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INFERENCE ON THE LIFETIME PERFORMANCE INDEX FOR THE GOMPERTZ DISTRIBUTION WITH THE PROGRESSIVELY FIRST-FAILURE CENSORED SAMPLES

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Abstract. *Process capability analysis is regarded as a utile and widely-spreading method to assess whether the quality of the product reaches the pre-specified level. In various industries, when the lower specification limit is determined, we often use the lifetime performance index to measure the product performance. In this study, we establish the statistical inferences to assess the lifetime performance index for the Gompertz distribution with progressively first-failure censored samples. The maximum likelihood estimator of the lifetime performance index is obtained to construct a hypothesis testing procedure and a confidence interval with given lower specification limit. Finally, we provide an example with real data as well as the Monte Carlo simulation to illustrate the proposed testing procedure under specified significance level.*
Keywords: Gompertz distribution, Progressive first-failure censoring, Lifetime performance index, Maximum likelihood estimator, Testing procedure

1. **Introduction.** Effectively measuring the operational process is very important in the industry of manufacturing or service, for it can increase the quality of the products, and meanwhile, reduce costs or time. Process capability analysis is an important and effective method in the aspect of quality monitor to determine whether the properties of the product meet the requirement, and managers can check the lifetime performance of the products by this means to make some adjustments on the production process. In [1, 2], authors suggested the conventional process capability index C_L to evaluate the lifetime performance of some electronic components, which shows the larger-the-better quality feature based on the time. Statistical inference for the lifetime performance index has been investigated by various authors under different distributions. Under the one-parameter exponential distribution, the uniformly minimum variance unbiased estimator for the lifetime performance index C_L was constructed in [3] and authors proposed a hypothesis testing procedure. A uniformly minimum variance unbiased estimator of C_L with the right type II censoring for the Pareto distribution has been considered in [4].

The Gompertz distribution was first developed and introduced by [5], an important distribution in the human death rate modeling and life table construction. In the book by [6], this distribution was proved as a special case from the type I extreme value distribution, and has been widely applied in the field of extreme-order statistics. The Gompertz distribution has been regarded as an effective model for the statistical analysis of lifetime data, showing a fine fit to data in the clinical trials and medicine science (see [7]). So far, there have been many biologically related products, such as pharmaceutical, meaning that

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the lifetime of a product could have Gompertz distribution and this distribution closely related to life span is important in industrial production, for it will also have potential application value in the future. Aiming at the Gompertz distribution, the lifetime performance index has been investigated by many scholars. In [8], authors considered the lifetime performance index for the Gompertz distribution with progressive type II censored samples and the scholars in [9] derived the testing procedure of the lifetime performance index for the Gompertz distribution based on the first-failure censored samples. Inspired by these two studies, our paper proposes the test procedures of the lifetime performance index for Gompertz distribution based on the progressive first-failure censoring, a more sophisticated censoring method.

The lifetime of a product sometimes can be quite long, so that the experiments often terminate before all units observed. A censoring plan, named the first-failure censoring, was introduced in [10]. Then, in [11], scholars described the progressive first-failure censoring in detail, which is an improved censoring plan based on the first-failure censoring. Instead of collecting the complete information of all items, the progressive first-failure censored sampling is often used in lifetime experiments to reduce time and cost. This censoring method has been used by many authors. For instance, the computational procedure of the lifetime performance index C_L for the Weibull distribution was developed with the progressively first-failure censored samples by [12]. Using the progressive first-failure censored sampling method, authors in [13] estimated C_L for the Weibull distribution based on the mean square error (MSE). The estimation of C_L for the two-parameter exponential lifetime distribution was considered in [14] with progressively first-failure censored samples. The literature investigated the lifetime performance index for several distributions, showing that progressive first-failure censoring is feasible to be applied in constructing test procedures based on different distributions, determining whether the lifetime performance index meets the requirements. We discuss the Gompertz distribution and establish the test procedures for the lifetime performance index with this censoring scheme.

We consider a more general case of progressive first-failure censoring schemes. Suppose there are N units in total in the life-testing experiment and divide them into n groups independently with k units in each group. When the first failure occurs, discard this corresponding group and R_1 groups from remaining $n-1$ groups randomly. When the second failure occurs in a group, discard this group and R_2 groups from remaining $n-R_1-2$ groups randomly, and so on. When the mth failure is observed, discard all remaining groups, and this procedure terminates. The number of the groups to be discarded $R = (R_1, R_2, \ldots, R_m)$ is pre-specified and called the censoring scheme. When $k = 1$ in each group, this censoring reduces to the progressive type II censoring and it becomes first-failure censoring when $R_m = n - m$ and $R_1 = R_2 = \cdots = R_{m-1} = 0$.

This paper aims to establish the hypothesis testing procedure based mainly on the maximum likelihood estimator of the lifetime performance index C_L under the progressive first-failure censoring, an improved censoring plan of first-failure censoring. The remainder of the paper is structured as follows. In Section 2, we present the lifetime performance index C_L for the Gompertz distribution and the conforming rate P_r. In Section 3, based on the progressive first-failure censoring, we derive the maximum likelihood estimator of the lifetime performance index C_L. In Section 4, a hypothesis testing process for the lifetime performance index is constructed. In Section 5, we derive a lower bound and a confidence interval for another testing procedure to evaluate the lifetime performance index. In Section 6, we analyze a real clinical example and conduct the Monte Carlo simulations. In Section 7, we draw the conclusions.

2. **The Lifetime Performance Index C_L and the Conforming Rate P_r.** The probability density function of the Gompertz distribution and the corresponding cumulative distribution function are given respectively by

$$f(x;\eta,\lambda) = \lambda \exp\left[\eta x - \frac{\lambda}{\eta}\left(e^{\eta x} - 1\right)\right], \quad x > 0 \tag{1}$$

and

$$F(x;\eta,\lambda) = 1 - \exp\left[-\frac{\lambda}{\eta}\left(e^{\eta x} - 1\right)\right], \quad x > 0 \tag{2}$$

where both the scale parameter λ and the shape parameter η are positive. And we have the hazard function of Gompertz distribution as

$$h(x) = \lambda e^{\eta x}, \quad x > 0 \tag{3}$$

The plots of the probability density functions and hazard functions of the Gompertz distribution with different parameters is shown in Figure 1 and Figure 2 respectively. Apparently, the plots show that the scale parameter λ stretches the plots only while the shape parameter η determines the shape of the plots. So we say the probability density function and hazard function are more affected by η. Thus, before determining the scale parameter λ, we apply the Gini statistic method to selecting the best shape parameter η.

Obviously, managers prefer that the lifetime of products can be longer, meaning a better quality of the product. So the lifetime has the larger-the-better type quality feature. In general, given L, the lower specification limit of the units, the lifetime is required to exceed it to meet the demand of investors. The scholar in [1] proposed a capability index C_L to appropriately assess the larger-the-better quality characteristic, which is given as

$$C_L = \frac{\mu - L}{\sigma} \tag{4}$$

where μ is the population mean of the process and σ is the standard deviation.

Suppose X has the Gompertz distribution with the probability density function given in (1). We do the data transformation $Y = e^{\eta X} - 1$. So the probability density function,

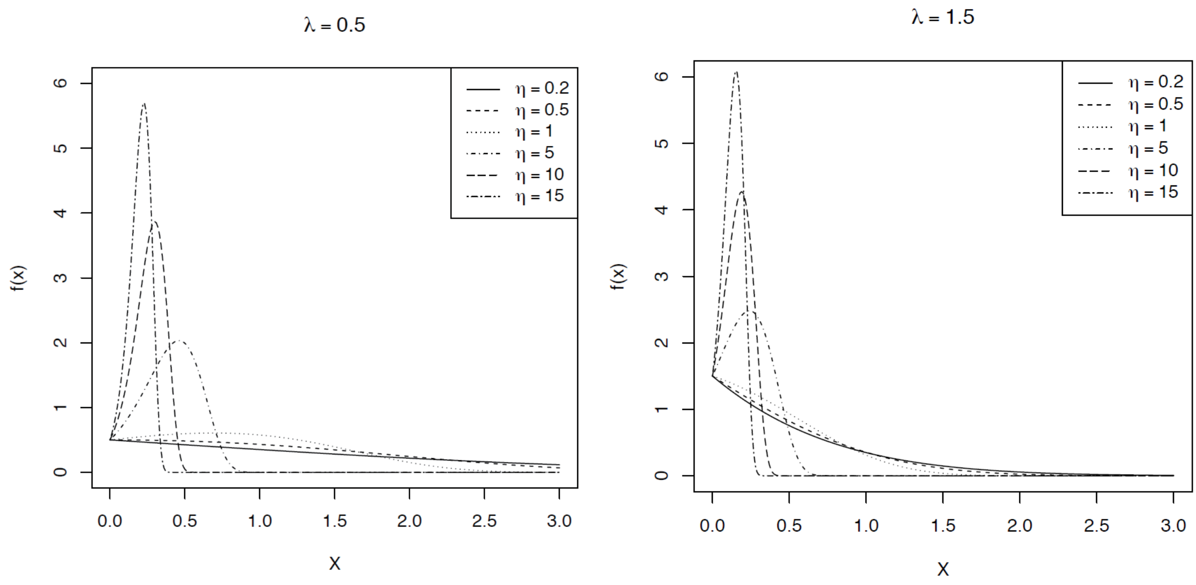

FIGURE 1. The probability density functions with different parameters for the Gompertz distribution

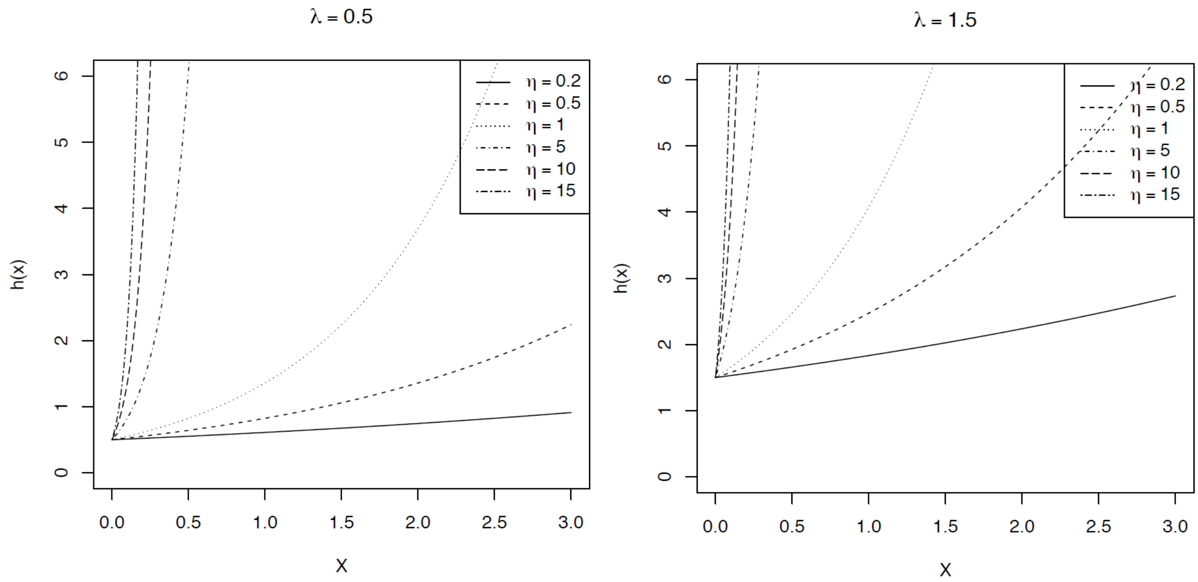

FIGURE 2. The hazard functions with different parameters for the Gompertz distribution

cumulative distribution function and the hazard function are shown respectively as follows:

$$f_Y(y) = \gamma e^{-\gamma y}, \quad y \geq 0 \tag{5}$$

$$F_Y(y) = 1 - e^{-\gamma y}, \quad y \geq 0 \tag{6}$$

$$h(y) = \frac{f_Y(y)}{1 - F_Y(y)} = \gamma \tag{7}$$

where $\gamma = \frac{\lambda}{\eta} > 0$.

Clearly, Y is an exponential random variable with mean γ. Hence, the population mean and the standard deviation are

$$\mu = E(Y) = \frac{1}{\gamma}, \quad \sigma = \sqrt{Var(Y)} = \frac{1}{\gamma} \tag{8}$$

According to Equation (4), the lifetime performance index C_L is

$$C_L = \frac{\mu - L}{\sigma} = 1 - \gamma L \tag{9}$$

There is a tendency that the value of the hazard function will be smaller while the lifetime performance index becomes bigger as γ decreases.

Moreover, when the lifetime of the product X for which $Y = e^{\eta X} - 1$ exceeds L, the lower specification limit, the product is regarded to be good. The conforming rate which represents the ratio of the conforming product is defined as

$$P_r = P(Y \geq L) = \int_L^{\infty} \gamma e^{-\gamma y} dy = e^{-\gamma L} = e^{C_L - 1}, \quad -\infty < C_L < 1 \tag{10}$$

Apparently, the conforming rate P_r and the lifetime performance index C_L have a strictly increasing one-to-one mathematical relationship. So we can use the lifetime performance index to infer the conforming rate as well. Table 1 shows the various values of C_L and the corresponding conforming rate P_r.

TABLE 1. The lifetime performance index C_L versus the corresponding conforming rate P_r

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000	0.025	0.37719	0.275	0.48432	0.525	0.62189	0.775	0.79852
−0.200	0.30119	0.050	0.38674	0.300	0.49659	0.550	0.63763	0.800	0.81873
−0.175	0.30882	0.075	0.39653	0.325	0.50916	0.575	0.65377	0.825	0.83946
−0.150	0.31664	0.100	0.40657	0.350	0.52205	0.600	0.67032	0.850	0.86071
−0.125	0.32465	0.125	0.41686	0.375	0.53526	0.625	0.68729	0.875	0.88250
−0.100	0.33287	0.150	0.42741	0.400	0.54881	0.650	0.70469	0.900	0.90484
−0.075	0.34130	0.175	0.43823	0.425	0.56270	0.675	0.72253	0.925	0.92774
−0.050	0.34994	0.200	0.44933	0.450	0.57695	0.700	0.74082	0.950	0.95123
−0.025	0.35880	0.225	0.46070	0.475	0.59156	0.725	0.75957	0.975	0.97531
0.000	0.36788	0.250	0.47237	0.500	0.60653	0.750	0.77880	1.000	1.00000

3. **The Maximum Likelihood Estimator of the Lifetime Performance Index C_L.** In practical application, the ture value of the lifetime performance index C_L is unknown, but we could obtain estimates with correlative known data. In this section, we derive the estimator of lifetime performance index by maximum likelihood extimation and show the feasibility of estimation methods.

Suppose the lifetime of products has the Gompertz distribution and there are N units in total but we need m units only. Denote $X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k}$ as a set of progressively first-failure censored data from the Gompertz distribution and the censoring scheme is pre-fixed by $(R_1, R_2, \ldots, R_m)$. According to [11, 15], the likelihood function based on the progressively first-failure censored sample is given by

$$L(\mu,\sigma;x_{1:m:n:k},\ldots,x_{m:m:n:k}) = Ck^m \prod_{i=1}^{m} f(x_{i:m:n:k};\mu,\sigma)\left\{1-F(x_{i:m:n:k};\mu,\sigma)\right\}^{k(R_i+1)-1} \tag{11}$$

where

$$C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1-\cdots-R_{m-1})$$

According to the functions in (1), (2) and (11), the likelihood function for the Gompertz distribution based on progressive first-failure censoring is given as follows (using x_i instead of $x_{i:m:n:k}$):

$$L(\eta,\lambda) = Ck^m\lambda^m \prod_{i=1}^{m} \exp\left[\eta x_i - \frac{k\lambda(R_i+1)(e^{\eta x_i}-1)}{\eta}\right] \tag{12}$$

The log-likelihood function is

$$l = \log L(\eta,\lambda) = \ln C + m\ln k + m\ln\lambda + \eta\sum_{i=1}^{m} x_i - \frac{\lambda}{\eta}\sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right] \tag{13}$$

Solve the likelihood equation for given η as follows:

$$\frac{\partial l}{\partial\lambda} = \frac{m}{\lambda} - \frac{1}{\eta}\sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right] = 0 \tag{14}$$

We express the maximum likelihood estimator (MLE) of λ as

$$\hat{\lambda} = \frac{m\eta}{\sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right]} \tag{15}$$

Thus,

$$\hat{\gamma} = \frac{\hat{\lambda}}{\eta} = \frac{m}{\sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right]} \tag{16}$$

According to the invariance principle of maximum likelihood function proposed by [16], the MLE of C_L is

$$\hat{C_L} = 1 - \hat{\gamma}L = 1 - \frac{mL}{\sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right]} = 1 - \frac{mL}{D} \tag{17}$$

where $D = \sum_{i=1}^{m}\left[k(R_i+1)(e^{\eta x_i}-1)\right]$.

We can show that $2\gamma D$ follows $\chi^2(2m)$, the chi-squared random variable with $2m$ degrees of freedom. Hence, the expected value of $\hat{C_L}$ is given by

$$E\left(\hat{C_L}\right) = E\left(1 - \frac{mL}{D}\right) = 1 - 2\gamma mL \cdot E\left(\frac{1}{2\gamma D}\right) = 1 - \frac{m}{m-1}\gamma L \tag{18}$$

Note that $E\left(\hat{C_L}\right) \neq C_L = 1 - \gamma L$, which means that the estimator is biased. When $m \to \infty$, $E\left(\hat{C_L}\right) \to C_L$, so that $\hat{C_L}$ is an asymptotically unbiased estimator but not an unbiased estimator of C_L. Further, we can show that $\hat{C_L}$ is consistent for C_L. So we could use $\hat{C_L}$ in Equation (17) to estimate the performance lifetime index C_L.

4. **Testing Procedure for the Lifetime Performance Index $\boldsymbol{C_L}$.** This section constructs a statistical testing process to evaluate whether the lifetime performance index adheres the specified level. Assume that c denotes the target value and the required index value of the lifetime performance is larger than c. We establish the null hypothesis $H_0 : C_L \le c$, and $H_1 : C_L > c$ is the alternative hypothesis. Using $\hat{C_L}$, the MLE of C_L, the rejection region can be written as $\left\{\hat{C_L} \middle| \hat{C_L} > C_0\right\}$. Suppose the significance level is α, and then the critical value is derived as

$$\begin{aligned}
& P\left(\hat{C_L} > C_0 \middle| C_L = c\right) = \alpha \\
\Rightarrow\ & P\left(1 - \frac{mL}{D} > C_0 \middle| 1 - \gamma L = c\right) = \alpha \\
\Rightarrow\ & P\left(2\gamma D > \frac{2\gamma mL}{1 - C_0} \middle| \gamma = \frac{1-c}{L}\right) = \alpha \\
\Rightarrow\ & P\left(2\gamma D > \frac{2m(1-c)}{1 - C_0}\right) = \alpha \\
\Rightarrow\ & \frac{2m(1-c)}{1 - C_0} = \chi^2_\alpha(2m) \\
\Rightarrow\ & C_0 = 1 - \frac{2m(1-c)}{\chi^2_\alpha(2m)}
\end{aligned} \tag{19}$$

Here, $\chi^2_\alpha(2m)$ denotes the upper αth quartile of the chi-squared distribution with $2m$ degrees of freedom. Hence, given the target value c, the significance level α and the number of observed failures m, the critical value is found to be

$$C_0 = 1 - \frac{2m(1-c)}{\chi^2_\alpha(2m)} \tag{20}$$

Clearly, we find that C_0 is independent of the censoring scheme $(R_1, R_2, \ldots, R_m)$ as well as the sample size n.

Let the lifetime of products have the Gompertz distribution. The testing procedure of C_L can be stated as follows.

- Step 1: Given the number of the observed failure times m, the sample size n, and the progressive censoring scheme $R = (R_1, R_2, \ldots, R_m)$ according to the actual production activities. Let $X_1, X_2, \ldots, X_m$ denote the observed data of the products under progressively first-failure censored sampling. Apply the Gini statistic method (see [17]) to selecting the shape parameter η.
- Step 2: Let the transformation $Y_i = e^{\eta X_i} - 1$, $i = 1, 2, \ldots, m$, where $X_1, X_2, \ldots, X_m$ is a set of progressively first-failure censored data.
- Step 3: Under the given lifetime lower specification limit L of the products and the target performance index value c, develop the null hypothesis $H_0 : C_L \le c$ and the alternative hypothesis $H_1 : C_L > c$.
- Step 4: Specify the significance level α.
- Step 5: Evaluate the maximum likelihood estimate $\hat{C}_L = 1 - \frac{mL}{\sum_{i=1}^m k(R_i+1)Y_i}$.
- Step 6: According to the given c, m and α, obtain the critical value C_0 from (20).
- Step 7: Make the decision as follows. If $\hat{C}_L > C_0$, reject the null hypothesis H_0 and the lifetime performance index of products reaches the required level.

Based on the progressively first-failure censored samples, the proposed testing procedure takes both mean and variance into account, and simplifies the calculation of the critical value by using pivot quantity and introducing chi-square distribution. This procedure can be effectively used in manufacturing or service industry to measure the quality of products subject to Gompertz distribution.

5. **Interval Estimation of the Lifetime Performance Index C_L.** Given an interval, we can accurately determine whether a measured value is within the normal range. So the confidence interval has been applied widely in many studies. For example, based on the confidence interval, authors in [18] investigated the improved variational linear regression.

In practice, the managers also can adopt the one-side confidence interval to determine the quality of products as well. We draw the conclusion that $2\gamma D \sim \chi^2(2m)$. So according to Equations (9) and (17), given the number of observed failure times m and the significance level α, the $100(1-\alpha)\%$ confidence interval for C_L is obtained as

$$\begin{aligned}
&P\left(2\gamma D \le \chi^2_{1-\alpha}(2m)\right) = 1-\alpha \\
\Rightarrow\ & P\left(\frac{2m(1-C_L)}{1-\hat{C}_L} \le \chi^2_{1-\alpha}(2m)\right) = 1-\alpha \\
\Rightarrow\ & P\left(C_L \ge 1 - \frac{\left(1-\hat{C}_L\right)\chi^2_{1-\alpha}(2m)}{2m}\right) = 1-\alpha
\end{aligned} \tag{21}$$

From Equation (21), we know that

$$C_L \ge 1 - \frac{\left(1-\hat{C}_L\right)\chi^2_{1-\alpha}(2m)}{2m} \tag{22}$$

where $\hat{C}_L = 1 - \frac{mL}{\sum_{i=1}^m k(R_i+1)(e^{\eta x_i}-1)}$.

Hence, the $100(1-\alpha)\%$ lower bound for C_L is derived as

$$\underline{LB} = 1 - \frac{\left(1-\hat{C}_L\right)\chi^2_{1-\alpha}(2m)}{2m} \tag{23}$$

And the $100(1-\alpha)\%$ one-side confidence interval can be obtained as

$$C.I. = (\underline{LB}, +\infty) \tag{24}$$

The testing procedure for the confidence interval of the lifetime performance index C_L can be established as follows.

- Step 1: Given the number of the observed failure times m, the sample size n and the progressive censoring scheme $R = (R_1, R_2, \ldots, R_m)$ according to the actual production situation. Let $X_1, X_2, \ldots, X_m$ denote the observed data of the products under progressively first-failure censored scheme. Use the Gini statistic method (see [17]) to select the shape parameter η.
- Step 2: Let the transformation $Y_i = e^{\eta X_i} - 1$, $i = 1, 2, \ldots, m$, where $X_1, X_2, \ldots, X_m$ is a set of the progressively first-failure censored data.
- Step 3: Under the given lifetime lower specification limit L of the products and the target performance index value c, develop the null hypothesis $H_0 : C_L \leq c$ and the alternative hypothesis $H_1 : C_L > c$.
- Step 4: Specify the significance level α.
- Step 5: Obtain the $100(1-\alpha)\%$ one-side confidence interval $(\underline{LB}, +\infty)$ for C_L, where $\underline{LB} = 1 - \frac{(1-\hat{C_L})\chi^2_{1-\alpha}(2m)}{2m}$ and $\hat{C_L} = 1 - \frac{mL}{\sum_{i=1}^{m} k(R_i+1)Y_i}$.
- Step 6: Make the decision as follows. If the target performance index c does not fall into $(\underline{LB}, +\infty)$, reject the null hypothesis H_0, and conclude that the lifetime performance index of products reaches the required level.

6. **Application.**

In this section, we will make numerical examples to show how the two proposed testing procedures work.

6.1. **Illustrative example with real data.**

Now, we consider a real example for illustration purpose.

A data set from a test of unsaturated diet was provided in [19] which has been proved to be from a Gompertz distribution by [20]. The data set shows the tumor-free days of 30 rats which have been fed with unsaturated diet, shown in Table 2.

TABLE 2. Tumor-free days of thirty rats which are given with unsaturated diet

60	63	63	63	66	66	66	68	70	70
77	77	84	91	91	94	98	101	105	108
109	112	112	115	126	143	153	161	164	178

To select the shape parameter η, we apply the Gini statistic, proposed by [17]. Suppose $X_1, X_2, \ldots, X_m$ is the sample from the Gompertz distribution. Use the data transformation $T_i = e^{\eta X_i} - 1$. The Gini statistic is given by

$$G_m = \frac{\sum_{i=1}^{m-1} iQ_{i+1}}{(m-1)\sum_{i=1}^{m} Q_i} \tag{25}$$

where $Q_i = (m-i+1)(T_i - T_{i-1})$, $T_0 = 0$, $i = 1, 2, \ldots, m$, and the distribution of G_m is

$$P(G_m \leq x) = 1 - \sum_{j=1}^{l} (c_j - x)^{m-1} \left\{ c_j \prod_{k \neq j}^{m-1} (c_j - c_k) \right\}^{-1} \tag{26}$$

where $c_j = \frac{m-j}{m-1}$ and l is the largest index subject to $x \leq c_l$. The p-value is

$$P\{|G_m - 0.5| > |g_m - 0.5|\} \tag{27}$$

where g_m is the observed value of G_m and α is the given significance level. Scholars in [17] also mentioned that the p-value can be derived as

$$P\left\{|Z| > \left|[12(m-1)]^{1/2} \cdot (g_m - 0.5)\right|\right\} \tag{28}$$

where Z follows the standard normal distribution $N(0,1)$.

Based on the Gini statistic, the various values of the shape parameter η and the corresponding p-value are listed in Table 3. We can find that when $\eta = 0.0223$, the p-value is 0.99240, which is the largest one among all other values. Assume the lower lifetime limit L is 30, and investors require that the conforming rate P_r must exceed 90 percent. According to Equation (10), we know that $C_L = \ln P_r + 1$, so the value of the lifetime performance index C_L is required to exceed 89.464 percent. Hence, the performance index value is supposed to be set at $c = 0.89464$. Thus, we develop the testing null hypothesis $H_0 : C_L \leq 0.89464$ and the alternative hypothesis $H_1 : C_L > 0.89464$. Here, we suppose the significance level α is 0.05.

Table 3. The values of beta versus the corresponding p-values

η	p-value	η	p-value	η	p-value	η	p-value	η	p-value
0.0211	0.72095	0.0217	0.85469	**0.0223**	**0.99240**	0.0229	0.87007	0.0235	0.73683
0.0212	0.74279	0.0218	0.87747	0.0224	0.98455	0.0230	0.84746	0.0236	0.71531
0.0213	0.76484	0.0219	0.90035	0.0225	0.96153	0.0231	0.82499	0.0237	0.69403
0.0214	0.78707	0.0220	0.92330	0.0226	0.93855	0.0232	0.80267	0.0238	0.67300
0.0215	0.80946	0.0221	0.94631	0.0227	0.91563	0.0233	0.78053	0.0239	0.65224
0.0216	0.83201	0.0222	0.96935	0.0228	0.89280	0.0234	0.75858	0.0240	0.63176

According to the hypothesis testing procedure in previous sections, we consider two cases to make assessments.

(I) Case I: We group the data into $n = 15$ sets randomly with $k = 2$ units in each set, shown in Table 4. And we consider three modes of the progressive first-failure censoring scheme in Table 5. Note that for convenience, we use short notations to denote different censoring schemes in this paper. For example, $(1 * 4)$ denotes $(1,1,1,1)$ and $((1,0) * 2)$ denotes $(1,0,1,0)$.

Table 4. Random grouping to the real data with 2 items each set for tumor-free days of thirty rats which are given with unsaturated diet

Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7	Set 8
60	63	63	66	66	68	70	77
70	63	66	91	84	101	70	105
Set 9	Set 10	Set 11	Set 12	Set 13	Set 14	Set 15	
91	94	98	108	112	112	143	
109	161	126	164	178	115	153	

Table 5. Different progressive first-failure censoring schemes based on tumor-free days of thirty rats which are given with unsaturated diet when $k = 2$

m	Censoring scheme	Progressive first-failure censored data
10	$R_1 = ((1,0) * 5)$	60, 63, 66, 68, 70, 77, 94, 108, 112, 143
10	$R_2 = (1 * 5, 0 * 5)$	60, 63, 63, 66, 68, 91, 108, 112, 112, 143
5	$R_3 = (2 * 5)$	60, 63, 66, 94, 98

We can obtain the corresponding $\hat{C}_L$, C_0 and $(\underline{LB}, +\infty)$ under different censoring schemes with different m shown in Table 6 and judge whether the lifetime of laboratory rats meets the requirement.

Table 6. The results of the hypothesis testing procedure under different censoring schemes when $k = 2$

Censoring scheme	m	$\hat{C}_L$	C_0	Judgment	$(\underline{LB}, +\infty)$	Judgment	Satisfied
R_1	10	0.9528657	0.8058016	$\hat{C}_L > C_0$	$(0.9259746, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes
R_2	10	0.9493694	0.8058016	$\hat{C}_L > C_0$	$(0.9204835, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes
R_3	5	0.9673065	0.7326078	$\hat{C}_L > C_0$	$(0.9401478, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes

(II) Case II: Now, we group the data into $n = 10$ sets randomly with $k = 3$ units in each set, shown in Table 7. And we consider four modes of the progressive first-failure censoring scheme in Table 8.

Table 7. Random grouping to the real data with 3 items each set for tumor-free days of thirty rats which are given with unsaturated diet

Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7	Set 8	Set 9	Set 10
60	63	63	66	66	68	77	91	105	112
63	94	70	108	91	70	109	143	161	126
66	101	84	112	98	77	115	178	164	153

Table 8. Different progressive first-failure censoring schemes based on tumor-free days of thirty rats which are given with unsaturated diet when $k = 3$

m	Censoring scheme	Progressive first-failure censored data
6	$R_4 = (1 * 4, 0 * 2)$	60, 63, 63, 66, 91, 105
5	$R_5 = (1 * 5)$	60, 63, 66, 77, 91
4	$R_6 = (2 * 3, 0)$	60, 63, 66, 112
4	$R_7 = (3, 1, 0, 2)$	60, 63, 77, 91

We can obtain the corresponding $\hat{C}_L$, C_0 and $(\underline{LB}, +\infty)$ under various censoring schemes with different observed failure times m shown in Table 9 and assess whether the lifetime meets the requirement.

Table 9. The results of the hypothesis testing procedure under different censoring schemes when $k = 3$

Censoring scheme	m	$\hat{C}_L$	C_0	Judgment	$(\underline{LB}, +\infty)$	Judgment	Satisfied
R_4	6	0.9531391	0.7580714	$\hat{C}_L > C_0$	$(0.9178916, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes
R_5	5	0.961139	0.7326078	$\hat{C}_L > C_0$	$(0.928857, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes
R_6	4	0.9673462	0.6915492	$\hat{C}_L > C_0$	$(0.9367035, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes
R_7	4	0.9696172	0.6915492	$\hat{C}_L > C_0$	$(0.9411056, \infty)$	$c \notin (\underline{LB}, +\infty)$	Yes

The results in Case I and Case II all show that under different censoring schemes, we have $\hat{C_L} > C_0$ and $c \notin (\underline{LB}, +\infty)$. Hence, we draw the conclusion that under the unsaturated diet, the tumor-free lifetime of these 30 rats meets the requirement.

From this illustrative example, we find that the proposed test procedure for assessing the lifetime performance index can be used not only in manufacturing service, but in medical community and even more fields.

6.2. **Illustrative example with simulated data.** Here, we illustrate the proposed hypothesis testing procedure by a Monte Carlo simulation study. First of all, using the algorithm given by [21], we can generate the progressively first-failure censored sample from the Gompertz distribution with $\lambda = 0.01$ and $\eta = 0.05$ respectively. Suppose the number of total units is 200, and we only need $m = 50$ units. Divide them into $n = 100$ groups with $k = 2$ units each group. Suppose the censoring scheme for the data is $R = (R_1, R_2, \ldots, R_m)$ where $R_i = 1$, $i = 1, 2, \ldots, m$. Thus, the sampling sample is given in Table 10.

TABLE 10. The data generated by Monte Carlo simulation study with progressive first-failure censoring

0.25	0.39	0.54	0.79	0.82	0.97	1.51	1.87	2.42	4.14
4.43	5.56	5.99	6.10	6.17	7.16	8.21	9.02	9.53	10.11
11.89	12.68	12.80	12.88	14.54	14.95	15.33	15.34	15.82	16.10
17.25	17.78	18.74	19.17	21.33	22.24	22.38	22.56	23.53	23.68
25.18	25.43	27.18	27.41	31.72	31.74	32.86	32.95	35.22	38.29

We take $L = 8.0$ and the conforming rate is expected to exceed 0.8. Hence, according to the proposed testing procedure, we can make the assessment for C_L as follows.

- Step 1: Let $X_1, X_2, \ldots, X_m$ denote the progressively first-failure censored sample of $n = 100$, $m = 50$ and $k = 2$ cases. Let the transformation $Y_i = e^{\eta X_i} - 1$, where $i = 1, 2, \ldots, m$.
- Step 2: We suppose the product lower lifetime limit L is 8.0. To meet the concerns of the managers and investors to the quality of products, the conforming rate P_r should exceed 80 percent. Hence, the C_L value is supposed to exceed 0.7769. So we set the performance index value c as 0.7769. We can establish the testing hypothesis as $H_0 : C_L \leq 0.7769$ versus $H_1 : C_L > 0.7769$.
- Step 3: Specify the significance level α.
- Step 4: Calculate the test statistic value $\hat{C_L} = 1 - \frac{mL}{\sum_{i=1}^{m} k(R_i+1)Y_i} = 0.9143653$, where R_i is from the censoring scheme R.
- Step 5: According to Equation (20), derive the critical value $C_0 = 0.7239476$ with $c = 0.7769$, $m = 50$ and $\alpha = 0.05$.
- Step 6: Since $\hat{C_L} = 0.9143653 > C_0 = 0.7239476$, we should reject H_0. So we determine the lifetime performance index of products reaches the required level.
- Step 7: Calculate the $\underline{LB} = 1 - \frac{(1-\hat{C_L})\chi^2_{1-\alpha}(2m)}{2m} = 0.8968408$. Thus, the $100(1-\alpha)\%$ one-side confidence interval $(\underline{LB}, +\infty)$ for C_L is $(0.8968408, +\infty)$.
- Step 8: Since $c = 0.7769 \notin (\underline{LB}, +\infty)$, we should reject H_0. Hence, we conclude that the lifetime performance index of products meets the required level.

In order to show the effectiveness of the proposed test procedures, we plot the density curve of $\hat{C_L}$ based on 10000 times simulation study and intervals $(\underline{LB}, +\infty)$ based on 200 times simulation study, shown in Figure 3 and Figure 4 respectively. We find that the simulation values of $\hat{C_L}$ are almost between 0.84 and 0.92, larger than $C_0 = 0.7239476$,

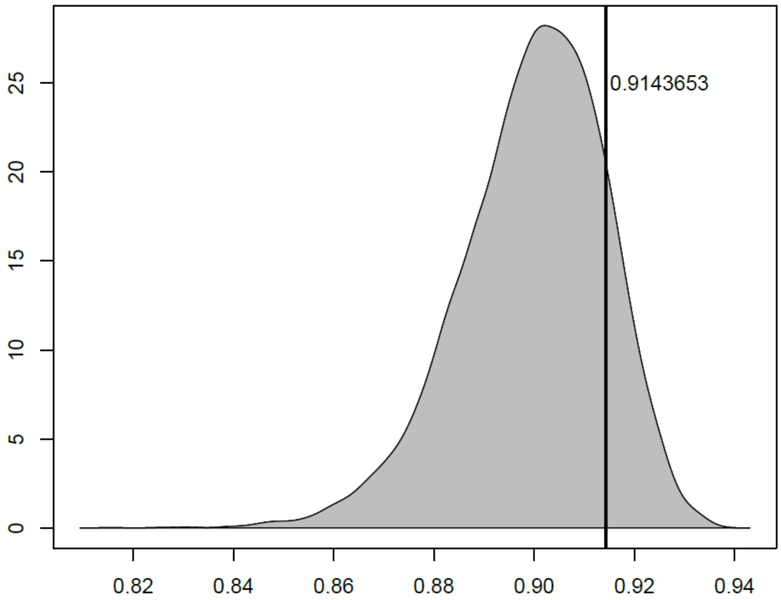

Figure 3. The density curve of $\hat{C}_L$ based on 10000 times simulation study

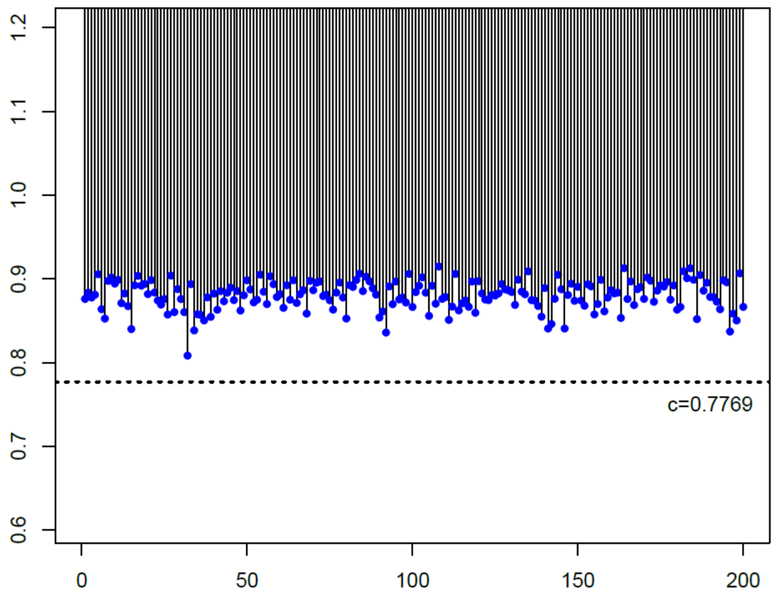

Figure 4. $(\underline{LB}, +\infty)$ calculated based on 200 times simulation study

and $\hat{C}_L = 0.9143653$ falls within this range. In Figure 4, $c = 0.7769$ does not fall into the simulation intervals. The simulation results show that the lifetime performance index of products meets the requirement.

This simulation study shows that based on the data collected in practice, the proposed test procedures can be used in many realms. It is easy to determine whether products, indices and even services are up to standard by assessing the lifetime performance index.

7. **Conclusions.** Process capability analysis is applied widely by many managers or producers to evaluating the process potential and performance. So far, many authors have discussed the estimation of lifetime performance index for common distributions. We focus on the Gompertz distribution, an important distribution in many fields, to investigate

the lifetime performance index. Compared with other studies, we consider the issue in the case of censoring schemes, and use maximum likelihood estimation and the pivotal quantity to construct test procedures. First, the paper shows that the conforming rate P_r of the products and the lifetime performance index C_L for the Gompertz distribution have a one-to-one mathematical relationship. Then, construct the maximum likelihood estimator and develop a testing procedure to assess the performance index with Gompertz distribution with progressively first-failure censored data. Also, we construct a one-side confidence interval to judge whether the products meet the requirement. Finally, we give numerical examples under progressive first-failure censoring. We use the Gini statistic to obtain the best fit value of the shape parameter β based on the real data set. And we present the Monte Carlo simulation study. The outcomes from the real data set as well as simulation data show that our proposed testing procedures have good characteristics. In conclusion, the proposed testing procedures in this paper can effectively determine whether the quality of products meets the required level.

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