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BBM ALGORITHM FOR SIGNED MIXED DOMINATION IN CARTESIAN PRODUCT OF CYCLES

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Abstract. *To determine the signed mixed domination number is NP-complete. The branch and bound method (BBM) is most well-known for solving NP-hard problems. However, branch and bound conditions are very different for different graphs. In this paper, we develop some effective branch and bound conditions for $C_m \Box C_n$ on the signed mixed domination problem. Based on them we design an algorithm, named BBM algorithm, to construct signed mixed dominating functions. Upon these dominating functions, we obtain upper bounds on the signed mixed domination number of $C_m \Box C_n$.*
Keywords: Branch and bound method, Signed mixed domination, Cartesian product, Cycles

1. **Introduction.** The problem of dominating queens [1] can be said to be the origin of the study of domination in graphs. Figure 1 illustrates a standard 8×8 chessboard on which is placed a queen. According to the rules of chess a queen can, in one move, advance any number of squares horizontally, vertically, or diagonally. Thus, the queen in Figure 1 can dominate all of the squares marked with an "X". The problem is to determine the minimum number of queens that can be placed on a chessboard so that all squares are either dominated by a queen or are occupied by a queen.

There are many useful applications of domination in graphs. For example, to install base station that can receive and send signals for a city's communication network. We would like to locate stations in somewhere of the city so that signals can be transmitted to all over the city. Since each station has a limited transmitting range, we must use several stations to make signals reach everywhere of the city. However, since stations are costly, we want to locate as few as possible. There are many such applications in real life.

We first introduce some terminologies and concepts used in this paper. Let G be a finite connected simple graph with a vertex set $V(G)$ and an edge set $E(G)$. For $v \in V(G)$, the neighborhood of v, denoted by $N(v)$, is a set $\{u|(u,v) \in E(G)\}$. And the closed neighborhood of v, denoted by $N[v]$, is a set $N(v) \cup \{v\}$. For $e \in E(G)$, the neighborhood of e, denoted by $N(e)$, is a set $\{g|g \in E(G)$ is adjacent to $e\}$. And the closed neighborhood of e, denoted by $N[e]$, is a set $N(e) \cup \{e\}$. For an element $x \in V(G) \cup E(G)$, the total closed neighborhood of x, denoted by $N_T[x]$, is $N_T[x] = \{y|y$

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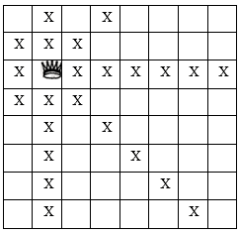

FIGURE 1. Dominating queens

is adjacent to x or y is incident with x, $y \in V(G) \cup E(G)\} \cup \{x\}$. If $v \in V$, then $E_G(v) = \{(u, v) \in E | u \in V\}$ is called the edge-neighborhood of v in G.

Signed domination and signed edge domination are introduced by Dunbar et al. [2] and Xu [3] respectively. Lv [4] extended the domain to $V(G) \cup E(G)$ and introduced total signed domination (which is also called signed mixed domination). A total signed dominating function of G is a function $f : V(G) \cup E(G) \to \{-1, 1\}$, such that $\sum_{y \in N_T[x]} f(y) \geq 1$ for each $x \in V(G) \cup E(G)$. The weight of f is $w(f) = \sum_{x \in V(G) \cup E(G)} f(x)$. The total signed domination number $\gamma_s^*(G)$ of G is the minimum weight of a total signed dominating function on G.

To determine the signed mixed domination number is NP-complete even for planar bipartite graphs [5]. However, signed domination of graphs has many applications in real life. In order to meet the needs of practical applications and promote the development of its theoretical research, scholars have studied the signed mixed domination number of many graphs, such as a complete graph with even order [6], a complete bipartite graph [7], and the Cartesian product of paths [8]. Many other variations of signed domination have also been studied, such as signed cycle domination number [9], twin signed total domination number [10], signed edge domination number [11], and signed Roman k-domination [12].

Since there are both vertices and edges in the neighborhood, signed mixed domination is more difficult than signed domination and signed edge domination. It is hard to give exact values or better bounds (which means there is a very small interval between upper and lower bounds) of the signed mixed domination number for graphs, especially for big graphs which have too many vertices and edges, such as the Cartesian product of two graphs.

In this paper, we use branch and bound method, develop some effective branch and bound conditions for $C_m \Box C_n$ on the signed mixed domination. Based on these branch and bound conditions we design an algorithm, named BBM algorithm, to construct signed mixed dominating functions. Upon these dominating functions, we obtain upper bounds on $\gamma_s^*(C_m \Box C_n)$. We also get lower bounds of $\gamma_s^*(C_m \Box C_n)$ by proof. Therefore, we present better bounds of $\gamma_s^*(C_m \Box C_n)$ for all integers $m, n \geq 3$.

The rest of this paper is organized as follows. In Section 2, we present some effective branch and bound conditions and design BBM algorithm to construct dominating functions. In Section 3, we describe the signed mixed dominating functions and present upper bounds on signed mixed domination number of $C_m \Box C_n$. In Section 4, we prove lower bounds on signed mixed domination number of $C_m \Box C_n$. The conclusions are given in Section 5.

2. BBM Algorithm.

To determine the signed mixed domination number is NP-complete, and the branch and bound method (BBM) is most well-known for solving NP-hard problems. However, branch and bound conditions are very different for different graphs.

For large graphs, if there is no valid branch and bound condition, then the huge amount of calculation will cause the computer to fail to work. $C_m \square C_n$ is large enough when m, n are bigger. Therefore, it is necessary to develop valid branch and bound conditions according to the speciality of $C_m \square C_n$. Table 1 shows branch and bound conditions we develop for $C_m \square C_n$ on signed mixed domination.

TABLE 1. Branch and bound conditions

for $C_m \square C_n$, $0 \le i \le n-1$, $0 \le j \le m-1$, mm_1 and nn_1 are the length and width of the loop body, mm and nn are the length and width of the part which can be looped, $(nn_1 \le i \le nn) \wedge (mm_1 \le j \le mm) \wedge (f(v_{i,j}) \ne f(v_{i-nn_1,j}) \vee f(v_{i,j}) \ne f(v_{i,j-mm_1}))$

Based on the branch and bound conditions in Table 1, we design BBM algorithm. The procedure of BBM algorithm is outlined in Algorithm 1.

Algorithm 1 BBM algorithm

Given a threshold *Smin* for signed mixed domination number;
Initialization: current position $l = 0$,
the consumed domination value $p_1 = 0$;
$l_0 : l++$; $jj = -3$;
$l_1 : jj+ = 2$; $jj > 1$; return l_2;
For current vertex, check whether it meets the following conditions:
if the consumed domination value $(p_1 + s[l]) \ge Smin$, **then**
  return l_1; $s[l]$ is the value on the l^{th} vertex, $s[l] = 1$ or $s[l] = -1$.
if current vertex cannot be dominated, **then**
  return l_1;
Query database of branch and bound conditions,
if meet the conditions, **then**
  return l_1;
$p_1 = p_1 + s[l]$; $jj_0[l] = jj$;
the current consumed domination value is stored in $jj_0[l]$.
if $l \le n_1$, **then**
  return l_0; n_1 is the number of vertexes
if the total consumed domination value is less than *Smin*, **then**
  replace *Smin* with the total consumed domination value;
for $(i = 1; i \le n_1; i++)$
  $f[i] = jj_0[i]$; f is the dominating function.
Endfor
$l++$;
$l_2 : l--$;
if $l \ge 1$, **then**
  $jj = jj_0[l]$;
  $j = f[l]$;
  $p_1 = p_1 - j$;
  return l_1;
end if
Output: *Smin* and the dominating function $f[1] \dots f[n_1]$.

3. **Upper Bounds on Signed Mixed Domination Number of Graph $C_m \square C_n$.** Using BBM algorithm (Algorithm 1), we can construct dominating functions. In this section, we describe some signed mixed dominating functions, and upon these functions we present upper bounds on the signed mixed domination number of $C_m \square C_n$ $(m, n \geq 3)$.

$C_m \square C_n$ means the Cartesian product of two cycles, C_m and C_n. Figure 2(a) shows the graph $C_4 \square C_3$. For clarity and convenience, we draw $C_4 \square C_3$ as Figure 2(b) and denote all of vertices and edges as $\{v_{0,0}, \ldots, v_{2,3}, e_{0,0}, \ldots, e_{2,3}, e'_{0,0}, \ldots, e'_{2,3}\}$.

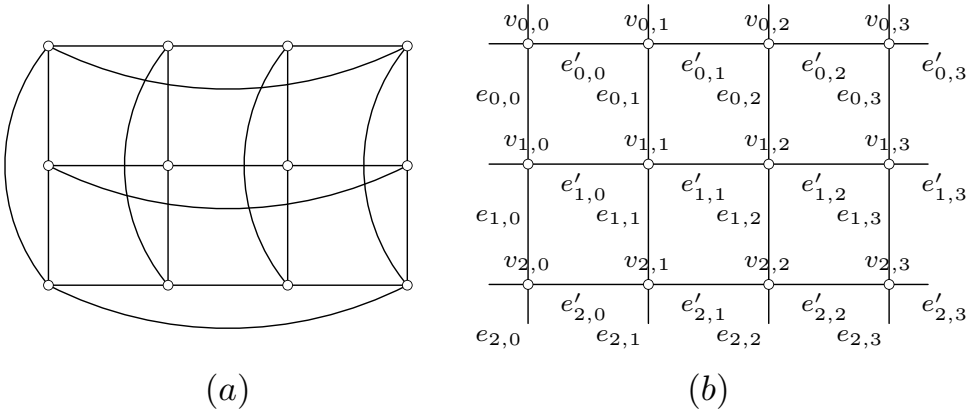

FIGURE 2. Graph $C_4 \square C_3$

Now, let $G = C_m \square C_n$ with $V(G) = \{v_{i,j} \mid 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$ and $E(G) = \{e_{i,j} \mid e_{i,j} = (v_{i,j}, v_{i+1,j}),\ 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\} \cup \{e'_{i,j} \mid e'_{i,j} = (v_{i,j}, v_{i+1,j}),\ 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$, where indices i and j are read modulo m and n respectively. Figure 3 shows the graph $C_m \square C_n$.

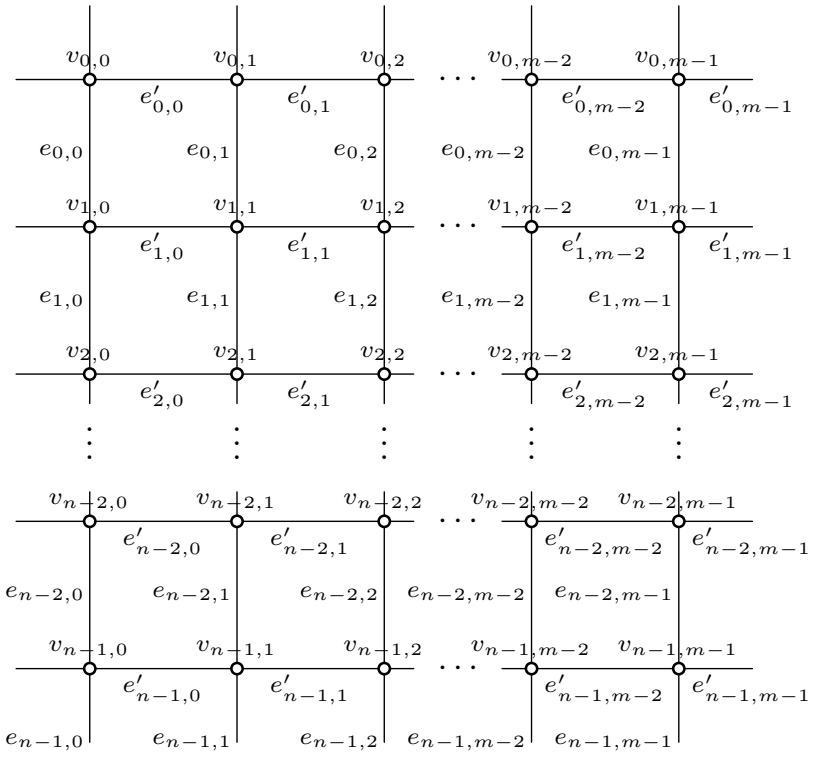

FIGURE 3. Graph $C_m \square C_n$

The following is the notation for signed mixed dominating function f:

$$f = \begin{pmatrix}
f(v_{0,0}) & f(e'_{0,0}) & f(v_{0,1}) & f(e'_{0,1}) & f(v_{0,2}) & \cdots & f(v_{0,m-1}) & f(e'_{0,m-1}) \\
f(e_{0,0}) & & f(e_{0,1}) & & f(e_{0,2}) & \cdots & f(e_{0,m-1}) & \\
f(v_{1,0}) & f(e'_{1,0}) & f(v_{1,1}) & f(e'_{1,1}) & f(v_{1,2}) & \cdots & f(v_{1,m-1}) & f(e'_{1,m-1}) \\
f(e_{1,0}) & & f(e_{1,1}) & & f(e_{1,2}) & \cdots & f(e_{1,m-1}) & \\
\vdots & & \vdots & & \vdots & & \vdots & \\
f(v_{n-2,0}) & f(e'_{n-2,0}) & f(v_{n-2,1}) & f(e'_{n-2,1}) & f(v_{n-2,2}) & \cdots & f(v_{n-2,m-1}) & f(e'_{n-2,m-1}) \\
f(e_{n-2,0}) & & f(e_{n-2,1}) & & f(e_{n-2,2}) & \cdots & f(e_{n-2,m-1}) & \\
f(v_{n-1,0}) & f(e'_{n-1,0}) & f(v_{n-1,1}) & f(e'_{n-1,1}) & f(v_{n-1,2}) & \cdots & f(v_{n-1,m-1}) & f(e'_{n-1,m-1}) \\
f(e_{n-1,0}) & & f(e_{n-1,1}) & & f(e_{n-1,2}) & \cdots & f(e_{n-1,m-1}) &
\end{pmatrix},$$

where $f(x) = 1$ or -1, $x \in V(G) \cup E(G)$.

For example, the following f is a signed mixed dominating function on $C_4 \square C_3$.

$$f = \begin{pmatrix}
-1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & & 1 & & 1 & & 1 & \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & 1 & & 1 & & 1 & \\
-1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & & 1 & & 1 & & 1 &
\end{pmatrix}.$$

The first line shows $f(v_{0,0}) = -1$, $f(e'_{0,0}) = 1$, $f(v_{0,1}) = -1$, $f(e'_{0,1}) = -1$, $f(v_{0,2}) = -1$, $f(e'_{0,2}) = 1$, $f(v_{0,3}) = -1$ and $f(e'_{0,3}) = -1$. The second line shows $f(e_{0,0}) = 1$, $f(e_{0,1}) = 1$, $f(e_{0,2}) = 1$ and $f(e_{0,3}) = 1$, so there are some null values. Figure 4 shows the above f and the corresponding graph $C_4 \square C_3$, where red vertex (edge) stands for $f(x) = 1$, and blue vertex (edge) stands for $f(x) = -1$.

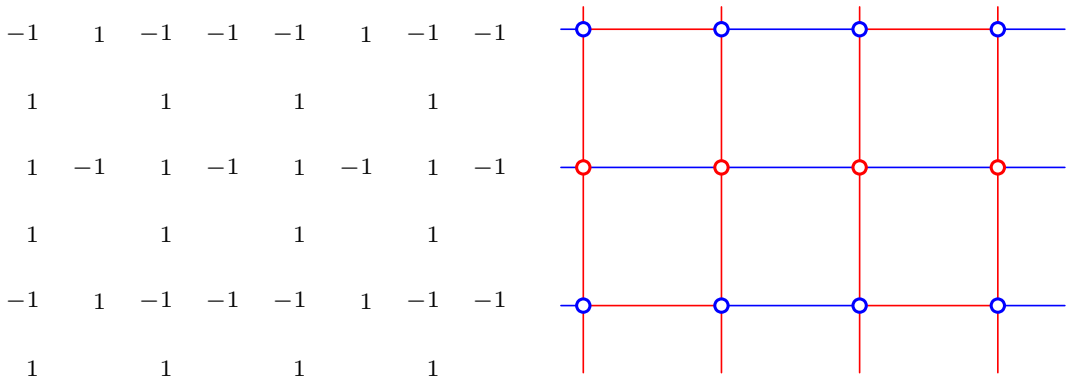

FIGURE 4. (color online) Dominating function f on $C_4 \square C_3$ and corresponding graph

Lemma 3.1. *For even $m \geq 4$ and even $n \geq 4$,*

$$\gamma_s^*(C_m \square C_n) \leq \frac{mn}{2}.$$

Proof: It is sufficient to define a function f with $w(f) = \frac{mn}{2}$. Let

$$f = \left(\begin{array}{cccc:c:cccc} -1 & 1 & -1 & -1 & \cdots & -1 & 1 & -1 & -1 \\ 1 & & 1 & & & 1 & & 1 & \\ 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ 1 & & 1 & & & 1 & & 1 & \\ \hdashline \vdots & & \vdots & & & \vdots & & \vdots & \\ \hdashline -1 & 1 & -1 & -1 & \cdots & -1 & 1 & -1 & -1 \\ 1 & & 1 & & & 1 & & 1 & \\ 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ 1 & & 1 & & & 1 & & 1 & \end{array}\right).$$

By adding all columns in each block, one can see the sum is 2, then $w(f) = 2\times\frac{m}{2}\times\frac{n}{2} = \frac{mn}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn}{2}$ for even $m \geq 4$ and even $n \geq 4$. Figure 5 shows the dominating function f on $C_4\Box C_4$ and the corresponding graph. □

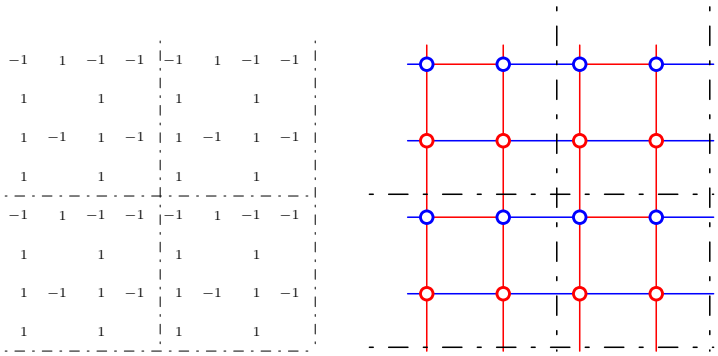

FIGURE 5. (color online) Dominating function f on $C_4\Box C_4$ and corresponding graph

Lemma 3.2. *For even $m \geq 4$ and odd $n \geq 3$,*

$$\gamma_s^*(C_m\Box C_n) \leq \begin{cases} \dfrac{mn}{2}, & m(\text{mod } 4) = 0 \ \wedge \ n(\text{mod } 2) = 1, \\ \dfrac{mn+2}{2}, & m(\text{mod } 4) = 2 \ \vee \ n(\text{mod } 2) = 1. \end{cases}$$

Proof:

Case 1. Suppose $m(\text{mod } 4) = 0$. It is sufficient to define a function f with $w(f) = \frac{mn}{2}$. Let

$$f = \left(\begin{array}{cccccccc:c:cccccccc} -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & \\ \hdashline \vdots & & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & \\ \hdashline -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & \\ \hdashline & & & & & & & & & & & & & & & & \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & \cdots & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & \end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 4 \times \frac{m}{4} \times \frac{n-1}{2} + 2 \times \frac{m}{4} = \frac{mn}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn}{2}$ for $m(\text{mod } 4) = 0$ and odd $n \geq 3$. Figure 6 shows the dominating function f on $C_8 \square C_5$ and the corresponding graph.

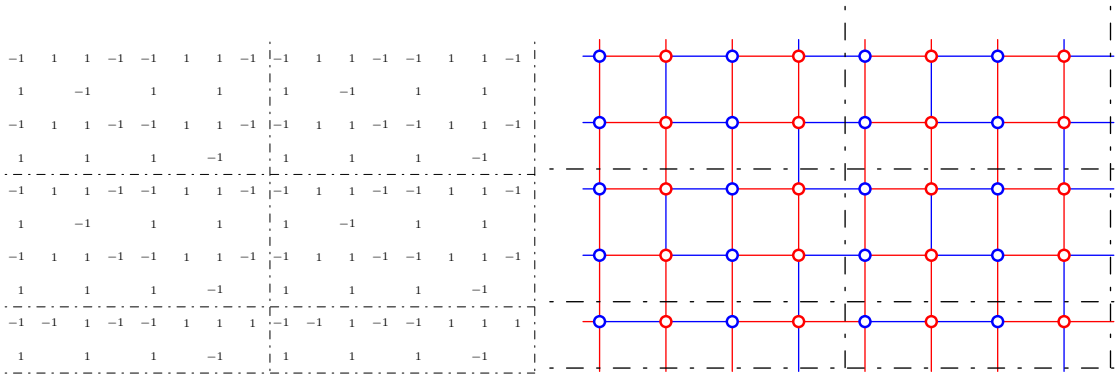

FIGURE 6. (color online) Dominating function f on $C_8 \square C_5$ and corresponding graph

Case 2. Suppose $m(\text{mod } 4) = 2$. It is sufficient to define a function f with $w(f) = \frac{mn+2}{2}$. Let

$$f = \left(\begin{array}{cccccccc:c:cccccccc:c:cccc}
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & & 1 & & 1 & \\
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\hdashline
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & & 1 & & 1 & \\
\hdashline
 & & & & & & & & & & & & & & & & & & & & \\
\hdashline
-1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & \cdots & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & & 1 & & 1 & & -1 & & & 1 & & 1 & & 1 & & -1 & & 1 & & 1 &
\end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 4 \times \frac{m-2}{4} \times \frac{n-1}{2} + 2 \times \frac{n-1}{2} + 2 \times \frac{m-2}{4} + 2 = \frac{mn+2}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn+2}{2}$ for $m(\text{mod } 4) = 2$ and odd $n \geq 3$. Figure 7 shows the dominating function f on $C_{10} \square C_5$ and the corresponding graph. □

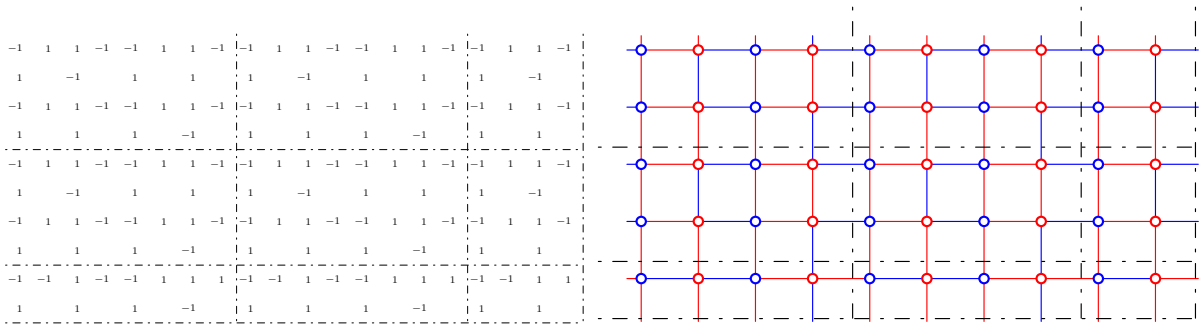

FIGURE 7. (color online) Dominating function f on $C_{10} \square C_5$ and corresponding graph

Lemma 3.3. *For odd $m \geq 3$ and odd $n \geq 3$,*

$$\gamma_s^*(C_m \Box C_n) \leq \begin{cases} \frac{mn+1}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n(\text{mod } 4) = 1, \\ \frac{3m-1}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n = 3, \\ \frac{mn+3}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n(\text{mod } 4) = 3 \ \wedge \ n \geq 7, \\ \frac{3m+1}{2}, & m(\text{mod } 4) = 3 \ \wedge \ n = 3, \\ \frac{mn+n-2}{2}, & m(\text{mod } 4) = 3 \ \wedge \ n(\text{mod } 4) = 3 \ \wedge \ n \geq 7. \end{cases}$$

Proof:

Case 1. Suppose $m(\text{mod } 4) = 1$ and $n(\text{mod } 4) = 1$. It is sufficient to define a function f with $w(f) = \frac{mn+1}{2}$. Let

$$f = \left(\begin{array}{cccccccc:c:cccccccc:c}
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 & \\
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\vdots & & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
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-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 & \\
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-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & 1 & & & 1 & & 1 & & -1 & & 1 & & 1 & \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & & 1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 &
\end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 8 \times \frac{m-1}{4} \times \frac{n-5}{4} + 2 \times \frac{n-5}{4} + 10 \times \frac{m-1}{4} + 3 = \frac{mn+1}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn+1}{2}$ for $m(\text{mod } 4) = 1$ and $n(\text{mod } 4) = 1$. Figure 8 shows the dominating function f on $C_9 \Box C_{13}$ and the corresponding graph.

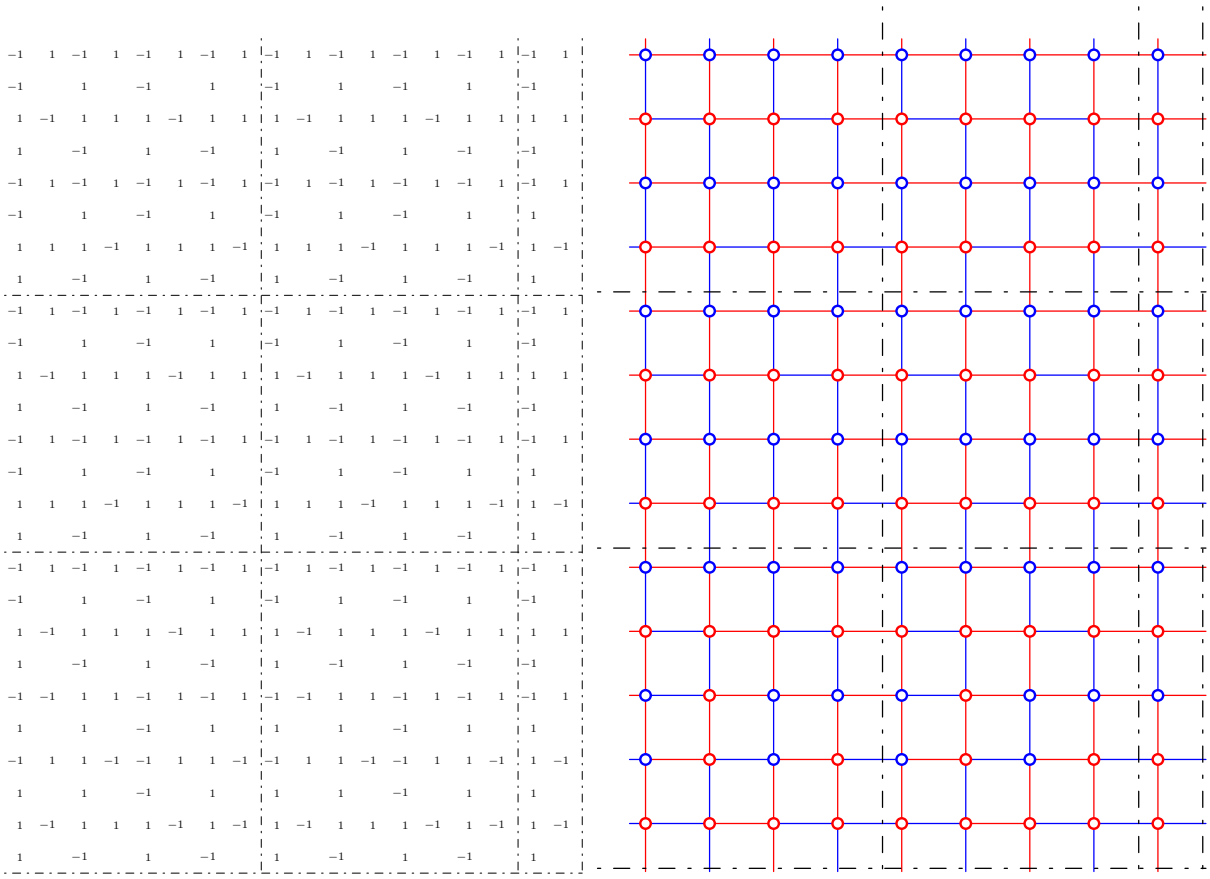

FIGURE 8. (color online) Dominating function f on $C_9 \square C_{13}$ and corresponding graph

Case 2. Suppose $m(\text{mod } 4) = 1$ and $n(\text{mod } 4) = 3$.

Case 2.1. Suppose $n = 3$. It is sufficient to define a function f with $w(f) = \frac{3m-1}{2}$. Let

$$f = \left(\begin{array}{cccccccccc|c|cccccccc|cc}
-1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & \cdots & & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & & -1 & & -1 & & -1 & & & & -1 & & -1 & & -1 & & -1 & & -1 & \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & \cdots & & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & & -1 & & 1 & & -1 & & & & 1 & & -1 & & 1 & & -1 & & 1 & \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & \cdots & & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & & 1 & & -1 & & 1 & & -1 & & 1 &
\end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 6 \times \frac{m-1}{4} + 1 = \frac{3m-1}{2}$. Hence, $\gamma_s^*(G) \leq \frac{3m-1}{2}$ for $m(\text{mod } 4) = 1$ and $n = 3$. Figure 9 shows the dominating function on $C_9 \square C_3$ and the corresponding graph.

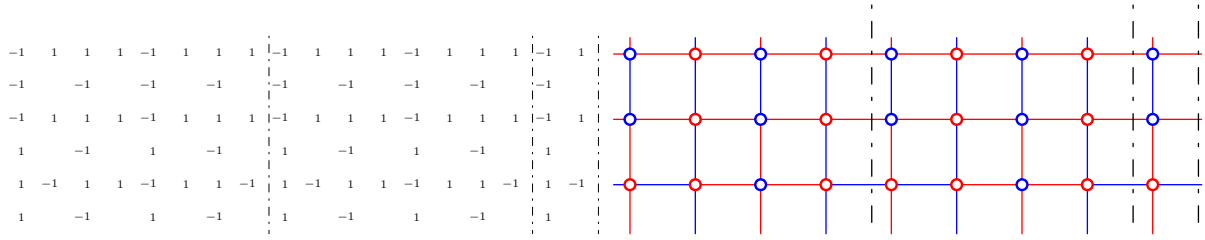

FIGURE 9. (color online) Dominating function f on $C_9 \square C_3$ and corresponding graph

Case 2.2. Suppose odd $n \geq 7$. It is sufficient to define a function f with $w(f) = \frac{mn+3}{2}$. Let

$$f = \left(\begin{array}{cccccccc:c:cccccccc:c}
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 & \\
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\vdots & & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\hdashline
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 & \\
\hdashline
 & & & & & & & & & & & & & & & & & & \\
\hdashline
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & 1 & & & 1 & & -1 & & 1 & & 1 & & 1 & \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & 1 & & 1 & & & 1 & & 1 & & 1 & & 1 & & -1 & \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & & 1 & & -1 & & 1 & & & -1 & & 1 & & -1 & & 1 & & -1 & \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & \cdots & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 &
\end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 8 \times \frac{m-1}{4} \times \frac{n-7}{4} + 2 \times \frac{n-7}{4} + 14 \times \frac{m-1}{4} + 5 = \frac{mn+3}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn+3}{2}$ for $m(\text{mod } 4) = 1$ and $n(\text{mod } 4) = 3$ and $n \geq 7$. Figure 10 shows the dominating function on $C_9 \Box C_{15}$ and the corresponding graph.

Case 3. Suppose $m(\text{mod } 4) = 3$ and $n(\text{mod } 4) = 3$.

Case 3.1. Suppose $n = 3$. It is sufficient to define a function f with $w(f) = \frac{3m+1}{2}$. Let

$$f = \left(\begin{array}{ccc:c:ccc:c:ccc}
-1 & 1 & 1 & \cdots & -1 & 1 & 1 & \cdots & -1 & 1 & -1 \\
1 & -1 & -1 & & 1 & -1 & -1 & & 1 & -1 & 1 \\
-1 & 1 & 1 & \cdots & -1 & 1 & 1 & \cdots & -1 & 1 & -1 \\
1 & -1 & 1 & & 1 & -1 & 1 & & 1 & -1 & 1 \\
1 & -1 & 1 & \cdots & 1 & -1 & 1 & \cdots & 1 & -1 & 1 \\
1 & -1 & 1 & & 1 & -1 & 1 & & 1 & -1 & 1
\end{array}\right).$$

By adding all columns in each block, one can see $w(f) = 6 \times \frac{m-3}{4} + 5 = \frac{3m+1}{2}$. Hence, $\gamma_s^*(G) \leq \frac{3m+1}{2}$ for $m(\text{mod } 4) = 3$ and $n = 3$. Figure 11 shows the dominating function on $C_{11} \Box C_3$ and the corresponding graph.

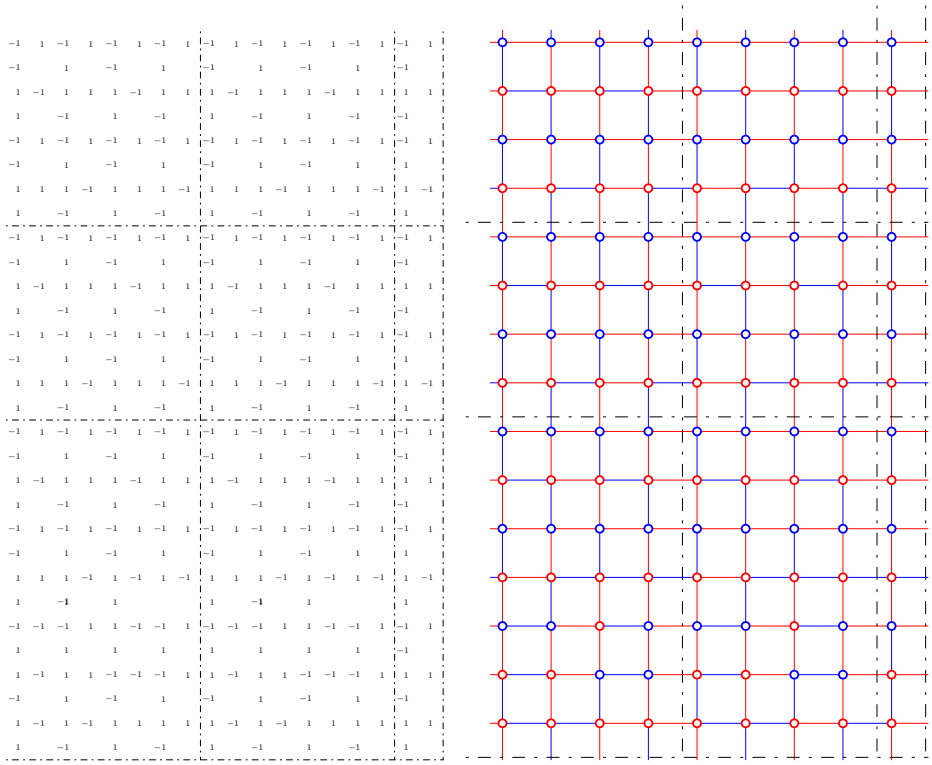

FIGURE 10. (color online) Dominating function f on $C_9 \square C_{15}$ and corresponding graph

Case 3.2. Suppose odd $n \geq 7$. It is sufficient to define a function f with $w(f) = \frac{mn+n-2}{2}$. Let

$$f = \left(\begin{array}{cccccc:cccccc:cccccc}
-1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & \\
1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & & -1 & & 1 & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & & -1 & & 1 & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & \\
\hdashline
\vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\hdashline
-1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & \\
1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & & -1 & & 1 & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & \\
-1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & & 1 & & -1 & & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & & -1 & & 1 & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & \\
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 & & & & & & & & & & & & & & & & & & \\
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-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & \cdots & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & & 1 & & -1 & & 1 & & & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & 1 & & & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & 1 & \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & -1 & & 1 & & -1 & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & \\
\end{array}\right).$$

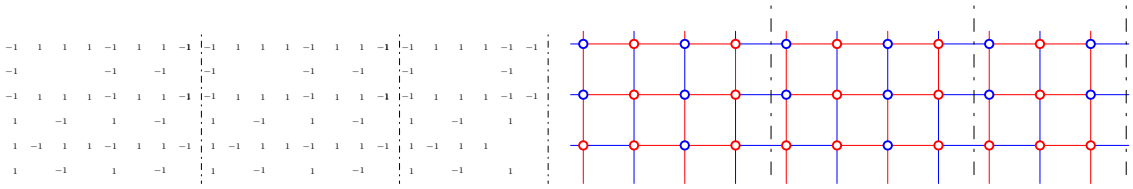

FIGURE 11. (color online) Dominating function f on $C_{11}\Box C_3$ and corresponding graph

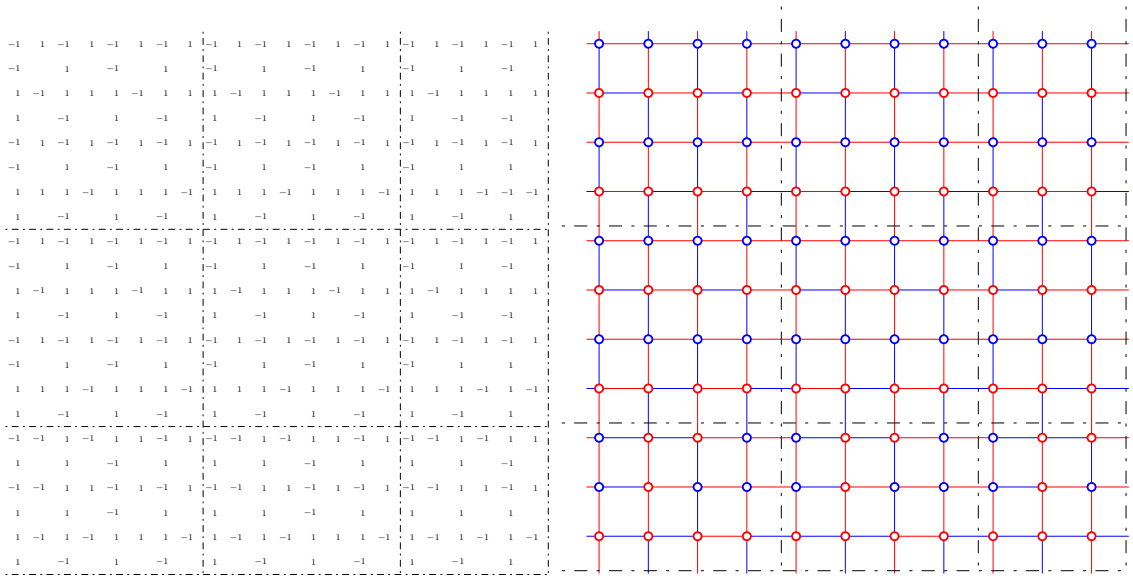

FIGURE 12. (color online) Dominating function f on $C_{11}\Box C_{11}$ and corresponding graph

By adding all columns in each block, one can see $w(f) = 8 \times \frac{m-3}{4} \times \frac{n-3}{4} + 8 \times \frac{n-3}{4} + 6 \times \frac{m-3}{4} + 5 = \frac{mn+n-2}{2}$. Hence, $\gamma_s^*(G) \leq \frac{mn+n-2}{2}$ for $m(\text{mod } 4) = 3$ and $n(\text{mod } 4) = 3$ and $n \geq 7$. Figure 12 shows the dominating function f on $C_{11}\Box C_{11}$ and the corresponding graph. □

By Lemma 3.1-Lemma 3.3, we have

Theorem 3.1. *For any integers $m, n \geq 3$,*

$$\gamma_s^*(C_m\Box C_n) \leq \begin{cases} \frac{mn}{2}, & m(\text{mod } 2) = 0 \ \wedge \ n(\text{mod } 2) = 0, \\ \frac{mn}{2}, & m(\text{mod } 4) = 0 \ \wedge \ n(\text{mod } 2) = 1, \\ \frac{mn+2}{2}, & m(\text{mod } 4) = 2 \ \wedge \ n(\text{mod } 2) = 1, \\ \frac{mn+1}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n(\text{mod } 4) = 1, \\ \frac{3m-1}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n = 3, \\ \frac{mn+3}{2}, & m(\text{mod } 4) = 1 \ \wedge \ n(\text{mod } 4) = 3 \ \wedge \ n \geq 7, \\ \frac{3m+1}{2}, & m(\text{mod } 4) = 3 \ \wedge \ n = 3, \\ \frac{mn+n-2}{2}, & m(\text{mod } 4) = 3 \ \wedge \ n(\text{mod } 4) = 3 \ \wedge \ n \geq 7, \end{cases}$$

that is,

$$\gamma_s^*(C_m \square C_n) \leq \frac{mn+n}{2}.$$

4. Lower Bounds on Signed Mixed Domination Number of Graph $C_m \square C_n$.

Lemma 4.1. *(Lv [4]) For any graph G, we have*

$$\gamma_s^*(G) \geq \left\lceil \frac{\delta(G) - \Delta(G) + 1}{\delta(G) + \Delta(G) + 1}(|E(G)| + |V(G)|) \right\rceil_{\mathcal{P}(|E(G)|+|V(G)|)}$$

and this bound is sharp, where $\mathcal{P}(s)$ is defined to be the parity of s, that is, $\mathcal{P}(s) = odd$ if s is odd and $\mathcal{P}(s) = even$ if s is even.

Theorem 4.1. *For any positive integers $m, n \geq 3$, we have*

$$\gamma_s^*(C_m \square C_n) \geq \left\lceil \frac{mn}{3} \right\rceil_{\mathcal{P}(3mn)}$$

and this bound is sharp, where $\mathcal{P}(s)$ is defined to be the parity of s, that is, $\mathcal{P}(s) = odd$ if s is odd and $\mathcal{P}(s) = even$ if s is even.

Proof: Let $G = C_m \square C_n$, then $\delta(G) = \Delta(G) = 4$, $|V(G)| = mn$ and $|E(G)| = 2mn$. Based on Lemma 4.1, we can obtain

$$\gamma_s^*(C_m \square C_n) \geq \left\lceil \frac{mn}{3} \right\rceil_{\mathcal{P}(3mn)}.$$

□

5. Conclusions.

By Theorems 3.1 and 4.1, we have

Theorem 5.1. *For any integers $m, n \geq 3$,*

$$\left\lceil \frac{mn}{3} \right\rceil_{\mathcal{P}(3mn)} \leq \gamma_s^*(C_m \square C_n) \leq \frac{mn+n}{2},$$

where $\mathcal{P}(s)$ is defined to be the parity of s, that is, $\mathcal{P}(s) = odd$ if s is odd and $\mathcal{P}(s) = even$ if s is even.

To determine signed domination number is NP-complete and signed domination of graphs has many applications in real life. In this paper, we develop valid branch and bound conditions, and design the BBM algorithm. Using the BBM algorithm, we construct signed mixed dominating functions. Upon these functions, we get upper bounds on $\gamma_s^*(C_m \square C_n)$. Based on upper bounds, together with lower bounds, we give better bounds on $\gamma_s^*(C_m \square C_n)$ for all integers $m, n \geq 3$. The research of this paper can not only enrich the results of signed domination on graphs, but also promote the application of signed domination, and it can be used to solve other NP-complete problems.

There are two directions in our future research. One is to study signed mixed domination on other graphs, such as circulant graphs, and generalized Petersen graphs. The other is to study other domination on graphs, such as Italian domination, and double Roman domination.

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