyapunov function is to be able to manipulate convergence time.

Theorem 3.1. *Let $\xi, \alpha > 0$. The T-S fuzzy system* (5) *is asymptotically stable, if there exist positive definite matrix P, and matrices G, U and N_i, for $i = 1, \dots, r$, such that* (10), (11) *and* (12) *hold.*

$$\Omega_{iii} < 0 \quad i = 1, \dots, r \tag{10}$$

$$\Omega_{iij} + \Omega_{iji} + \Omega_{jii} < 0 \quad i, j = 1, \dots, r \quad i \neq j \tag{11}$$

$$\Omega_{ijl} + \Omega_{ilj} + \Omega_{jil} + \Omega_{jli} + \Omega_{lij} + \Omega_{lji} < 0 \quad 1 \leq i \neq j \neq l \leq r \tag{12}$$

where

$$\Omega_{ijl} = \begin{bmatrix} -G - G^T & * & * \\ \Omega_{ijl}^{21} & \Omega_{ijl}^{22} & * \\ \Omega_{ijl}^{31} & \Omega_{ijl}^{32} & -\xi U - \xi U^T \end{bmatrix} \tag{13}$$

$$\begin{aligned} \Omega_{ijl}^{21} &= \left(A_i^T + \frac{\alpha}{2} I \right) G^T + P - G + C_l^T N_j^T B_i^T \\ \Omega_{ijl}^{22} &= sym \left\{ G \left(A_i + \frac{\alpha}{2} I \right) + B_i N_j C_l \right\} \\ \Omega_{ijl}^{31} &= \xi \left(B_i^T G^T - U^T B_i^T \right) \\ \Omega_{ijl}^{32} &= \xi \left(B_i^T G^T - U^T B_i^T \right) + N_j C_l \end{aligned} \tag{14}$$

The gain matrices are given by $F_i = U^{-1} N_i$, $i = 1, \dots, r$.

Proof: Suppose that inequalities (10)-(12) hold. The feasible solution of this inequality satisfies $-\xi U - \xi U^T < 0$, which implies that matrix U is nonsingular. Obviously, the LMI conditions (10)-(12) can be rewritten as follows:

$$\sum_{i=1}^{r} h_i^3 \Omega_{iii} + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ i \neq j}}^{r} h_i^2 h_j (\Omega_{iij} + \Omega_{iji} + \Omega_{jii})$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} \sum_{l=j+1}^{r} h_i h_j h_l (\Omega_{ijl} + \Omega_{ilj} + \Omega_{jil} + \Omega_{jli} + \Omega_{lij} + \Omega_{lji}) \tag{15}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i h_j h_l \Omega_{ijl} < 0$$

which is verified if

$$\Omega_{ijl} = \begin{bmatrix} T_{ijl} & * \\ \xi Q_i^T + UW_{jl} & -\xi U - \xi U^T \end{bmatrix} < 0 \tag{16}$$

Applying Lemma 2.1 with

$$T_{ijl} = \begin{bmatrix} -G - G^T & * \\ \Omega_{ijl}^{21} & \Omega_{ijl}^{22} \end{bmatrix}, \quad Q_i = \begin{bmatrix} GB_i - B_i U \\ GB_i - B_i U \end{bmatrix}, \quad W_{jl} = \begin{bmatrix} 0 & N_j C_l \end{bmatrix} \tag{17}$$

where Ω_{ijl}^{21} and Ω_{ijl}^{22} are defined in (14).

Obviously, (16) can be written as

$$T_{ijl} + sym\left\{ \begin{bmatrix} GB_i - B_i U \\ GB_i - B_i U \end{bmatrix} U^{-1} \begin{bmatrix} 0 & N_j C_l \end{bmatrix} \right\} < 0 \tag{18}$$

From (18), we have

$$T_{ijl} + \begin{bmatrix} 0 & (GB_i - B_i U)U^{-1}N_j C_l \\ * & (GB_i - B_i U)U^{-1}N_j C_l \end{bmatrix} < 0 \tag{19}$$

Substituting T_{ijl} in (17) into (19) and applying the change of variables $N_j = UF_j$, we obtain

$$\begin{bmatrix} -G - G^T & * \\ -G + \bar{A}_{ijl}^T G^T + P & G\bar{A}_{ijl} + \bar{A}_{ijl}^T G^T \end{bmatrix} < 0 \tag{20}$$

where $\bar{A}_{ijl} = A_i + B_i F_j C_l + \frac{\alpha}{2} I$.

We can verify that (20) is equivalent to

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + sym\left(\begin{bmatrix} G \\ G \end{bmatrix} \begin{bmatrix} -I & \bar{A}_{ijl} \end{bmatrix} \right) < 0 \tag{21}$$

By Lemma 2.2 with

$$\Sigma = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \quad X = I, \quad Y = \begin{bmatrix} G \\ G \end{bmatrix}, \quad Z = \begin{bmatrix} -I & \bar{A}_{ijl} \end{bmatrix}$$

inequality (21) can guarantee

$$\begin{bmatrix} \bar{A}_{ijl}^T & I \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_{ijl} \\ I \end{bmatrix} < 0 \tag{22}$$

Defining $\bar{A}_{ijl} = \tilde{A}_{ijl} + \frac{\alpha}{2} I$ inequality (22) can be rewritten as

$$P\tilde{A}_{ijl} + \tilde{A}_{ijl}^T P + \frac{\alpha}{2}P + \frac{\alpha}{2}P < 0 \tag{23}$$

Let us consider the Lyapunov function given by [16]

$$V(x(t)) = x(t)^T P x(t) \tag{24}$$

For the decay rate control design, the condition is defined as follows

$$\dot{V}(x(t)) \le -\alpha V(x(t)) \tag{25}$$

The derivative of (24) with respect to time satisfies

$$\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i h_j h_l \left\{ x(t)^T \left[\tilde{A}_{ijl}^T P + P\tilde{A}_{ijl} \right] x(t) \right\} \tag{26}$$

where $\tilde{A}_{ijl} = A_i + B_i F_j C_l$.

Since (23) holds, it can be easily seen that

$$\dot{V}(x(t)) + \alpha V(x(t)) = x(t)^T \left[\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} h_i h_j h_l \left\{ \tilde{A}_{ijl}^T P + P\tilde{A}_{ijl} + \alpha P \right\} \right] x(t) < 0 \quad (27)$$

End proof. □

Remark 3.1. *Pre- and post-multiplying both sides of* (23) *by* P^{-1} *and applying the change of variables* $Q^{-1} = P$*, we get another LMI problem which is reformulated in the next theorem.*

Theorem 3.2. *Let* $\xi, \alpha > 0$*. The closed-loop fuzzy system* (5) *is asymptotically stable, if there exist positive definite matrix* Q*, and matrices* G*,* U *and* N_i*, for* $i = 1, \ldots, r$*, fulfilling the following set of LMIs.*

$$\Gamma_{iii} < 0 \quad i = 1, \ldots, r \quad (28)$$

$$\Gamma_{iij} + \Gamma_{iji} + \Gamma_{jii} < 0 \quad i, j = 1, \ldots, r \quad i \neq j \quad (29)$$

$$\Gamma_{ijl} + \Gamma_{ilj} + \Gamma_{jil} + \Gamma_{jli} + \Gamma_{lij} + \Gamma_{lji} < 0 \quad 1 \leq i \neq j \neq l \leq r \quad (30)$$

with

$$\Gamma_{ijl} = \begin{bmatrix} -G - G^T & * & * \\ \Gamma_{ijl}^{21} & \Gamma_{ijl}^{22} & * \\ \Gamma_{ijl}^{31} & \Gamma_{ijl}^{32} & -\xi U - \xi U^T \end{bmatrix} \quad (31)$$

and

$$\begin{aligned} \Gamma_{ijl}^{21} &= Q - G + \left(A_i + \frac{\alpha}{2} I \right) G^T + B_i K_j C_l \\ \Gamma_{ijl}^{22} &= sym\left\{ \left(A_i + \frac{\alpha}{2} I \right) G^T + B_i K_j C_l \right\} \\ \Gamma_{ijl}^{31} &= C_l G^T - U C_l \\ \Gamma_{ijl}^{32} &= \xi K_j^T B_i^T + C_l G^T - U C_l \end{aligned} \quad (32)$$

The gain matrices are given by $F_i = K_i U^{-1}$*.*

Proof: Similar to that of Theorem 3.1, we suppose that inequalities (28)-(30) hold. So we can rewrite (28)-(30) as (15) with $\Omega_{ijl} = \Gamma_{ijl}$.

By using Lemma 2.1 with

$$\begin{aligned} \tilde{T}_{ijl} &= \begin{bmatrix} -G - G^T & * \\ \Gamma_{ijl}^{21} & \Gamma_{ijl}^{22} \end{bmatrix}, \quad \tilde{Q}_{ij} = \begin{bmatrix} 0 \\ B_i K_j \end{bmatrix} \\ \tilde{W}_l &= U^{-1} \begin{bmatrix} C_l G^T - U C_l & C_l G^T - U C_l \end{bmatrix} \end{aligned} \quad (33)$$

where Γ_{ijl}^{21} and Γ_{ijl}^{22} are defined in (32).

Inequality (31) is equivalent to

$$\tilde{T}_{ijl} + \begin{bmatrix} 0 & * \\ B_i K_j U^{-1}(C_l G^T - U C_l) & sym\{B_i K_j U^{-1}(C_l G^T - U C_l)\} \end{bmatrix} < 0 \quad (34)$$

Substituting $\tilde{T}_{ijl}$ in (33) into (34) and applying the change of variables $K_j = F_j U$, we have

$$\begin{bmatrix} -G - G^T & * \\ -G + Q + \bar{A}_{ijl} G^T & \bar{A}_{ijl} G^T + G \bar{A}_{ijl}^T \end{bmatrix} < 0 \quad (35)$$

We can verify that (35) is equivalent to

$$\begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix} + sym\left(\begin{bmatrix} G \\ G \end{bmatrix} \begin{bmatrix} -I & \bar{A}_{ijl}^T \end{bmatrix} \right) < 0 \quad (36)$$

By Lemma 2.2 with

$$\Sigma = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}, \quad X = I, \quad Y = \begin{bmatrix} G \\ G \end{bmatrix}, \quad Z = \begin{bmatrix} -I & \bar{A}_{ijl}^T \end{bmatrix}$$

inequality (36) can guarantee

$$\begin{bmatrix} \bar{A}_{ijl} & I \end{bmatrix} \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_{ijl}^T \\ I \end{bmatrix} < 0 \tag{37}$$

Defining $\bar{A}_{ijl} = \tilde{A}_{ijl} + \frac{\alpha}{2} I$, we can verify that (37) is equivalent to

$$\tilde{A}_{ijl} Q + Q \tilde{A}_{ijl}^T + \frac{\alpha}{2} Q + \frac{\alpha}{2} Q < 0 \tag{38}$$

Pre- and post-multiplying both sides of (38) by Q^{-1} and applying the change of variables $Q^{-1} = P$, we obtain inequality (23).

This completes the proof. □

Remark 3.2. *An iterative method based on LMI decomposition has been presented in* [24] *for SOF fuzzy controller. The major drawback of this approach is that design of SOF fuzzy controller depends on the initial values. To avoid such drawback, an LMI design method is provided in this work without the need to impose any constraints on system matrices.*

Remark 3.3. *By using the concept of decay rate, we have obtained a sufficient condition for stability conditions. The advantage of using the concept of decay rate is to be able to manipulate convergence time.*

4. Computer Simulations.

In order to show the efficiency of the proposed method, we present two illustrative examples. The first one is used here to compare the proposed results in terms of conservatism. The second example is provided to show the practical application of the proposed controller design methodology for the permanent magnetic synchronous motor (PMSM).

Example 4.1. *In this section, we consider the nonlinear system* (39) *borrowed from* [16]*, to demonstrate the validity of the proposed method.*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_2(t) + \sin x_3(t) - 0.1x_4(t) + \left(x_1^2(t) + 1\right) u(t) \\ \dot{x}_2(t) &= x_1(t) - 2x_2(t) \\ \dot{x}_3(t) &= x_1(t) + x_1^2 x_2(t) - 0.3x_3(t) \\ \dot{x}_4(t) &= \sin x_3(t) - x_4(t) \\ y_1(t) &= x_2(t) + \left(x_1^2(t) + 1\right) x_4(t) \\ y_2(t) &= x_1(t) \end{aligned} \tag{39}$$

Assume $x_1(t) \in [-a, a]$, $x_3(t) \in [-b, b]$, where a and b are positive numbers. The nonlinear system (39) *can be represented by T-S fuzzy system as follows:*

Plant Rule 1: If $x_1(t)$ is M_1^1 and $x_3(t)$ is M_2^1

$$\text{Then} \begin{cases} \dot{x}(t) = A_1 x(t) + B_1 u(t) \\ y(t) = C_1 x(t) \end{cases}$$

Plant Rule 2: If $x_1(t)$ is M_1^1 and $x_3(t)$ is M_2^2

$$\text{Then} \begin{cases} \dot{x}(t) = A_2 x(t) + B_2 u(t) \\ y(t) = C_2 x(t) \end{cases}$$

Plant Rule 3: If $x_1(t)$ *is* M_1^2 *and* $x_3(t)$ *is* M_2^1

$$\text{Then} \begin{cases} \dot{x}(t) = A_3x(t) + B_2u(t) \\ y(t) = C_3x(t) \end{cases}$$

Plant Rule 4: If $x_1(t)$ *is* M_1^2 *and* $x_3(t)$ *is* M_2^2

$$\text{Then} \begin{cases} \dot{x}(t) = A_4x(t) + B_4u(t) \\ y(t) = C_4x(t) \end{cases}$$

where the premise membership functions are taken as the same as those used in [16]

$$\begin{aligned} & M_1^1(x_1) = \frac{x_1^2}{a^2} \\ & M_1^2(x_1) = 1 - M_1^1(x_1) \\ & M_2^1(x_3) = \begin{cases} \dfrac{b\sin x_3 - x_3 \sin b}{x_3(b - \sin b)} & x_3 \neq 0 \\ 1 & x_3 = 0 \end{cases} \\ & M_2^2(x_3) = 1 - M_2^1(x_1) \end{aligned} \tag{40}$$

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & a^2 & -0.3 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & \frac{\sin b}{b} & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & a^2 & -0.3 & 0 \\ 0 & 0 & \frac{\sin b}{b} & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -0.3 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & \frac{\sin b}{b} & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -0.3 & 0 \\ 0 & 0 & \frac{\sin b}{b} & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 + a^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = B_1, \quad B_4 = B_3,$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 & 1 + a^2 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_3 = C_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The feasible area of the above system is checked by using Theorem 3.1 for $a \in [0, 8]$, $b \in [0, 8]$, *and different values of* ξ *and* α. *Figure 1 shows the feasible area given by* $\alpha = -3.4$ *and different values of* ξ, *(*$\xi = 0.055$, $\xi = 0.118$ *and* $\xi = 1.5$*). Figure 2 shows the feasible area given by* $\xi = 5$ *and different values of* α, *(*$\alpha = 0$, $\alpha = -3$ *and* $\alpha = -5$*). As shown in Figure 1 and Figure 2, the feasible area became larger as the value of* ξ *increases or* α *decreases for Theorem 3.1.*

Now, we use fminsearch, to find the optimal value of ξ *such that the LMIs of Theorem 3.1 have a feasible solution.*

For $a = 4$ *and* $b = 5$, *there is no feasible solution by the results of* [16, 17], *but Theorem 3.1 can still find a feasible solution, with* $\alpha = -5$, *we can obtain the optimal value of* $\xi = 2.0135$ *and the following SOF gain matrices:*

$$F_1 = \begin{bmatrix} -0.0287 & -1.2096 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.0130 & -1.6700 \end{bmatrix}$$
$$F_3 = \begin{bmatrix} -0.0481 & -3.7705 \end{bmatrix}, \quad F_4 = \begin{bmatrix} -0.0221 & -2.2552 \end{bmatrix}$$

With the above gain matrices and $x(0) = [-1.2,\ 0.5,\ 0.7,\ -0.6]^T$, *the state response curves for the closed-loop system are given in Figure 3. From Figure 3, it is possible to conclude that the closed-loop system is asymptotically stable.*

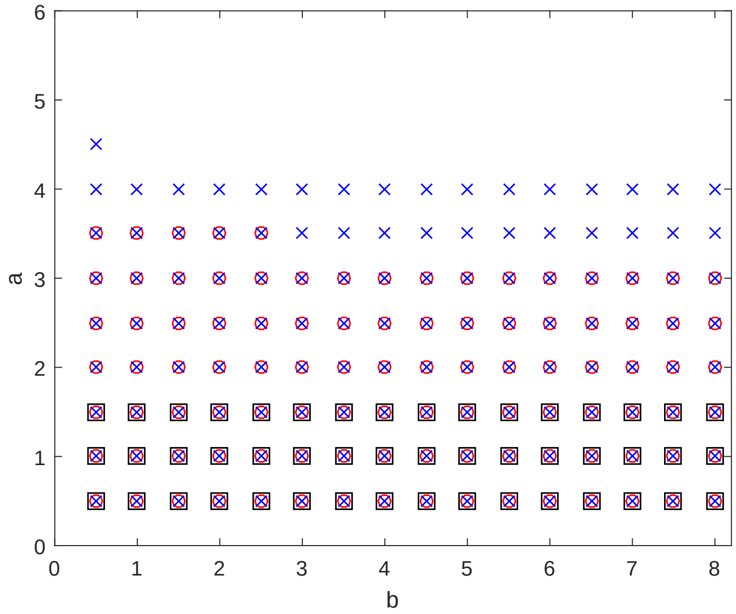

FIGURE 1. Feasible region provided by Theorem 3.1 with $\xi = 1.5$ marked with ($\times$), $\xi = 0.118$ marked with ($\circ$), $\xi = 0.055$ marked with ($\square$)

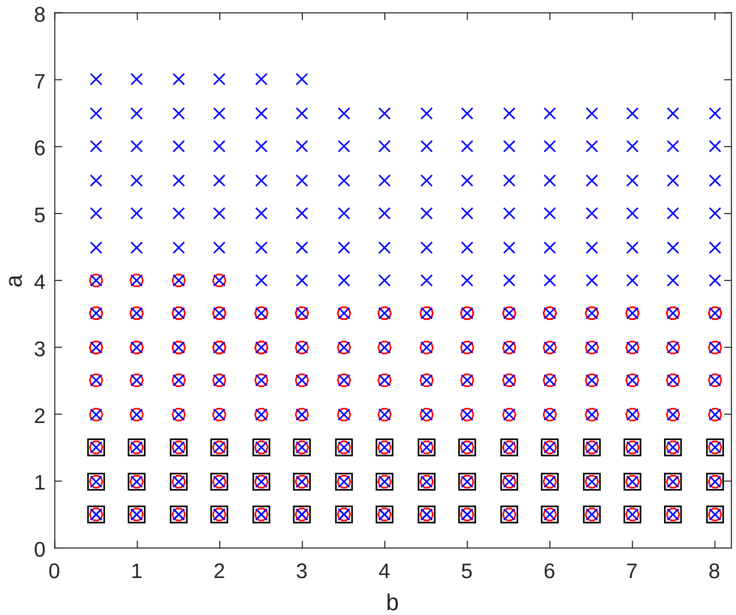

FIGURE 2. Feasible region provided by Theorem 3.1 with $\alpha = -5$ marked with ($\times$), $\alpha = -3$ marked with ($\circ$), $\alpha = 0$ marked with ($\square$)

Now, if we replace

$$B_1 = \begin{bmatrix} 1 + a^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = B_1, \quad B_4 = B_3$$

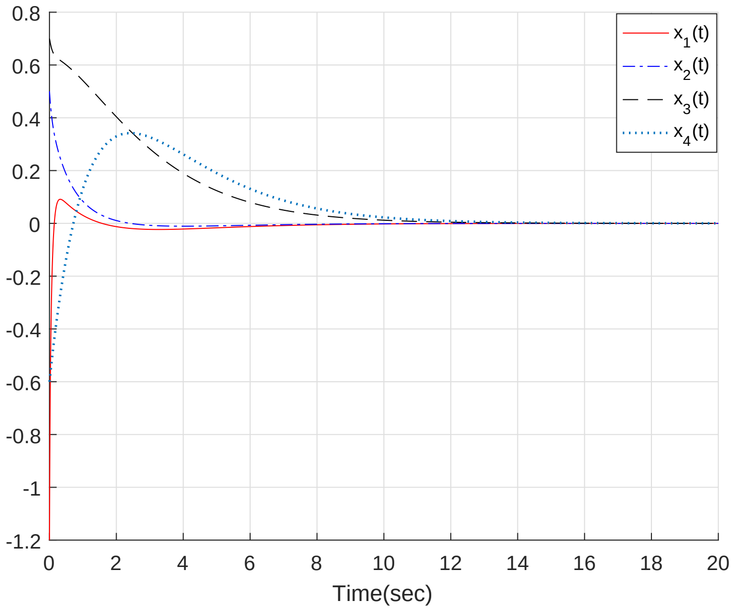

FIGURE 3. State response curves for the closed-loop system

by

$$B_1 = \begin{bmatrix} (1+a^2) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} (1+a^2) \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

we can easily compare our results with [17, 24]. *For* $a = 2$ *and* $b = 1$, *there is no feasible solution by the results of* [17], *but Theorem 3.2 can still find a feasible solution, with* $\xi = 0.01$. *Using fminsearch, we obtain the optimal value of* $\alpha = -1.8171$ *and the following SOF gain matrices:*

$$F_1 = \begin{bmatrix} -0.0797 & -0.9902 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.0598 & -0.8473 \end{bmatrix}$$
$$F_3 = \begin{bmatrix} -0.1233 & -1.7719 \end{bmatrix}, \quad F_4 = \begin{bmatrix} -0.0841 & -2.3423 \end{bmatrix}$$

Figures 4-7 show the time responses for the closed-loop system, by using Theorem 1 in [24], *and Theorem 3.2, and the initial values of the state vector,* $x(0) = [-1.2,\ 0.5,\ 0.7,\ -0.6]^T$. *From Figures 4-7, it is clear that the convergence of the trajectories of the system states with our approach is faster than with the existing one in* [24].

Example 4.2. *In this example, let us consider a permanent magnetic synchronous motor (PMSM)* [27]. *The fuzzy model has been modeled by a two-rule fuzzy model* [14] *as follows:*

$$\begin{cases} \dot{i}_d(t) = -\dfrac{R}{L} i_d(t) + n_p i_q(t) W(t) + \nu_d(t) \\ \dot{i}_q(t) = -\dfrac{R}{L} i_d(t) + n_p i_d(t) W(t) - \dfrac{\Psi}{L} W(t) + \nu_q(t) \\ \dot{W}(t) = \dfrac{\Psi}{L} i_q(t) - \dfrac{\tau}{j} W(t) + 2\omega(t) \end{cases} \tag{41}$$

where $i_q(t)$ *and* $i_d(t)$, *are the quadrature and direct current respectively.* $W(t)$ *is the motor angular velocity.* $\nu_q(t)$ *and* $\nu_d(t)$ *are the quadrature and direct input voltages.* $n_p = 1$ *and* $\Psi = 0.031 Nm/A$ *denote the number of pole-pairs and the permanent-magnet flux*

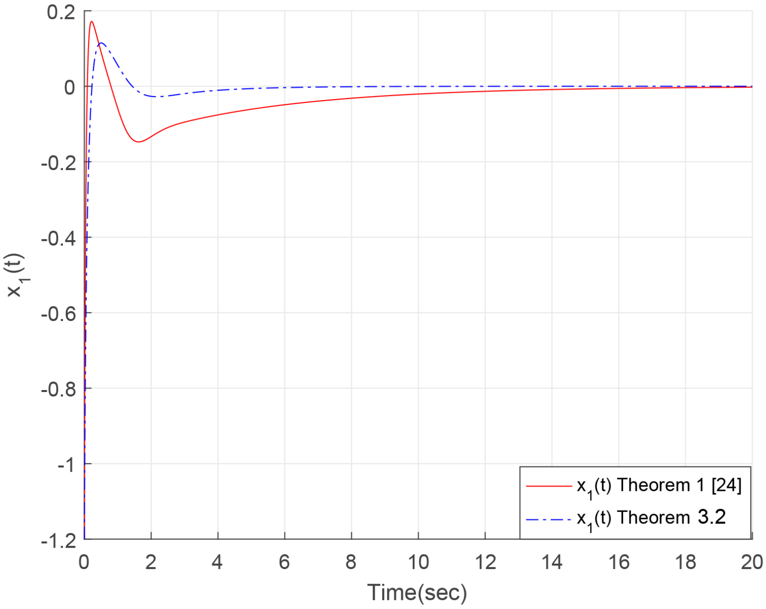

FIGURE 4. Time response of the system state $x_1(t)$

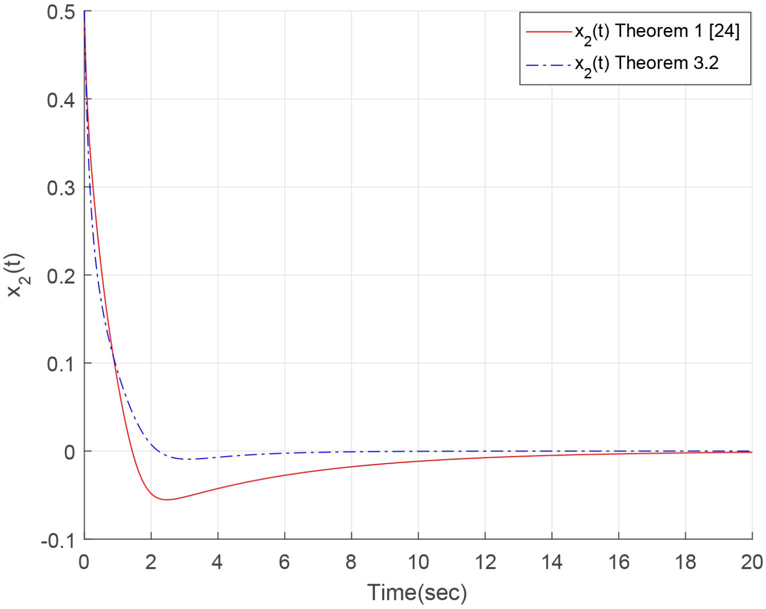

FIGURE 5. Time response of the system state $x_2(t)$

respectively, $L = 0.01425H$ *is the direct quadrature-axis stator inductors,* $R = 0.9\Omega$ *is the stator winding resistance, and* $j = 4.5 \times 10^{-5} kg{\cdot}m^2$ *is the polar moment of inertia,* $\tau = 0.0162N/rad/s$ *is the viscous damping coefficient. In this paper we assume* $\omega(t) = 0$*. The nonlinear terms satisfy the conditions* $W(t) \in [\theta_1, \theta_2]$ *and then we can obtain the following parameters:*

$$A_1 = \begin{bmatrix} -\frac{R}{L} & n_p\theta_1 & 0 \\ -n_p\theta_1 & -\frac{R}{L} & -\frac{\Psi}{L} \\ 0 & \frac{\Psi}{L} & -\frac{\tau}{j} \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{R}{L} & n_p\theta_2 & 0 \\ -n_p\theta_2 & -\frac{R}{L} & -\frac{\Psi}{L} \\ 0 & \frac{\Psi}{L} & -\frac{\tau}{j} \end{bmatrix}, \quad B_{21} = B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where $x(t) = \left[i_d^T(t),\ i_q^T(t),\ W^T(t)\right]^T$ *and* $u(t) = \left[\nu_d^T(t),\ \nu_q^T(t)\right]^T$.

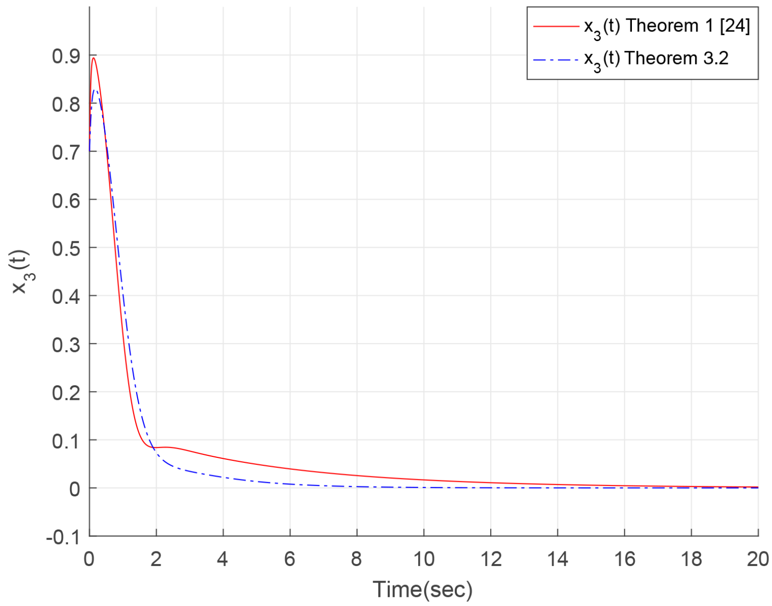

FIGURE 6. Time response of the system state $x_3(t)$

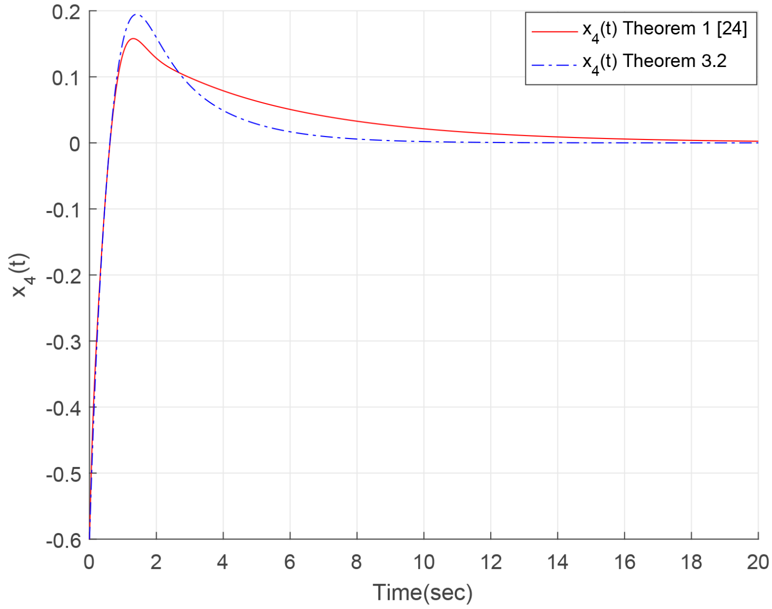

FIGURE 7. Time response of the system state $x_4(t)$

The membership functions are represented as

$$h_1(W(t)) = \frac{\theta_2 - W(t)}{\theta_2 - \theta_1}, \quad h_2(W(t)) = 1 - h_1(W(t)) \tag{42}$$

Assume that only $i_q(t)$ *and* $W(t)$ *are measurable variables and the measurable output is the nonlinear function:*

$$\begin{cases} y_1(t) = 2(1+\partial)i_d(t) + 4i_q(t) + 4W(t) + i_q(t)W(t) - W(t)^2 + 0.1\omega(t) \\ y_2(t) = W(t) \end{cases}$$

with ∂ being an uncertain parameter, in this paper we assume $\partial = 0$, and $\theta_1 = -1$, $\theta_2 = 1$, then

$$C_{11} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

In [14], the robust H_∞ control is used to study the problem of SOF for uncertain systems. The methodologies of finding LMIs in [14] and our study are different. Hence, theoretically, it is difficult to make a comparison with this paper. However, we can draw some improvements. A linear matrix equality constraint is inserted on a Lyapunov matrix in [14]. On the other hand, the SOF condition in this paper is made independent of any transformation matrices and any equality constraint.

In order to find the optimal value of ξ such that the LMIs of Theorem 3.1 have a feasible solution, we use fminsearch. The conditions (10)-(12) in Theorem 3.1 were solved using LMI optimization toolbox in Matlab, with $\alpha = 0$, we obtain the optimal values of $\xi = 2.0995$, and the following SOF gain matrices:

$$F_1 = \begin{bmatrix} 0.7697 & -3.733 \\ 1.3640 & -6.6964 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.3536 & -0.9387 \\ 0.9579 & -2.7390 \end{bmatrix}$$

Suppose the initial state $x(0) = [1\ 0.5\ -0.5]^T$, the simulation results are shown in Figures 8-10. Figure 8 shows the state responses of the close-loop fuzzy system (5). Figures 9 and 10 illustrate the response of direct input voltage $\nu_d(t)$ and the quadrature input voltage $\nu_q(t)$ respectively. According to the simulation results, the proposed method stabilizes the permanent magnetic synchronous motor. This shows that the proposed method is effective to control the practical system.

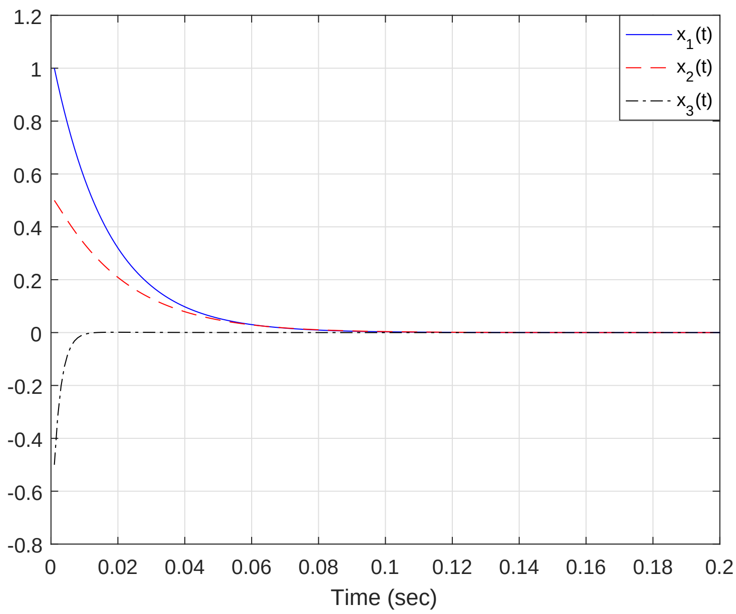

FIGURE 8. Trajectories for the PMSM Example 4.2

5. **Conclusion.** To sum up, we have presented in this paper the SOF controller design for the continuous-time T-S fuzzy system. The proposed results not only overcome the drawback of the iterative algorithms based on LMI approach but also can provide a less conservative design. Simulation examples have shown the effectiveness and merits of the proposed design method.

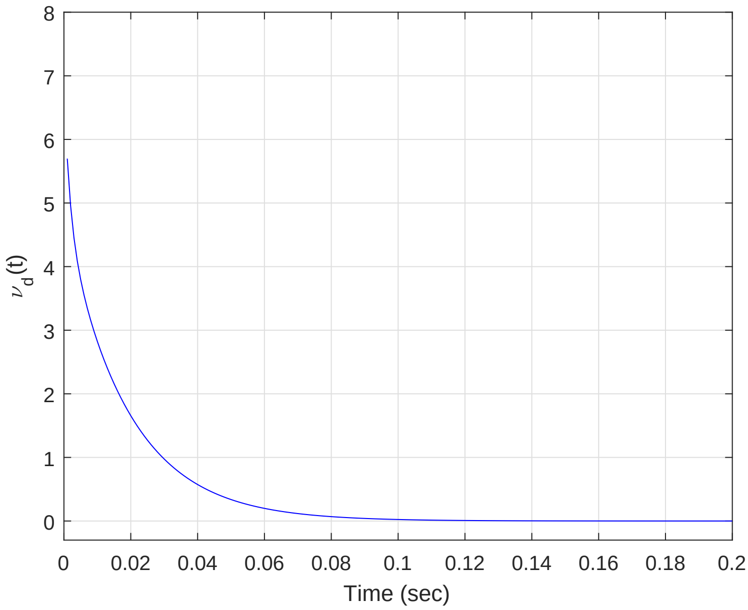

FIGURE 9. Trajectory of direct input voltage $\nu_d(t)$ Example 4.2

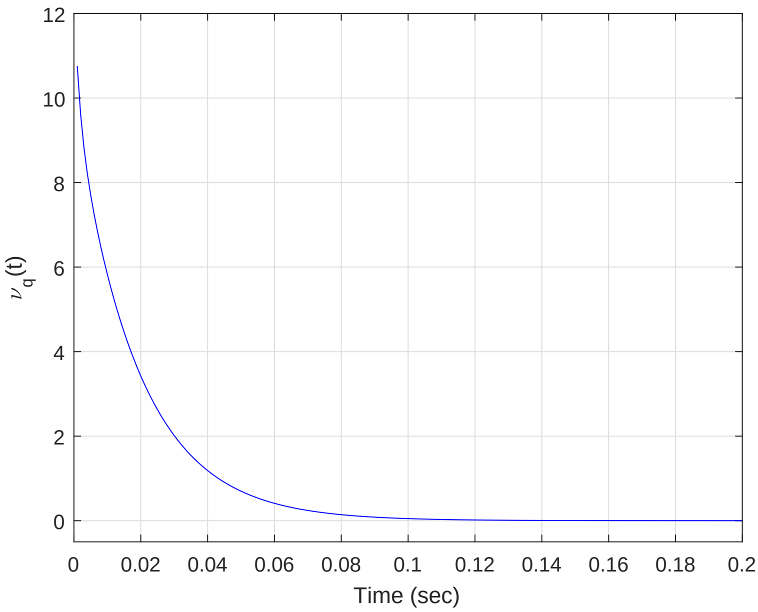

FIGURE 10. Trajectory of quadrature input voltage $\nu_q(t)$ Example 4.2

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