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CAPACITY OF CONTROL FOR STOCHASTIC DYNAMICAL SYSTEMS PERTURBED BY MIXED FRACTIONAL BROWNIAN MOTION WITH DELAY IN CONTROL

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Abstract. *In this paper, we discuss the relationships between capacity of control in entropy theory and intrinsic properties in control theory for a class of finite dimensional stochastic dynamical systems described by the linear stochastic differential equations driven by mixed fractional Brownian motion with delay in control. Stochastic dynamical systems can be described as an information channel between the space of control signals and the state space. We study this control to state information capacity of this channel in continuous time. We turned out that, the capacity of control depends on a time of final state in dynamical systems. By using the analysis and representation of fractional Gaussian process, the mathematical derivation of a closed form continuous optimal control law is derived. The reached optimal control law maximizes the mutual information between control signals and future state over a finite time horizon. The results obtained here are motivated by control to state information capacity for linear systems in both types, deterministic and stochastic models that are widely used to understand information flows in wireless network information theory. We discover some new relationships between control theory and entropy theoretic properties of stochastic dynamical systems with delay in control. Finally, we present two examples that serve to illustrate the relationships between capacity of control and intrinsic properties in control theory.*
Keywords: Controllability, Mutual information, Capacity of control, Stable, Optimal control, Stochastic control systems, Delay in control

1. **Introduction.** One of the fundamental results in the entropy theory is the capacity of control for communication system channels. The capacity of control or empowerment plays an important role in many applied fields in engineering, computer science, economy, physics, chemistry and other sciences.

Since Shannon in 1948 [1], information theoretic limits of capacity of channels have been studied extensively. The capacity of channels of single or multi input signals was introduced by the authors in [1-3] respectively. In [4], a limiting expression for the capacity regions of multi-access channels with memoryis was obtained.

In this work, we introduce an alternative view of some concepts in control theory such as reachability, stability and controllability by studying the relationship between the control process and the result states in terms of information theory.

Therefore, the stochastic control systems perturbed by mixed fractional Brownian motion with delay in control are considered as information channels between the random

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vectors (control and future output state). There are several definitions of capacity of control and some of them differ in formulation. Capacity of control or empowerment means the maximization of mutual information between the control process and output state over all input probability distributions of the channel of the control system. In a series of theoretical and practical studies such as [5-9], the suitability of capacity of control to implement a style controller was explored.

The researchers in [10] characterized the capacity region of a Gaussian multiple access channel with vector inputs and a vector output with or without inter symbol interference. Ranade and Sahai in [11] discussed the relationship between the stability and capacity of control for discrete linear dynamical system perturbed by Gaussian process and developed a notation of capacity of control that gives such a fundamental limit on the rate at which a controller can dissipate the uncertainty from a system. In [12], an optimal Gaussian controller is obtained for unstable dynamical systems by using the analysis of Gaussian process. In a recent important work Tiomkin et al. in [13] studied the problem of the control to state information capacity of stochastic dynamic systems driven by a Gaussian process with a single control in continuous time, when the states are observed only partially and also they derived an efficient solution for computing the control to state information channel. Our work focuses on studying a new relationship between capacity of control in entropy theory and intrinsic properties in control theory for linear systems with delay in control.

The contributions of this paper are as follows. We derive an efficient method for computing the information capacity between control process and final state of stochastic control systems driven by mixed fractional Brownian motion, and discover some properties which related between control theory and information theory. In particular, we demonstrate that the capacity of the control does not grow without limiting the length of control process. We show that the time delay contributes effectively to determination of capacity of control for stochastic control systems. Also, we indicate that the information capacity between control-state processes depends upon both final times and delay of this system.

The paper is organized as follows. Section 2 contains the mathematical model of stochastic control system which is described as an information channel between the space of the control process and the state space. In Section 3 the explicit formula for the mutual information between the resulting state and the control process is derived. We represent and analyze the control process in Section 4. In Section 5 the controllability is studied. In Section 6, we formulate the optimality conditions for control process. The relationship between capacity of control and final time is discussed in Section 7. In Section 8, two examples are presented to illustrate the relationship between capacity of control and the intrinsic properties of stochastic dynamic systems. Conclusion of this paper is shown in Section 9.

2. **System Description.**

In this paper, we use the following notations: Let $(\Omega, \Im_t, P)$ be a complete probability space with probability measure P on a simple space Ω defined on a filtration $\{\Im_t, t \in [0,T]\}$. $\Im_t$ is the σ-algebra generated by the random variables $\left\{W^H_{(s)}, W_{(s)}, s \in [0,t]\right\}$ and the P-null sets. Let $L^2(\Omega, \Im_t, R^n)$ denote the Hilbert space of all $\Im_t$ measurable square integrable random variables with values in R^n.

Definition 2.1. [14]*: Let h be a constant belonging to $(0,1)$. A one-dimensional fractional Brownian motion $W^h = \left\{W^h_{(t)}, t \geqslant 0\right\}$ of the Hurst index h is a continuous and centered Gaussian process with covariance function*

$$E\left(W^h_{(t)} W^h_{(s)}\right) = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right), \text{ for } t, s \geqslant 0.$$

- *If $h = \frac{1}{2}$, then the increments of W^h are non-correlated, and consequently independent. So W^h is a Wiener process which we denote further by W.*
- *If $h \in \left(\frac{1}{2}, 1\right)$ then the increments are positively correlated.*
- *If $h \in \left(0, \frac{1}{2}\right)$ then the increments are negatively correlated.*

The integral representation of W^h appears as:

$$W^h_{(t)} = \int_0^t K_h(t,s) dW_{(s)}$$

where W is a Wiener process and the kernel $K_h(t,s)$ defined as

$$K_h(t,s) = \mathrm{ch}\, s^{\frac{1}{2}-h} \int_s^t (u-s)^{h-\frac{3}{2}} u^{h-\frac{1}{2}} du$$

$$\frac{\partial K}{\partial t}(t,s) = \mathrm{ch} \left(\frac{t}{s}\right)^{h-\frac{1}{2}} (t-s)^{h-\frac{3}{2}}$$

where $\mathrm{ch} = \left[\frac{h(2h-1)}{\beta\left(2-2h, h-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$, *$t > s$ and β is a beta function.*

In this section, we consider the description of a channel for a linear communication system in the following form:

$$\left.\begin{aligned} \frac{d}{dt}x(t) = Ax(t) + B_1 u(t) + B_2 u(t-h) + G\frac{d}{dt}W(t) + \sigma_1 \frac{d}{dt} W^{H^1}(t) \\ y(t) = Dx(t) + \sigma_2 W^{H^2}(t),\ t \in [0,T] \\ x(0) = x_0 \\ u(t) = 0,\ t \in [-h, 0] \end{aligned}\right\} \tag{1}$$

where $x(t) \in C\left([0,T]; L^2\left(\Omega, \Im_t, R^n\right)\right)$ represents the state of stochastic control system with the dynamics matrix $A \in R^{n\times n}$ and x_0 is a centered Gaussian random variable defined on a probability space $(\Omega, \Im_0, P)$. The control process $u(.)$ is p-dimensional centered Gaussian process over times $t \in [0,T]$ distributed according to the density function $p(u(t))$, which is restricted to the space of Gaussian distributions and B_1, B_2 are constant matrices belonging to $R^{n\times p}$. $W(t)$, $t \in [0,T]$ is an n-dimensional standard Brownian motion (Wiener process) defined on $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, P)$ with scaling matrix $G \in R^{n\times n}$. The gain matrix $\sigma_1 \in R^{n\times n}$ scaling the white n-multivariate fractional Brownian motion $W^{H^1}(t)$, $t \in [0,T]$ with Hurst parameter $H^1 \in (0,1)^n$ defined in a complete probability space $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, P)$. $W^{H^2}(t)$, $t \in [0,T]$ is a p-multivariate fractional Brownian motion with Hurst parameter $H^2 \in (0,1)^p$ defined in a complete probability space $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, P)$ and the matrices D and σ_2 belong to $R^{p\times n}$.

Assume that the following conditions are satisfied

1) For any $t \in [0,T]$ the components w_k, $k = 1, 2, \ldots, n$ of process $W(t)$ are independent one to another and in the same condition with reference to the processes $W^{H^1}(t)$ and $W^{H^2}(t)$.
2) The control processes $u(t)$, $t \leqslant T-h$ and $u(t)$, $t > T-h$ are independent where h is the time of delay.

Remark 2.1. *An information channel which is given by the conditional probability distribution $p(y(T)|u(t), t \in [0,T])$ is induced by the stochastic dynamical system in (1) such that system input is perturbed by the noise which results from mixed fractional Brownian motion and the system output is perturbed by the noise of fractional Brownian motion.*

The linear stochastic dynamical system with initial conditions in (1) has the unique solution $x(t) \in L^2(\Omega, \Im_t, R^n)$ for any $t \in [0, T]$ see [2], which can be represented in the following integral equation:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)} \left[B_1 u(s) + B_2 u(s-h)\right] ds + \int_0^t e^{A(t-s)} G dW(s) \\ &\quad + \int_0^t e^{A(t-s)} \sigma_1 dW^{H^1}(s) \end{aligned} \tag{2}$$

For any $t \in [0, h]$, the solution of Equation (2) has the following form:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} B_1 u(s) ds + \int_0^t e^{A(t-s)} G dW(s) + \int_0^t e^{A(t-s)} \sigma_1 dW^{H^1}(t) \tag{3}$$

If we take $t > h$, then Equation (2) can be written as:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)} B_1 u(s) ds + \int_0^{t-h} e^{A(t-s-h)} B_2 u(s) ds + \int_0^t e^{A(t-s)} G dW(s) \\ &\quad + \int_0^t e^{A(t-s)} \sigma_1 dW^{H^1}(s) \end{aligned} \tag{4}$$

Assuming that $T > h$ then Equation (4) is equivalent to the integral equation:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^{t-h} \left[e^{A(t-s)} B_1 + e^{A(t-s-h)} B_2\right] u(s) ds + \int_{t-h}^t e^{A(t-s)} B_1 u(s) ds \\ &\quad + \int_0^t e^{A(t-s)} G dW(s) + \int_0^t e^{A(t-s)} \sigma_1 dW^{H^1}(s) \end{aligned} \tag{5}$$

3. **Mutual Information.** In this section, we formulate and show the explicit formula for the mutual information between the resulting state signal at the final time T and the control signals over time $t \in [0, T]$, by using some properties of entropy theory and fractional stochastic analysis theory. The mutual information between any two continuous random variables x and y is defined by the difference between the differential entropy of x and the conditional differential entropy of x with respect to y. In other words the mutual information between x and y is defined as:

$$I(x; y) = H(x) - H(x|y)$$

Mathematically, the mutual information between the stochastic processes $y(T)$ and $u(t)$, $t \geqslant 0$ is defined as:

$$I(y(T); u(t)) = H(y(T)) - H(y(T)|u(t)), \quad t \in [0, T] \tag{6}$$

Because the processes $u(t)$, $W(t)$ and $W^{H^1}(t)$ are independent for any $t \in [0, T]$ by assumptions and for any one of these processes is Gaussian, then Equation (5) shows that if the initial condition in (1) is the multivariate Gaussian random variable or deterministic vector then $x(t)$ is also multivariate Gaussian for any $t \in [0, T]$. The random vector $X \in R^n \sim N(0, S_x)$ with zero mean and covariance matrix S have the differential entropy of X in the following form [15]: $H(x) = \frac{1}{2} \ln(2\pi e)^n \det(S_x)$. Therefore, the differential entropy of multivariate Gaussian random vector depends only on its covariance matrix. Hence, the differential entropy of future observed state $y(T)$ at terminal time T is also depending on its covariance matrix $S_y(T)$.

$$H(y(T)) = \frac{1}{2} \ln(2\pi e)^n \det(S_y(T)) \tag{7}$$

Consequently, under the independence assumptions of processes $u(t)$, $W(t)$, $W^{H^1}(t)$ and $W^{H^2}(t)$ then the final state $x(T)$ is independent with multivariate fractional Brownian motion W^{H^2} at time T. Furthermore, from the control system in (1) the covariance matrix of $y(T)$ appears as:

$$S_y(T) = DS_x(T)D' + \sigma_2 S_{W^{H^2}}(T)\sigma_2' \tag{8}$$

Therefore, from Equation (5) and the independence assumptions of processes $u(t)$, $W(t)$ and $W^{H^1}(t)$ we get that, for every $t \in [0, T]$ with initial condition x_0 the covariance matrix of final state $x(T)$ is given in the following form:

$$S_x(T) = S_{x_0}(T) + S_u(T-h) + S_u(T) + S_{dW}(T) + S_{dW^{H^1}}(T) \tag{9}$$

where

$$\begin{aligned}
S_{x_0}(T) &= e^{AT}C_{x_0}e^{A'T} \\
S_u(T-h) &= \int_0^{T-h}\int_0^{T-h}\Big[e^{A(T-t_1)}B_1 + e^{A(T-t_1-h)}B_2\Big]C_u(t_1,t_2)\Big[e^{A(T-t_2)}B_1 \\
&\quad + e^{A(T-t_2-h)}B_2\Big]'dt_1dt_2 \\
S_u(T) &= \int_{T-h}^{T}\int_{T-h}^{T}\Big[e^{A(T-t_1)}B_1\Big]C_u(t_1,t_2)\Big[e^{A(T-t_2)}B_1\Big]'dt_1dt_2 \\
S_{dW}(T) &= \int_0^T\int_0^T e^{A(T-t_1)}GC_{dW}(t_1,t_2)G'e^{A'(T-t_2)}dt_1dt_2 \\
S_{dW^{H^1}}(T) &= \int_0^T\int_0^T e^{A(T-t_1)}\sigma_1 C_{dW^{H^1}}(t_1,t_2)\sigma_1' e^{A'(T-t_2)}dt_1dt_2
\end{aligned}$$

wherein $C_X(t_1, t_2)$ represents the covariance matrix for any two states $x(t_1)$ and $x(t_2)$.

Lemma 3.1. [16]*: Let $V[0,T]$ be the class of functions such that $f : [0,T] \times \Omega \to R$, f is measurable, $\Im_t$-adapted and $E\left[\int_0^T (f(t,\omega))^2 dt\right] \leqslant \infty$. Then for every $f \in V[0,T]$*

$$E\left[\int_0^T f(t,\omega)dw(t)\right]^2 = E\left[\int_0^T (f(t,\omega))^2 dt\right] \tag{10}$$

where $w(t)$ is a standard Wiener process. Consequently, the covariance matrix $S_{dW}(T)$ can be written as

$$S_{dW}(T) = \int_0^T\int_0^T e^{A(T-t_1)}GG'e^{A'(T-t_2)}dt_1dt_2 \tag{11}$$

Lemma 3.2. [14]*: Let $\frac{1}{2} < h \leqslant 1$ then for any functions Φ, $\varphi \in L^2[0,T] \cap L^1[0,T]$, we have*

(i) $E\left(\int_0^T \Phi(t)dw^h_{(t)}\int_0^T \varphi(s)dw^h_{(s)}\right) = h(2h-1)\int_0^T \cdot \int_0^T \Phi(t)\varphi(s)|t-s|^{2h-2}dsdt$

(ii) $E\left(dw^h_{(t)}dw^h_{(s)}\right) = h(2h-1)|t-s|^{2h-2}dsdt$

Therefore, by using above lemma and the independence assumptions of processes $w^{h_k}_{H^1}(t)$, $k = 1, 2, \ldots, n$ for any $t \in [0, T]$, we obtain:

$$S_{dW^{H^1}}(T) = \int_0^T\int_0^T e^{A(T-t_1)}\sigma_1 R(t_1,t_2)\sigma_1' e^{A'(T-t_2)}dt_1dt_2 \tag{12}$$

where $R(t_1, t_2)$ is an $n \times n$ diagonal matrix whose kk-th entry is specified by the parameter h_k

$$r_{kr}(t_1, t_2) = \begin{cases} h_k(2h_k - 1)|t_1 - t_2|^{2h_k - 2}, & \text{for } r = k \\ 0, & \text{for } r \neq k \end{cases}$$

The covariance function of the input signal is defined as $C_u(t_1, t_2) = E\left[u(t_1)u'(t_2)\right]$.

Now, to find an explicit formula of the covariance matrix of $W^{H^2}(T)$. By the independence assumptions of the components $w_{H^2}^{\tilde{h}_k}(t)$, $k = 1, 2, \ldots, p$ for any $t \in [0, T]$ then the covariance matrix $S_{W^{H^2}}(T)$ of $W^{H^2}(T)$ is diagonal, whose kk-th entry is specified by the parameters $\tilde{h}_k$ of the multivariate fractional process W^{H^2} at final time T appearing as follows

$$\left[S_{W^{H^2}}(T)\right]_{kk} = T^{2\tilde{h}_k}, \quad k = 1, 2, \ldots, p \tag{13}$$

Consequently, the covariance matrix of output state $y(T)$ is given as

$$\begin{aligned} S_y(T) &= DS_{x_0}(T)D' + DS_u(T-h)D' + DS_u(T)D' + DS_{dW}(T)D' \\ &+ DS_{dW^{H^1}}(T)D' + \sigma_2 \begin{bmatrix} T^{2\tilde{h}_1} & & 0 \\ & \ddots & \\ 0 & & T^{2\tilde{h}_P} \end{bmatrix}_{P\times P} \sigma_2' \end{aligned} \tag{14}$$

By using the independence assumption of control process $u(t)$, $t \leqslant T - h$ with the last h of time, we study the mutual information between $y(T)$ and the following control signal $\{u(t), t \leqslant T\}$.

Firstly, we need to compute the differential entropy of a random vector $y(T)$ conditional on the control process $u(t)$, $t \in [0, T]$. Denote the total covariance of uncontrolled noise which it comes from the power of mixed fractional Brownian motion by S_{Total}. Under the independence assumption of $W(t)$, $W^{H^1}(t)$, $W^{H^2}(t)$ and the initial condition x_0 then the total covariance is the following form:

$$\begin{aligned} S_{Total}(T) &= DS_{x_0}(T)D' + DS_{dW}(T)D' + DS_{dW^{H^1}}(T)D' \\ &+ \sigma_2 \begin{bmatrix} T^{2\tilde{h}_1} & & 0 \\ & \ddots & \\ 0 & & T^{2\tilde{h}_P} \end{bmatrix}_{P\times P} \sigma_2' \end{aligned} \tag{15}$$

The covariance of the future observed state $y(T)$ given the control process $u(t)$, $t \in [0, T]$ is also the total covariance of uncontrolled noise with initial random vector x_0

$$S_{Total}(T) = S_{(y|u(t), t\in[0,T])}(T) \tag{16}$$

Hence,

$$H(y(T)|u(t), t \in [0, T]) = \frac{1}{2}\ln(2\pi e)^n \det(S_{Total}(T)) \tag{17}$$

Therefore, the mutual information between $y(T)$ and the control signal $u(t)$, $t \leqslant T$ can be written as

$$I(y(T); u(t), t \in [0, T]) = \frac{1}{2}\ln\left\{\det\left(I_{n\times n} + S_{Total}^{-1}(T)(S_u(T-h) + S_u(T))\right)\right\} \tag{18}$$

4. **Representation and Analysis of Control Process.** In this section we represent and analyze the control process by using orthonormal expansion of Gaussian process.

Lemma 4.1. ***(Karhunen-Loeve Theorem)*** [17]*: Let $x(t)$, $t \in [a, b]$ be a central square integrable stochastic process defined over a probability space $(\Omega, \Im, P)$ with continuous covariance function C_x then $x(t) = \sum_{j=1}^{\infty} z_j e_j(t)$, where e_j, $j = 1, 2, \ldots$ is a set of orthonormal basis on $L^2([a, b])$ and the random variables z_j, $j = 1, 2, \ldots$ are independent and have* *zero mean. From this above lemma for any component $u_k(t)$, $k = 1, 2, \dots, p$ of the control signal $u(t) \in L^2\left(\Omega, \Im_t, L^2\left(0, T-h, R^P\right)\right)$ may be represented by appropriate choice of $\{e_{j,k}(t)\}_{j=1}^{\infty}$, such that the set of functions $\{e_{j,k}(t)\}_{j=1}^{\infty}$ is countable of real orthonormal functions in $L^2\left([0, T-h]\right)$ and $u_{j,k}$, $j = 1, 2, \dots$ is a sequence of independent Gaussian random variables as the following*

$$u_k(t) = \sum_{j=1}^{\infty} u_{j,k} e_{j,k}(t) \tag{19}$$

Also, for any component $u_k(t)$, $k = 1, 2, \dots, p$ of the control signal $u(t) \in L^2\big(\Omega, \Im_t, L^2\big(T-h, T, R^P\big)\big)$ can be written as:

$$u_k(t) = \sum_{j=1}^{\infty} \hat{u}_{j,k} \theta_{j,k}(t) \tag{20}$$

wherein $\{\theta_{j,k}(t)\}_{j=1}^{\infty}$ is countable of real orthonormal functions in $L^2(T-h, T, R^P)$ and $\hat{u}_{j,k}$, $j = 1, 2, \dots$ is a sequence of independent Gaussian random variables. Now, for any $t \in [0, T-h]$ the p-dimensional control process can be represented as follows:

$$u(t) = \begin{bmatrix} \sum_{j=1}^{\infty} u_{j,1} e_{j,1}(t) \\ \sum_{j=1}^{\infty} u_{j,2} e_{j,2}(t) \\ \vdots \\ \sum_{j=1}^{\infty} u_{j,p} e_{j,p}(t) \end{bmatrix} \tag{21}$$

where $\{e_{j,k}(t)\}_{j=1}^{\infty}$ is countable of real orthonormal functions in $L^2(0, T-h, R^P)$. By independence assumption of components of control process $u(t)$ then its covariance function is diagonal, whose kk-th entry is specified by the parameters σ_{jk} of the control process as follows

$$[C_u(t_1, t_2)]_{kk} = \sum_{j=1}^{\infty} \sigma_{jk} e_{j,k}(t_1) e_{j,k}(t_2), \quad k = 1, 2, \dots \tag{22}$$

Also, for any $t \in [T-h, T]$ the p-dimensional control process can be represented as:

$$u(t) = \begin{bmatrix} \sum_{j=1}^{\infty} \hat{u}_{j,1} \theta_{j,1}(t) \\ \sum_{j=1}^{\infty} \hat{u}_{j,2} \theta_{j,2}(t) \\ \vdots \\ \sum_{j=1}^{\infty} \hat{u}_{j,p} \theta_{j,p}(t) \end{bmatrix} \tag{23}$$

and

$$[C_u(t_1, t_2)]_{kk} = \sum_{j=1}^{\infty} \omega_{jk} \theta_{j,k}(t_1) \theta_{j,k}(t_2), \quad k = 1, 2, \dots$$

Assume that $B = B_1 + e^{-Ah} B_2$ then the covariance of $u(t)$, $t \leqslant T-h$ can be written as

$$S_u(T-h) = \int_0^{T-h} \int_0^{T-h} e^{A(T-t_1)} \sum_{k=1}^{p} \sum_{j=1}^{\infty} b_k \sigma_{jk} e_{j,k}(t_1) e_{j,k}(t_2) b_k' e^{A'(T-t_2)} dt_1 dt_2 \tag{24}$$

where $b_k = B[\colon, k]$ is a k-th column of a matrix B.

Similarity,

$$S_u(T) = \int_{T-h}^{T} \int_{T-h}^{T} e^{A(T-t_1)} \sum_{k=1}^{p} \sum_{j=1}^{\infty} b_{1^k} \omega_{jk} \theta_{j,k}(t_1) \theta_{j,k}(t_2) b_{1^k}' e^{A'(T-t_2)} dt_1 dt_2 \tag{25}$$

where $b_{1^k} = B_1[\colon, k]$ is a k-th column of matrix B_1. By using the assumption that $u(t) \in L^2\left(\Omega, \Im_t, L^2\left(0, T, R^P\right)\right)$, then the power constraints give

$$\int_0^{T-h} ||u(t)||^2_{L^2(\Omega,\Im_t,L^2(0,T,R^P))} dt \leqslant M_1 \tag{26}$$

$$\int_{T-h}^{T} ||u(t)||^2_{L^2(\Omega,\Im_t,L^2(0,T,R^P))} dt \leqslant M_2 \tag{27}$$

such that M_1 and M_2 are constants. This impels

$$Tr\left\{\int_0^{T-h} C_u(t,t)dt\right\} = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \int_0^{T-h} \sigma_{jk} e_{j,k}(t) e_{j,k}(t) dt = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \sigma_{jk} \leqslant M_1 \tag{28}$$

$$Tr\left\{\int_{T-h}^{T} C_u(t,t)dt\right\} = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \int_{T-h}^{T} \omega_{jk} \theta_{j,k}(t) \theta_{j,k}(t) dt = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \omega_{jk} \leqslant M_2 \tag{29}$$

5. **Controllability.** In this section, we study the concept of controllability of the dynamical system in (1) by using the properties of deterministic linear system with delay in control and controllable matrix. Consider the deterministic linear system with delay control as

$$\left.\begin{aligned} \frac{d}{dt}x(t) = Ax(t) + B_1 u(t) + B_2 u(t-h),\ t \in [0,T] \\ x(0) = x_0 \\ u(t) = 0,\ t \in [-h, 0] \end{aligned}\right\} \tag{30}$$

Define the linear control operator $\mathcal{L}_0^T u$: $L^2\left(\Omega, \Im_t, L^2\left(0, T, R^P\right)\right) \to L^2\big(\Omega, \Im_t, L^2\big(0, T, R^n\big)\big)$ as

$$\mathcal{L}_0^T u = \int_0^{T-h} \left[e^{A(T-s)} B_1 + e^{A(T-s-h)} B_2\right] u(s) ds + \int_{T-h}^{T} e^{A(T-s)} B_1 u(s) ds$$

It is clear that $\mathcal{L}_0^T u$ is bounded. The adjoint operator $\left(\mathcal{L}_0^T u\right)^*$: $L^2\left(\Omega, \Im_t, L^2\left(0, T, R^n\right)\right) \to L^2\left(\Omega, \Im_t, L^2\left(0, T, R^p\right)\right)$ is defined by

$$\left(\mathcal{L}_0^T\right)^* z = \begin{cases} \left(e^{A(T-t)} B\right)' E(z), & t \in [0, T-h] \\ \left(e^{A(t-t)} B_1\right)' E(z), & t \in [T-h, T] \end{cases}$$

Also, to define the controllability operator G_T associated with control operator $\mathcal{L}_0^T$ in the deterministic case of Equation (30) as [18]

$$G_T = \mathcal{L}_0^T \left(\mathcal{L}_0^T\right)^* = \int_0^{T-h} e^{A(T-t)} B B' e^{A'(T-t)} dt + \int_{T-h}^{T} e^{A(T-t)} B_1 B_1' e^{A'(T-t)} dt \tag{31}$$

Lemma 5.1. [19-21]*: The following conditions are equivalent*

i) The deterministic control system in (30) is controllable on $[0,T]$*.*

ii) The stochastic control system in (1) is controllable on $[0,T]$*.*

iii) The controllability matrix G_T in (31) is nonsingular.
iv) The matrix $[B_1, B_2, AB_1, AB_2, \ldots, A^{n-1}B_1, A^{n-1}B_2]$ is full rank.

Suppose that the following notations are used in this paper

$$g_{A,b_k}(t) = e^{A(T-t)}b_k \in R^n \tag{32}$$

$$g_{A,b_{1^k}}(t) = e^{A(T-t)}b_{1^k} \in R^n \tag{33}$$

$$v_{ik}(T) = \int_0^{T-h} (g_{A,b_k}(t)e_{j,k}(t))\,dt \tag{34}$$

$$z_{ik}(T) = \int_{T-h}^{T} g_{A,b_{1^k}}(t)\theta_{j,k}(t)dt \tag{35}$$

Hence, the covariance matrices $S_u(T)$, $S_u(T-h)$ and controllability matrix can be written as

$$S_u(T) = \sum_{k=1}^{p}\sum_{j=1}^{\infty} \omega_{jk} z_{ik}(T)z'_{ik}(T) \tag{36}$$

$$S_u(T-h) = \sum_{k=1}^{p}\sum_{j=1}^{\infty} \sigma_{jk} v_{ik}(T)v'_{ik}(T) \tag{37}$$

$$G_T(b_k, b_{1^k}) = \int_0^{T-h} g_{A,b_k}(t)g'_{A,b_k}(t)dt + \int_{T-h}^{T} g_{A,b_{1^k}}(t)g'_{A,b_{1^k}}(t)dt \in R^{n\times n} \tag{38}$$

Assume that the deterministic linear control system corresponding to stochastic control system in (1) is controllable. This means that the matrix $G_T(b_k, b_{1^k})$ is positive definite for any $k = 1, 2, \ldots, p$.

In the next section we shall formulate and prove the conditions for optimal control in other word we find the expression of control signal which maximizes the mutual information between control signals and future observe stat for the stochastic dynamic system in (1).

6. **Optimality.**

Consider the following constraint optimization problem as

$$\left.\begin{array}{l}
\max\limits_{p(u(t),t\in[0,T])} I(y(T); u(t),\, t\in[0,T]) \\
\text{subject to} \\
\dfrac{d}{dt}x(t) = Ax(t) + B_1u(t) + B_2u(t-h) + G\dfrac{d}{dt}W(t) + \sigma_1\dfrac{d}{dt}W^{H^1}(t) \\
y(t) = Dx(t) + \sigma_2 W^{H^2}(t),\ t\in[0,T] \\
x(0) = x_0 \\
u(t) = 0,\ t\in[-h,0]
\end{array}\right\} \tag{39}$$

By using the representation of control process in Section 4, Equation (18) can be written in the following form:

$$\begin{aligned}
&I(y(T); u(t), t\in[0,T]) \\
&= \frac{1}{2}\ln\left(\det\left(I_{n\times n} + S^{-1}_{Total}(T)\left\{\begin{array}{c} \sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\sigma_{jk}Dv_{ik}(T)v'_{ik}(T)D' \\ +\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\omega_{jk}Dz_{ik}(T)z'_{ik}(T)D'\end{array}\right\}\right)\right)
\end{aligned} \tag{40}$$

The constraint conditions (28) and (29) with orthonormality conditions imply to a new expression of optimization problem in (39), which gives

$$\left.\begin{aligned}
&\max_{\sigma,\omega,\left\{\theta_{j,k}(t)\right\}_{jk},\left\{e_{j,k}(t)\right\}_{jk}} \frac{1}{2}\ln\left(\det\left(I_{n\times n}\right.\right.\\
&\left.\left.+\,S_{Total}^{-1}(T)\left\{\begin{array}{l}\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk}Dv_{ik}(T)v'_{ik}(T)D'\\ +\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk}Dz_{ik}(T)z'_{ik}(T)D'\end{array}\right\}\right)\right)\\
&\text{subject to}\\
&\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk}=M_1\\
&\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk}=M_2\\
&\sigma_{jk}\geqslant 0\\
&\omega_{jk}\geqslant 0\\
&\int_0^{T-h}e_{j,k}(t)e_{i,k}(t)dt=\delta_{ji}\\
&\int_{T-h}^{T}\theta_{j,k}(t)\theta_{i,k}(t)dt=\delta_{ji}
\end{aligned}\right\}\tag{41}$$

where,

$$\delta_{ji}=\left\{\begin{array}{ll}0, & \text{if } i=j\\ 1, & \text{if } i\neq j\end{array}\right., \ \sigma=\{\sigma_{jk}\}_{jk} \text{ and } \omega=\{\omega_{jk}\}_{jk}$$

Therefore, the Lagrange function can be written as

$$\begin{aligned}
L=&\frac{1}{2}\ln\left(\det\left(I_{n\times n}+S_{Total}^{-1}(T)\left\{\begin{array}{l}\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk}Dv_{ik}(T)v'_{ik}(T)D'\\ +\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk}Dz_{ik}(T)z'_{ik}(T)D'\end{array}\right\}\right)\right)\\
&-\gamma_1\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk}-M_1\right)-\gamma_2\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk}-M_2\right)-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\lambda_{jk}\sigma_{jk}\\
&-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\mu_{jk}\omega_{jk}-\sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\beta_{k,j,i}\left(\int_0^{T-h}e_{j,k}(t)e_{i,k}(t)dt-\delta_{ji}\right)\\
&-\sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\alpha_{k,j,i}\left(\int_{T-h}^{T}\theta_{j,k}(t)\theta_{i,k}(t)dt-\delta_{ji}\right)
\end{aligned}\tag{42}$$

where γ_1, γ_2, λ_{jk}, μ_{jk}, $\beta_{k,j,i}$ and $\alpha_{k,j,i}$ are the Lagrange multipliers.

Consequently, the corresponding KKT optimality conditions of optimization problem in (41) are in the following form, for any $t\in[0,T]$, $k=1,2,\ldots,p$ and $j,i=1,2,\ldots$

$$\text{(i)} \quad \left.\begin{aligned} \frac{\delta L}{\delta e_{j,k}}(t) &= 0 \\ \frac{\delta L}{\delta \theta_{j,k}}(t) &= 0 \\ \frac{\partial L}{\partial \sigma_{jk}} &= 0 \\ \frac{\partial L}{\partial \omega_{jk}} &= 0 \end{aligned}\right\} \tag{43}$$

$$\text{(ii)} \quad \left.\begin{aligned} \sum_{k=1}^{p}\sum_{j=1}^{\infty} \sigma_{jk} - M_1 &= 0 \\ \sum_{k=1}^{p}\sum_{j=1}^{\infty} \omega_{jk} - M_2 &= 0 \\ \sigma_{jk} &\geqslant 0 \\ \omega_{jk} &\geqslant 0 \\ \int_0^T e_{j,k}(t) e_{i,k}(t) dt - \delta_{ji} &= 0 \\ \int_{T-h}^T \theta_{j,k}(t)\theta_{i,k}(t) dt - \delta_{ji} &= 0 \end{aligned}\right\} \tag{44}$$

$$\text{(iii)} \quad \left.\begin{aligned} \lambda_{jk}\sigma_{jk} &= 0 \\ \mu_{jk}\omega_{jk} &= 0 \end{aligned}\right\} \tag{45}$$

$$\text{(iv)} \quad \left.\begin{aligned} \lambda_{jk} &\geqslant 0 \\ \mu_{jk} &\geqslant 0 \end{aligned}\right\} \tag{46}$$

Clearly, the nonlinear optimization problem in (41) is a convex with respect to $\{\sigma_{jk}\}_{jk}$ and $\{\omega_{jk}\}_{jk}$ for a given set of the expansion functions $\{e_{j,k}(t)\}_{jk}$ and $\{\theta_{j,k}(t)\}_{jk}$. The partial control signal without the rm-th control component for Gaussian process representation in Equations (19) and (20) is defined as

$$\left.\begin{aligned} u_{\widetilde{rm}}(t) &= \sum_{jk \neq rm} u_{j,k} e_{j,k}(t), \quad t \in [0, T-h] \\ u_{\widetilde{rm}}(t) &= \sum_{jk \neq rm} \hat{u}_{j,k}\theta_{j,k}(t), \quad t \in (T-h, T] \end{aligned}\right\} \tag{47}$$

Therefore, the covariance matrices of control signals in (24) and (25) can be written as

$$\left.\begin{aligned} S_u(T-h) &= \sum_{jk \neq rm} \sigma_{jk} v_{jk}(T) v'_{jk}(T) + \sigma_{rm} v_{rm}(T) v'_{rm}(T) \\ S_u(T) &= \sum_{jk \neq rm} \omega_{jk} z_{jk}(T) z'_{jk}(T) + \omega_{rm} z_{rm}(T) z'_{rm}(T) \end{aligned}\right\} \tag{48}$$

Consequently, the covariance matrix of partial control signal $u_{\widetilde{rm}}(t)$ appears as

$$\left.\begin{aligned} S_{u_{\widetilde{rm}}}(T-h) &= S_u(T-h) - \sigma_{rm} v_{rm}(T) v'_{rm}(T) \\ S_{u_{\widetilde{rm}}}(T) &= S_u(T) - \omega_{rm} z_{rm}(T) z'_{rm}(T) \end{aligned}\right\} \tag{49}$$

The covariance matrix of a final observable state conditional by rm-th control process component can be written as

$$S_{(y|u_{rm})}(T) = DS_{u_{\widetilde{rm}}}(T-h)D' + DS_{u_{\widetilde{rm}}}(T)D' + S_{Total}(T) \tag{50}$$

Substituting $S_{Total}(T)$ in Lagrange function (42), we get

$$\begin{aligned} L = {} & \frac{1}{2}\ln\det\Bigg\{ S_{(y|u_{rm})}(T) - DS_{u_{\widetilde{rm}}}(T-h)D' \\ & - DS_{u_{\widetilde{rm}}}(T)D' \sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk}Dv_{ik}(T)v'_{ik}(T)D' \\ & + \sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk}Dz_{ik}(T)z'_{ik}(T)D'\Bigg\} - \frac{1}{2}\ln\det(S_{Total}(T)) \\ & - \gamma_1\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk} - M_1\right) - \gamma_2\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk} - M_2\right) \\ & - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\lambda_{jk}\sigma_{jk} - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\mu_{jk}\omega_{jk} \\ & - \sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\beta_{k,j,i}\left(\int_0^{T-h} e_{j,k}(t)e_{i,k}(t)dt - \delta_{ji}\right) \\ & - \sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\alpha_{k,j,i}\left(\int_{T-h}^{T}\theta_{j,k}(t)\theta_{i,k}(t)dt - \delta_{ji}\right) \end{aligned} \tag{51}$$

Using equations in (49), we get

$$\begin{aligned} L = {} & \frac{1}{2}\ln\det\left\{S_{(y|u_{rm})}(T) + \sigma_{rm}Dv_{rm}(T)v'_{rm}(T)D' + \omega_{rm}Dz_{rm}(T)z'_{rm}(T)D'\right\} \\ & - \frac{1}{2}\ln\det(S_{Total}(T)) - \gamma_1\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\sigma_{jk} - M_1\right) - \gamma_2\left(\sum_{k=1}^{p}\sum_{j=1}^{\infty}\omega_{jk} - M_2\right) \\ & - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\lambda_{jk}\sigma_{jk} - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\mu_{jk}\omega_{jk} - \sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\beta_{k,j,i}\left(\int_0^{T-h} e_{j,k}(t)e_{i,k}(t)dt - \delta_{ji}\right) \\ & - \sum_{k=1}^{p}\sum_{j,i=1}^{\infty}\alpha_{k,j,i}\left(\int_{T-h}^{T}\theta_{j,k}(t)\theta_{i,k}(t)dt - \delta_{ji}\right) \end{aligned} \tag{52}$$

Now, to compute the ordinary derivative of a Lagrange function with respect to σ_{rm} for each $r = 1, 2, \dots$ and $m = 1, 2, \dots, p$

$$\begin{aligned} \frac{\partial L}{\partial \sigma_{rm}} = {} & \frac{v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T)}{1 + \sigma_{rm}v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T) + \omega_{rm}z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}Dz_{rm}(T)} \\ & - \gamma_1 - \lambda_{rm} \end{aligned} \tag{53}$$

Equating the ordinary derivative of Lagrange function to zero for each $r = 1, 2, \dots$, $m = 1, 2, \dots, p$, and by using the KKT conditions (44)-(46), we have $\sigma_{rm} = 0$ leads to

$$\frac{v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T)}{1 + \omega_{rm}z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}Dz_{rm}(T)} - \gamma_1 - \lambda_{rm} = 0 \tag{54}$$

Or $\sigma_{rm} > 0$ leads to

$$\frac{v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T)}{1 + \sigma_{rm}Dv_{rm}(T)S^{-1}_{(y|u_{rm})}v'_{rm}(T)D' + \omega_{rm}z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}Dz_{rm}(T)} - \gamma_1 = 0 \tag{55}$$

Hence, for any $k = 1, 2, \ldots, p$ and $j, i = 1, 2, \ldots$, we get

$$\begin{aligned}\sigma_{jk} = \max\Bigg\{0, \frac{1}{\gamma_1} - \frac{1}{v'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dv_{jk}(T)} \\ -\frac{\omega_{jk}z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)}{v'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dv_{jk}(T)}\Bigg\}\end{aligned} \tag{56}$$

From KKT condition in (44), we have

$$\begin{aligned}\sum_{k=1}^{p}\sum_{j=1}^{\infty}\max\Bigg\{0, \frac{1}{\gamma_1} - \frac{1}{v'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dv_{jk}(T)} \\ -\frac{\omega_{jk}z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)}{v'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dv_{jk}(T)}\Bigg\} = M_1\end{aligned} \tag{57}$$

Similarity, the ordinary derivative of a Lagrange function with respect to ω_{rm} for each $r = 1, 2, \ldots$ and $m = 1, 2, \ldots, p$ appears as:

$$\begin{aligned}\frac{\partial L}{\partial \omega_{rm}} = & \frac{z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dz_{rm}(T)}{1 + \sigma_{rm}v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T) + \omega_{rm}z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}Dz_{rm}(T)} \\ & - \gamma_2 - \mu_{rm}\end{aligned} \tag{58}$$

If we choose $\omega_{rm} = 0$ this gives

$$\frac{z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dz_{rm}(T)}{1 + \sigma_{rm}v'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dv_{rm}(T)} - \gamma_2 - \mu_{rm} = 0 \tag{59}$$

And if $\omega_{rm} > 0$ this gives

$$\frac{z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}(T)Dz_{rm}(T)}{1 + \sigma_{rm}Dv_{rm}(T)S^{-1}_{(y|u_{rm})}v'_{rm}(T)D' + \omega_{rm}z'_{rm}(T)D'S^{-1}_{(y|u_{rm})}Dz_{rm}(T)} - \gamma_2 = 0 \tag{60}$$

Therefore, for any $k = 1, 2, \ldots, p$ and $j, i = 1, 2, \ldots$, we get

$$\omega_{jk} = \max\Bigg\{0, \frac{1}{\gamma_2} - \frac{1}{z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)} - \frac{\sigma_{jk}Dv_{jk}(T)S^{-1}_{(y|u_{jk})}v'_{jk}(T)D'}{z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)}\Bigg\} \tag{61}$$

and

$$\begin{aligned}\sum_{k=1}^{p}\sum_{j=1}^{\infty}\max\Bigg\{0, \frac{1}{\gamma_2} - \frac{1}{z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)}\Bigg\} \\ -\frac{\sigma_{jk}Dv_{jk}(T)S^{-1}_{(y|u_{jk})}v'_{jk}(T)D'}{z'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dz_{jk}(T)} = M_2\end{aligned} \tag{62}$$

Now, we derive the optimality conditions for the expansion functions $\{e_{j,k}(t)\}$ by computing the derivative of the Lagrange function in (42). A variation of any function g can be represented as $\delta g(t) = \varepsilon\varphi(t)$, the quantity ε is an infinitesimal number and φ is a test function. The ordinary or partial derivative of functional $G(g)$ can be defined via the variation δG which results from the variation of g by δG

$$\delta G = G(g + \delta g) - G(g)$$

Using the Taylor expansion of $G(g + \delta g) = G(g + \varepsilon\varphi)$, we get

$$G(g+\varepsilon\varphi) = G(g) + \frac{d}{d\varepsilon}G(g+\varepsilon\varphi)\bigg|_{\varepsilon=0}\varepsilon + \frac{1}{2}\frac{d^2}{d\varepsilon^2}G(g+\varepsilon\varphi)\bigg|_{\varepsilon=0}\varepsilon^2 + \cdots$$

The functional derivative $G(g)$ with respect to g is defined as

$$\frac{d}{d\varepsilon}G(g+\varepsilon\varphi)\bigg|_{\varepsilon=0} = \int \frac{\delta G(g)}{\delta g(t)}\varphi(t)dt$$

Hence,

$$\frac{\delta L}{\delta e_{r,m}} = Tr\left\{S_y^{-1}(T)\sigma_{rm}D\left(g_{A,b_m}(t)v'_{rm}(T) + v_{rm}(T)g'_{A,b_m}(t)\right)D'\right\} + 2\sum_{i=1}^{\infty}\beta_{m,r,i}e_{i,m}(t) \quad (63)$$

The optimality condition in (43) implies to the following equation:

$$\begin{aligned}&\sigma_{rm}Tr\left\{S_y^{-1}(T)Dg_{A,b_m}(t)v'_{rm}(T)D'\right\} + \sigma_{rm}Tr\left\{S_y^{-1}(T)Dv_{rm}(T)g'_{A,b_m}(t)D'\right\}\\ &+ 2\sum_{i=1}^{\infty}\beta_{m,r,i}e_{i,m}(t) = 0\end{aligned} \quad (64)$$

Using the property that the trace of matrices is symmetric and the symmetry of $S_y^{-1}(T)$, we obtain

$$\sigma_{rm}Tr\left\{S_y^{-1}(T)Dv_{rm}(T)g'_{A,b_m}(t)D'\right\} = -\sum_{i=1}^{\infty}\beta_{m,r,i}e_{i,m}(t) \quad (65)$$

This equation is true for any $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots$ and $t \in [0, T-h]$; therefore,

$$\sigma_{jk}Tr\left\{S_y^{-1}(T)Dv_{jk}(T)g'_{A,b_k}(t)D'\right\} = -\sum_{i=1}^{\infty}\beta_{k,j,i}e_{i,k}(t) \quad (66)$$

Multiplying both sides by $e_{\tau,k}(t)$, $\tau = 1, 2, \ldots$ and taking an integral over $t \in [0, T-h]$, we have

$$\sigma_{jk}Tr\left(S_y^{-1}(T)Dv_{jk}(T)\int_0^{T-h}e_{\tau,k}(t)g'_{A,b_k}(t)dtD'\right) = -\sum_{i=1}^{\infty}\beta_{k,j,i}\int_0^{T-h}e_{\tau,k}(t)e_{i,k}(t)dt \quad (67)$$

Using orthonormality of a set of the functions $\{e_{j,k}(t)\}$, we have

$$\sigma_{jk}Tr\left(S_y^{-1}(T)Dv_{jk}(T)v'_{\tau k}(T)D'\right) = -\beta_{k,j,\tau} \quad (68)$$

Substituting $\beta_{k,j,\tau}$ in Equation (67), we get

$$\sigma_{jk}Tr\left\{S_y^{-1}(T)Dv_{jk}(T)g'_{A,b_k}(t)D'\right\} = \sum_{i=1}^{\infty}\sigma_{jk}Tr\left(S_y^{-1}(T)Dv_{jk}(T)v'_{ik}(T)D'\right)e_{i,k}(t) \quad (69)$$

Multiplying above equation by $g_{A,b_k}(t)$ and taking an integral over $t \in [0, T-h]$, we have

$$\sigma_{jk}\int_0^{T-h}g_{A,b_k}(t)g'_{A,b_k}(t)dtD'S_y^{-1}(T)Dv_{jk}(T) = \sigma_{jk}\sum_{i=1}^{\infty}v_{ik}(T)v'_{ik}(T)D'S_y^{-1}(T)Dv_{jk}(T)$$

Therefore, the above equation can be written as

$$\sigma_{jk}\left[\int_0^{T-h}g_{A,b_k}(t)g'_{A,b_k}(t)dt - \sum_{i=1}^{\infty}v_{ik}(T)v'_{ik}(T)\right]D'S_y^{-1}(T)Dv_{jk}(T) = 0 \quad (70)$$

Assume that

$$\widehat{G}_1 = G_1(T) - \sum_{i=1}^{\infty}v_{ik}(T)v'_{ik}(T) \quad (71)$$

where $G_1(b_k) = \int_0^{T-h} g_{A,b_k}(t) g'_{A,b_k}(t)dt$.

Equation (70) is satisfied if at least one of the following cases is held

i) $\sigma_{jk} = 0$

ii) $v_{jk}(T) = 0$

iii) $\widehat{G}_1 = 0$

iv) $D' S_y^{-1}(T) D v_{jk}(T)$ belong to null $\left(\widehat{G}_1\right)$.

Lemma 6.1. *Assume that the matrix D is nonsingular and there exist r, m such that $\sigma_{rm} > 0$, but $v_{rm}(T) = 0$. Then*

$$I\left(y(T)|u(t), t \in (T-h, T]; u(t), t \in [0, T-h]\right) = 0$$

Proof: Suppose that there exist r, m such that $\sigma_{rm} > 0$ and $v_{rm}(T) = 0$. Since $S^{-1}_{(y|u_{rm})}(T)$ is positive definite and $v_{rm}(T) = 0$ then

$$v'_{rm}(T) D' S^{-1}_{(y|u_{rm})}(T) D v_{rm}(T) = 0$$

However, $\sigma_{rm} > 0$ by assumption of this lemma, then from equation in (55), we get $\gamma_1 = 0$.

Consequently, for all j, k

$$v'_{jk}(T) D' S^{-1}_{(y|u_{jk})}(T) D v_{jk}(T) = 0$$

This means that $v_{jk}(T) = 0$, in other words,

$$\sum_{j,k:\sigma_{jk}>0} \sigma_{jk} D v_{jk}(T) v'_{jk}(T) D' = 0 \tag{72}$$

If $\sigma_{jk} = 0$, then

$$\sum_{j,k:\sigma_{jk}=0} \sigma_{jk} D v_{jk}(T) v'_{jk}(T) D' = 0 \tag{73}$$

Adding Equation (72) to Equation (73), we get

$$\sum_{j,k} \sigma_{jk} D v_{jk}(T) v'_{jk}(T) D' = 0$$

This completes the proof.

Now, since there exists at least one of a set $\{\sigma_{jk}\}_{jk}$ which is greater than zero. Therefore, if $\sigma_{rm} > 0$ for some r, m then from lemma above, the mutual information between $y(T)$ conditional by $u(t)$, $t \in (T-h, T]$ and the control signal $u(t)$, $t \in [0, T-h]$ becomes zero and $v_{jk}(T) = 0$, for any j, k when $v_{rm}(T) = 0$. Therefore, $v_{rm}(T) \neq 0$. Since $S_y^{-1}(T)$ is positive definite, then $D' S_y^{-1}(T) D v_{jk}(T) \neq 0$. In third case the solution of an equation in (70) is a decomposition of the matrix $G_1(b_k)$ into the sum of one rank matrices

$$G_1(b_k) = \sum_{i=1}^{\infty} v_{ik}(T) v'_{ik}(T) \tag{74}$$

Similarity,

$$G_2(b_{1^k}) = \sum_{i=1}^{\infty} z_{ik}(T) z'_{ik}(T) \tag{75}$$

where,

$$G_2(b_{1^k}) = \int_{T-h}^{T} g_{A,b_{1^k}}(t) g'_{A,b_{1^k}}(t)dt \tag{76}$$

Theorem 6.1. *Assume that both matrices $G_1(b_k)$ and $G_2(b_{1^k})$ are positive definite, then the optimal sets $\{\sigma_{jk}\}_{jk}$ and $\{\omega_{jk}\}_{jk}$ satisfy the following equations*

$$\sigma_{jk} = \frac{a - cb}{1 - db} \tag{77}$$

$$\omega_{jk} = \frac{c - ad}{1 - bd} \tag{78}$$

where,

$$a = \frac{1}{\gamma_1} - \frac{1}{g_{jk}(T)s'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Ds_{jk}(T)} \tag{79}$$

$$b = \frac{d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)}{g_{jk}(T)s'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Ds_{jk}(T)} \tag{80}$$

$$c = \frac{1}{\gamma_2} - \frac{1}{d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)} \tag{81}$$

$$d = \frac{g_{jk}(T)Ds_{jk}(T)S^{-1}_{(y|u_{jk})}s'_{jk}(T)D'}{d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)} \tag{82}$$

Proof: Since the matrices $G_1(b_k)$ and $G_2(b_{1^k})$ are positive definite, then the matrix $G_T(b_k, b_{1^k})$ is also positive definite for any k. Therefore, the controllability matrix G_T is nonsingular. This implies that the system in (1) is controllable from Lemma 5.1. Also, since the decomposition of any positive definite matrix to a sum of one rank matrices is not unique, if we take the eigenvalues decomposition of $G_1(b_k)$, we have

$$G_1(b_k) = S_k(T)O_k(T)S'_k(T)$$

such that, $O_k(T)$ is a diagonal matrix of eigenvalues $g_{jk}(T)$, $j = 1, 2, \dots, n$ of $G_1(b_k)$ and $S_k(T)S'_k(T) = I_{n\times n}$. Then,

$$G_1(b_k) = \sum_{j=1}^{n} g_{jk}(T)s_{jk}(T)S'_k(T)$$

$s_{jk}(T)$, $j = 1, 2, \dots, n$ are the columns of a matrix $S_k(T)$. The matrix $G_1(b_k)$ is positive definite by assumption, then $g_{jk}(T) > 0$ for any j. If we choose $v_{jk}(T)$ as eigenvectors of the controllability matrix $G_1(b_k)$ then

$$v_{jk}(T) = \sqrt{g_{jk}(T)}s_{jk}(T), \text{ for every } j \in \{1, 2, \dots, n\}$$

$$v_{jk}(T) = 0, \text{ for every } j \in \{n+1, n+2, \dots\}$$

In the similar method, we take the eigenvalue decomposition of $G_2(b_{1^k})$, we have

$$G_2(b_{1^k}) = R_k(T)D_k(T)R'_k(T)$$

where $D_k(T)$ is a diagonal matrix of eigenvalues $d_{jk}(T)$, $j = 1, 2, \dots, n$ of $G_2(b_{1^k})$ and $R_k(T)R'_k(T) = I_{n\times n}$. Then,

$$G_2(b_{1^k}) = \sum_{j=1}^{n} d_{jk}(T)r_{jk}(T)r'_k(T)$$

$r_{jk}(T)$, $j = 1, 2, \dots, n$ are the columns of a matrix $R_k(T)$. However, the matrix $G_2(b_{1^k})$ is positive definite by assumption, then $d_{jk}(T) > 0$ for any j. We choose $z_{jk}(T)$ as eigenvectors of the controllability matrix $G_2(b_{1^k})$ then

$$z_{jk}(T) = \sqrt{d_{jk}(T)}r_{jk}(T), \text{ for every } j \in \{1, 2, \dots, n\}$$

$$z_{jk}(T) = 0, \text{ for every } j \in \{n+1, n+2, \dots\}$$

From this, we obtain that for every $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, p$, the expansion variances can be written as

$$\begin{aligned} \sigma_{jk} = \max\left\{0, \frac{1}{\gamma_1} - \frac{1}{g_{jk}(T)s'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Ds_{jk}(T)}\right. \\ \left. - \frac{\omega_{jk}d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)}{g_{jk}(T)s'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Ds_{jk}(T)}\right\} \end{aligned} \tag{83}$$

and σ_{jk} satisfies the power constraint in (41). Also, Equation (61) can be written as

$$\begin{aligned} \omega_{jk} = \max\left\{0, \frac{1}{\gamma_2} - \frac{1}{d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)}\right. \\ \left. - \frac{\sigma_{jk}g_{jk}(T)Ds_{jk}(T)S^{-1}_{(y|u_{jk})}s'_{jk}(T)D'}{d_{jk}(T)r'_{jk}(T)D'S^{-1}_{(y|u_{jk})}(T)Dr_{jk}(T)}\right\} \end{aligned} \tag{84}$$

ω_{jk} satisfy the constraint in Equation (44).

From assumption notations in this theorem, the proof is completed.

7. Relationship between Capacity of Control and Final Time.

In this section, we show that in general the capacity of control depends on the final time T. Precisely, we show that the capacity of the control is bounded by the value of T in the following cases.

First case: infinite final time

The total covariance of uncontrolled noise matrix with the resulting matrix from initial random vector x_0, $S_{Total}(T)$ diverges for $T \to \infty$ the unlimited growth appears from the covariance matrix of noise W^{H^2} with time T, which means that the capacity of control would go to zero as $T \to \infty$. We found that the capacity of the control is finite both in asymptotically stable systems and in an unstable system (all the eigenvalues of A are positive real part).

1) **Stable systems:**

The stochastic control system in (1) is stable, when the total covariance of uncontrolled noise matrix $S_{Total}(T)$ and controllability matrix G_T are finite for $T \to \infty$ [22]. By solving the following Lyapunov equations for any $k = 1, 2, \dots, p$

$$AG_1(b_k, \infty) + G_1(b_k, \infty)A' = -b_k b'_k \tag{85}$$

$$AS_{dW}(\infty) + S_{dW}(\infty)A' = GG' \tag{86}$$

$$AS_{dW^{H^1}} + S_{dW^{H^1}}A' = \sigma_1\sigma'_1 \tag{87}$$

we obtain

$$\begin{aligned} \sigma_{jk} = \max\left\{0, \frac{1}{\gamma_1} - \frac{1}{g_{jk}(\infty)s'_{jk}(\infty)D'S^{-1}_{(y|u_{jk})}(\infty)Ds_{jk}(\infty)}\right. \\ \left. - \frac{\omega_{jk}d_{jk}(\infty)r'_{jk}(\infty)D'S^{-1}_{(y|u_{jk})}(\infty)Dr_{jk}(\infty)}{g_{jk}(\infty)s'_{jk}(\infty)D'S^{-1}_{(y|u_{jk})}(\infty)Ds_{jk}(\infty)}\right\} \end{aligned} \tag{88}$$

$$\begin{aligned} \omega_{jk} = \max\left\{0, \frac{1}{\gamma_2} - \frac{1}{d_{jk}(\infty)r'_{jk}(\infty)D'S^{-1}_{(y|u_{jk})}(\infty)Dr_{jk}(\infty)}\right. \\ \left. - \frac{\sigma_{jk}g_{jk}(\infty)Ds_{jk}(\infty)S^{-1}_{(y|u_{jk})}s'_{jk}(\infty)D'}{d_{jk}(\infty)r'_{jk}(\infty)D'S^{-1}_{(y|u_{jk})}(\infty)Dr_{jk}(\infty)}\right\} \end{aligned} \tag{89}$$

This provides implies that the capacity of control $C(\infty)$ is finite for any power constraints M_1 and M_2, where the total covariance matrix and the controllability matrix G_T completely defines the capacity of control.

2) **Unstable systems:**

In this case we show that capacity of control is finite as well, if all eigenvalues of a matrix A are positive real part.

Lemma 7.1. *The following controllability matrices and process noise variance matrix*

$$\widetilde{G}_1(b_k) = \int_0^{T-h} e^{-At} b_k b_k' e^{-A't} dt \tag{90}$$

$$\widetilde{G}_2(b_{1^k}) = \int_{T-h}^{T} e^{-At} b_{1^k} b_{1^k}' e^{-A't} dt \tag{91}$$

$$\begin{aligned}\widetilde{S}_{Total}(T) = DC_{x_0}D' + D\int_0^T e^{-At}GG'e^{-A't}dtD' + DH(2H-1)\\ \cdot \int_0^T e^{-At}\sigma_1|T-t|^{2H-2}{\sigma_1}'e^{-A't}dtD' + \sigma_2 \begin{bmatrix} T^{2\widetilde{h}_1} & & 0 \\ & \ddots & \\ 0 & & T^{2\widetilde{h}_P}\end{bmatrix}_{P\times P} \sigma_2'\end{aligned} \tag{92}$$

Define the same mutual information objective function in (41), as the original matrices, $G_1(b_k)$, $G_2(b_{1^k})$ and $S_{Total}(T)$.

Proof: From the optimal sets of $\{v_{jk}(T)\}$ and $\{z_{jk}(T)\}$, the objective function in (41) can be written as

$$\begin{aligned}&I(y(T); u(t), t \in [0,T])\\ &= \ln\left\{\det\left[I_{n\times n} + S_{Total}^{-1}(T)\sum_{k=1}^{p} DG_1(b_K)D' + S_{Total}^{-1}(T)\sum_{k=1}^{p} DG_2(b_{1^k})D'\right]\right\}\end{aligned} \tag{93}$$

where $S_{Total}^{-1}(T)$, $G_1(b_K)$ and $G_2(b_{1^k})$ can be represented by

$$S_{Total}^{-1}(T) = \left[e^{A'T}\right]^{-1}\left(\widetilde{S}_{Total}(T)\right)^{-1}\left[e^{AT}\right]^{-1}$$

$$G_1(b_K) = \left[e^{AT}\right]\widetilde{G}_1(b_k)\left[e^{A'T}\right]$$

$$G_2(b_{1^k}) = \left[e^{AT}\right]\widetilde{G}_2(b_{1^k})\left[e^{A'T}\right]$$

Therefore, the expression in (93) is equivalent to:

$$\begin{aligned}I\left(y(T); u(t), t \in [0,T]\right) = \ln\Bigg\{\det\Bigg[I_{n\times n} + \left(\widetilde{S}_{Total}(T)\right)^{-1}\sum_{k=1}^{p} D\widetilde{G}_1(b_K)D'\\ + \left(\widetilde{S}_{Total}(T)\right)^{-1}\sum_{k=1}^{p} D\widetilde{G}_2(b_{1^k})D'\Bigg]\Bigg\}\end{aligned} \tag{94}$$

when all eigenvalues of A are positive (the system is unstable) then the following equations are held

$$A\widetilde{S}_{dW}(\infty) + \widetilde{S}_{dW}(\infty)A' = GG'$$

$$A\widetilde{S}_{dW^{H^1}} + \widetilde{S}_{dW^{H^1}}A' = \sigma_1\sigma_1'$$

$$A\widetilde{G}_1(b_k,\infty) + \widetilde{G}_1(b_k,\infty)A' = -b_k b_k'$$

$$A\widetilde{G}_2(b_{1^k},\infty) + \widetilde{G}_2(b_{1^k},\infty)A' = -b_{1^k} b_{1^k}'$$

By solving the corresponding Lyapunov equations above to get the solution by replacing

$$G_1(b_k,\infty)=\widetilde{G}_1(b_k,\infty)$$
$$S_{Total}(T)=\widetilde{S}_{Total}(T)$$

Second case: infinitesimal final time

In this case, we show that the capacity of control becomes a linear function of final time T, when T goes to zero with constant coefficient that depends on its parameters of the system. In particular the capacity of control is vanished for $T\to 0$. By approximating integral in Equation (15) for $T\to 0$, we get

$$S_{Total}(T)=(DGG'D'+D\sigma_1\sigma_1'D'+\sigma_2\sigma_2')T \tag{95}$$

Similarity, for T takes values approaching to zero, the effect of h is not almost existent, so it can be assumed zero. Then the control process covariance matrix $S_u(T)$ can be written as

$$S_u(T)=\sum_{k=1}^{p}\sum_{j=1}^{n}\sigma_{jk}b_ke_{j,k}(0)b_k'e_{j,k}(0)T^2 \tag{96}$$

Therefore, the capacity of control for T near to zero appears as

$$C(T)=\ln\left\{\det\left[I_{n\times n}+S_{Total}^{-1}(T)\sum_{k=1}^{p}DG_1(b_K)D'+S_{Total}^{-1}(T)\sum_{k=1}^{p}DG_2(b_{1^k})D'\right]\right\} \tag{97}$$

Substituting (95) and (96) in Equation (97), we get

$$C(T)=\ln\{\det[I_{n\times n}+Q_{n\times n}T]\} \tag{98}$$

Hence,

$$C(T)=Tr\{Q_{n\times n}\}T \tag{99}$$

where $Q_{n\times n}=(DGG'D'+D\sigma_1\sigma_1'D'+\sigma_2\sigma_2')^{-1}\sum_{k=1}^{p}\sum_{j=1}^{n}\sigma_{jk}b_ke_{j,k}(0)b_k'e_{j,k}(0)$ is constant matrix.

8. **Illustrative Examples.** As the simple illustrative examples, explain the relationship between capacity of control and the intrinsic properties of stochastic dynamic systems perturbed by mixed fractional Brownian motion with delay in control. Consider the stochastic control system in the form (1) defined in a given time interval $[0,T]$, assume that $T>h$.

Example 8.1. *Consider the following constant matrices*

$$A=\begin{bmatrix}-1&0&1\\0&-2&0\\0&0&-1\end{bmatrix},\quad B_1=\begin{bmatrix}0&1\\1&1\\1&-1\end{bmatrix},\quad B_2=\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix},\quad G=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\sigma_1=\begin{bmatrix}1&1&0\\-1&1&0\\2&0&1\end{bmatrix},\quad D=\begin{bmatrix}1&-1&0\\-3&1&0\\2&0&1\end{bmatrix},\quad \sigma_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad H_1=\begin{bmatrix}0.75\\0.75\\0.75\end{bmatrix}$$

$$H_2=\begin{bmatrix}0.75\\0.75\\0.75\end{bmatrix},\quad C_{x_0}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Here, $n=3$, $p=2$ and assume that $T=2$ with constant point delay $h=1$. Moreover,

$$B=B_1+e^{-Ah}B_2=\begin{bmatrix}1&1.3679\\8.389056&2.0000\\2&1.7183\end{bmatrix}$$

If $M_1 + M_2 \leqslant 1$ then

$$G_1(b_1) = \begin{bmatrix} 329.4882 & 62.0411 & 176.6757 \\ 62.0411 & 12.2614 & 34.4356 \\ 176.6757 & 34.4356 & 97.1219 \end{bmatrix}, \; G_1(b_2) = \begin{bmatrix} 111.8777 & 32.9990 & 38.7258 \\ 32.9990 & 10.2647 & 12.0753 \\ 38.7258 & 12.0753 & 14.2068 \end{bmatrix}$$

$$G_2(b_{1^1}) = \begin{bmatrix} 7.6311 & 3.7827 & 4.3504 \\ 3.7827 & 2.1101 & 2.3812 \\ 4.3504 & 2.3812 & 2.6966 \end{bmatrix}, \; G_2(b_{1^2}) = \begin{bmatrix} 0.4545 & 0.1169 & -0.2368 \\ 0.1169 & 0.2454 & 0.5479 \\ -0.2368 & 0.5479 & 1.9039 \end{bmatrix}$$

Note that the matrices $G_1(b_1)$, $G_1(b_2)$, $G_2(b_{1^1})$ and $G_2(b_{1^2})$ are positive definite then the controllability matrix G_T is nonsingular. Hence, by Lemma 5.1 the system is controllable and this mean that the capacity of control is not equal to zero. The optimal vectors are

$$v_{11}(T) = \begin{bmatrix} 18.1368 \\ 3.4458 \\ 9.7879 \end{bmatrix}, \quad v_{21}(T) = \begin{bmatrix} 0.7370 \\ -0.6188 \\ -1.1478 \end{bmatrix}, \quad v_{31}(T) = \begin{bmatrix} 0.0051 \\ 0.0683 \\ -0.0335 \end{bmatrix}$$

$$v_{12}(T) = \begin{bmatrix} 10.5681 \\ 3.1473 \\ 3.6952 \end{bmatrix}, \quad v_{22}(T) = \begin{bmatrix} 0.4383 \\ -0.5991 \\ -0.7431 \end{bmatrix}, \quad v_{32}(T) = \begin{bmatrix} 0.0000122 \\ -0.0009262 \\ 0.0007539 \end{bmatrix}$$

$$z_{11}(T) = \begin{bmatrix} 2.7430 \\ 1.4170 \\ 1.6188 \end{bmatrix}, \quad z_{21}(T) = \begin{bmatrix} 0.3271 \\ -0.3189 \\ -0.2750 \end{bmatrix}, \quad z_{31}(T) = \begin{bmatrix} 0.0020 \\ 0.0200 \\ -0.0208 \end{bmatrix}$$

$$z_{12}(T) = \begin{bmatrix} 0.1711 \\ -0.3986 \\ -1.3796 \end{bmatrix}, \quad z_{22}(T) = \begin{bmatrix} 0.6515 \\ 0.2870 \\ -0.0021 \end{bmatrix}, \quad z_{32}(T) = \begin{bmatrix} 0.0285 \\ -0.0646 \\ 0.0222 \end{bmatrix}$$

$$S_{Total}(T) = \begin{bmatrix} 77.6392 & -235.9072 & 170.4309 \\ -235.9072 & 825.5392 & -619.2593 \\ 170.4309 & -619.2593 & -619.2593 \end{bmatrix}$$

$$Capacity = 1.9071 nat$$

Example 8.2. *Consider the control system in (1) is defined in a given time interval $[0, T]$, assume that $T > 0$ with the following constant matrices*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.75 \\ 0.75 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.75 \\ 0.75 \end{bmatrix}, \quad C_{x_0} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Hence, $n = 2$, $p = 1$.

As seen in Figure 1, the graphs of capacity of control $C_h(M_1, M_2 = 1, T = 1)$ become more similar to each other when the time delay h decreases. In contrast working when the time delay h increases the capacity of control $C_h(M_2, M_1 = 1, T = 1)$ becomes more similar to each other. Also, we observe that the capacity of control $C_h(M_2, M_1 = 1, T = 1)$ tends approximate to be constant for any M_2 when h converges to zero.

As shown in Figure 2, for any T, the capacity of control $C_h(M_1 = 1, M_2 = 1, T)$ converges to the finite value, for decreasing h. This behavior is qualitatively similar for different power constraints M_1 and M_2 of dynamical system. Also, we observe that, there is a relationship between the value of capacity of control and the values of T and h, $C_h(M_1 = 1, M_2 = 1, T)$ converges to 0.9118 for h converges to zero this is true when $T = 0.6300$ on the other hand, as always the entropy of observe state $H(y(T))$ is increasing

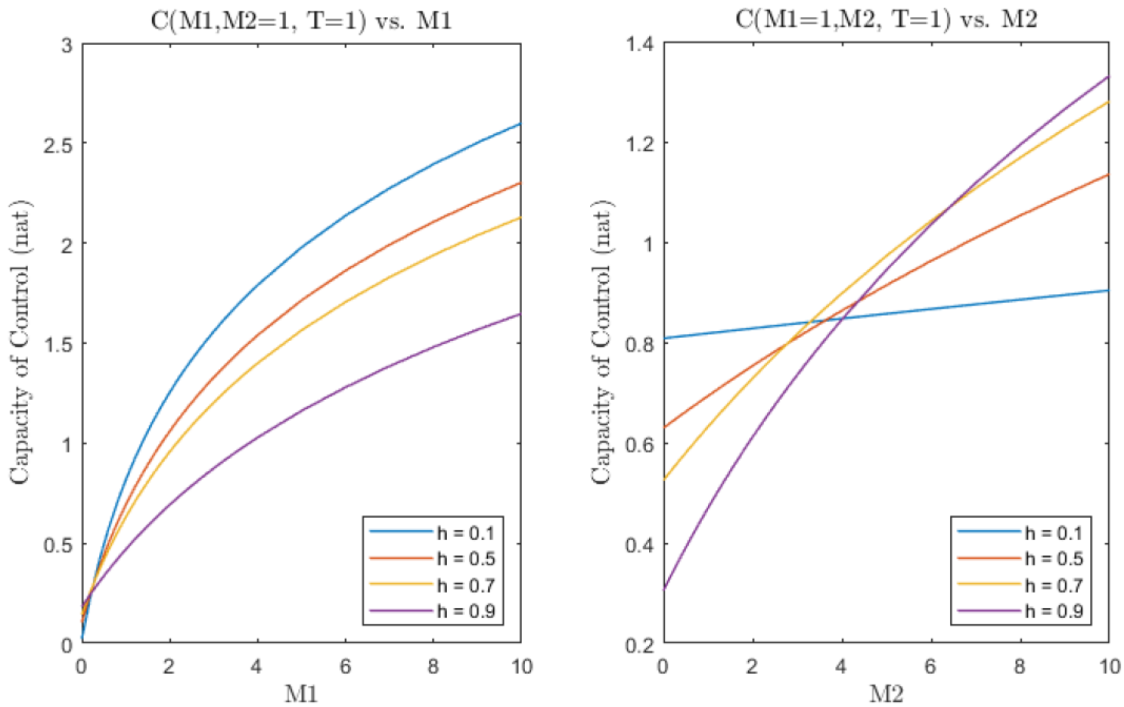

FIGURE 1. (color online) The relationship between the different power constraints M_1, M_2 and the capacity of control. Here the delay times are chosen arbitrary in an interval $[0, T]$.

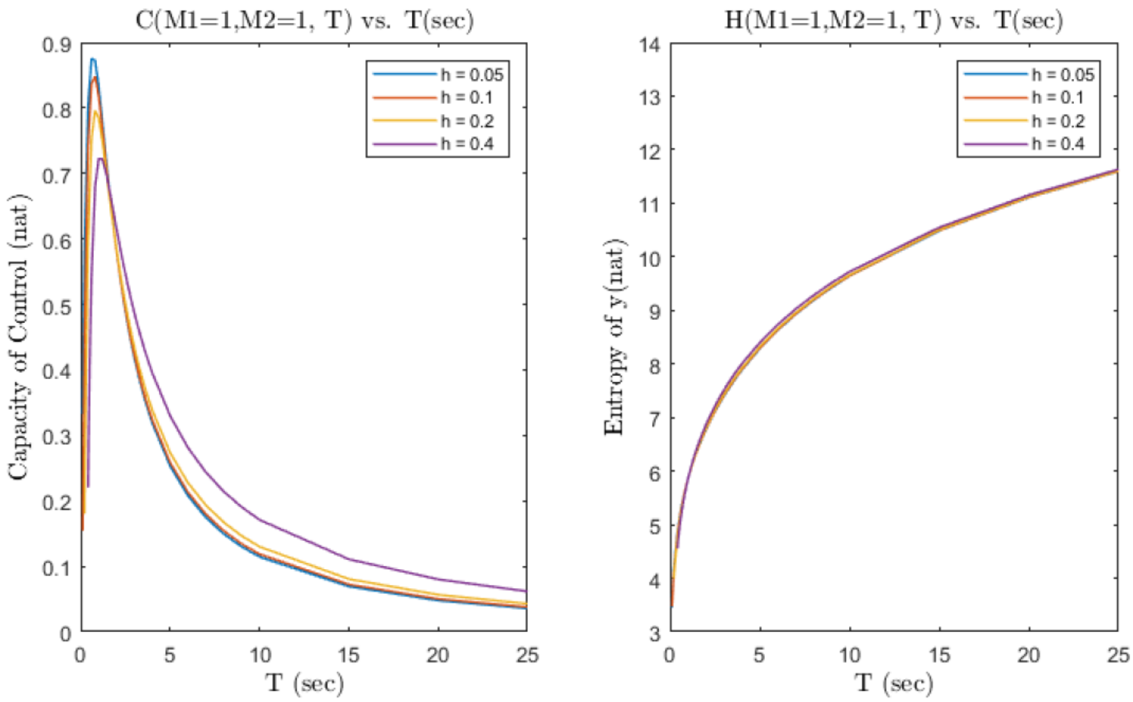

FIGURE 2. (color online) The relationship between the capacity of control and entropy for observing state of stochastic control system with the constant time T. Also, here the delay times are chosen arbitrary in an interval $[0, T]$.

when T increasing. Mathematically, the ratio of the noise matrix to the controllability matrix in Equation (94) converges to a constant for T converges to zero.

9. **Conclusion.** The information channel between the control process to output state for stochastic control systems driven by mixed fractional Brownian motion is explored. The efficient method is derived for computing the capacity of control of these systems. The effective contribution of time delay to determination of capacity of control is shown explicitly. We demonstrate that the capacity of control does not grow without limiting the length of the control process and the value of delay time. An expression of optimal control is found which maximizes the mutual information of the control process and the system future output.

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