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MANUFACTURE ALLOCATION OF THE BATCH PROCESS AND THE CYCLIC FLOW PROCESS VIA STOCHASTIC ANALYSIS

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Abstract. *We report the manufacturing allocation which types are the batch processes (hereafter BP) and the cyclic flow processes (hereafter CFP), in a steady-state production system. It is the most important for the efficient production of small- and medium-sized manufacturing companies to secure profits. We analyze how to determine these two allocations through a stochastic theory and determine the optimal allocation. First, we construct the mathematical models in terms of potential energy for BP and CFP. Next, we discuss the potential in the production field based on the cost potentials. We define for each cost variable and the marginal utility function is derived. However, we discuss the allocation in a steady-state production system by consideration of actual operation. Finally, we report the numerical results for the allocation method.*
Keywords: Manufacture allocation, Potential energy, Fluctuations, Stochastic differential equation, Lagrange function

1. **Introduction.** In small- and medium-sized enterprises, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, with respect to human intervention with outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of human intervention [1, 2, 3, 4]. Naturally, the effect of human behavior is not just a problem with small- and medium-sized companies; it must be regarded as one of the major problems that may occur when humans directly intervene in the production process [5, 6, 7, 8].

Previously, we have reported that by creating a state in which the production density of each process corresponds to physical propagation, the manufacturing process is most appropriately described using a diffusion equation [2]. In other words, if the potential of the production field (stochastic field) is minimized, the equation is defined by the production density function $S_i(x, t)$ and the constraint is described using an advective diffusion equation to determine the transportation speed ρ [2, 10].

To enable efficient application to a production system, we have proposed a mathematical model that focuses on the selection process and production lead time adaptation mechanism. To model the throughput time for a production demand/manufacturing system

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in the manufacturing stage, the dynamic behavior is derived using a lognormal stochastic differential equation. Using this model, the evaluation equation for the compatibility condition production lead time is defined using the risk-neutral integral, and the evaluation formula for the above conditions is calculated. Furthermore, by performing the synchronization process, the throughput for the manufacturing process is reduced [3, 4].

We have been studying throughput improvements and factors in production processes from the viewpoint of physical and mathematical properties. In this study, we represented the analysis of the throughput (lead time) fluctuation in a production system by applying a phase-field model. The factors of this fluctuation are as follows.

- Uncertainty of logistics
- Uncertainty of production planning
- Stochastic characteristics of the order and start time series

Previously, we have described the differences between the synchronous and asynchronous models under CFP and have shown that the throughput of a manufacturing process depends on volatility. Synchronization implies that the machines and assembly lines manufacture the required production volumes in accordance with timing requirements. Moreover, to understand the difference between the asynchronous method, which causes a delay in the manufacturing process, and the synchronous method, which reduces the process throughput time in manufacturing processes, we manufactured equipment [5].

"Synchronization with preprocess" is a manufacturing method used to increase throughput. Synchronization reduces volatility from the start of production until it finishes. The automotive industry has adopted the synchronization process to reduce volatility. We showed that by using our proposed physical approach, we can obtain results similar to those obtained by the synchronization process.

The synchronization process is the best method available. However, because it is difficult to apply in real-world situations, we have proposed a realistic method termed "Synchronization with preprocess". "Synchronization with preprocess" means that by carrying out the reclassification of the working process, it is a method for smoothing the volatility of the working time. In general, the lead times of processes should be set equal to the same value. However, in the "Synchronization with preprocess" method, before starting the manufacturing process, we analyze a particular process and select different lead times. Using this approach, the "Synchronization with preprocess" method can achieve a much better total throughput [3].

In recent years, for example, business models of plant factories have increased significantly [9]. The point of this business model is how to bring out the advantages of plant factories. It is reported that there are cases where the potential of plant factories cannot be exploited. The points of the business model are as follows.

- Efficiency to cope with the price compared to hydroponic vegetables, considering the edible ratio.
- Stability should do the business plan according to the demand fluctuation of the seller, not always a constant production volume.

Typically, the efficiency and stability of production are required in the manufacturing field. We aim to be able to provide customers with stable production at a reasonable cost. In a manufacturing site, the case where it is compelled to use a manufacturing process together according to a change of a worker occurs. Further, we sometimes use a combination of two production methods to maintain efficient production activities. We may be forced to use this combination due to company circumstances.

We describe the two types of production method in Section 2. In this paper, we represent the both stochastic models for BP and CFP and report the manufacturing allocation in a steady-state system. First, we construct the mathematical models in terms of potential energy for BP ($S(t)$) and CFP ($C(t)$). Next, we discuss the potential in the production field based on the cost potentials of $S(t)$ and $C(t)$. We represent the optimal allocation for BP and CFP using Bellman's dynamic programming. We define for each cost variable and the marginal utility function is derived. Therefore, we formulate using Lagrange's undetermined multiplier method to obtain a dynamic optimal allocation. However, we do not seek the dynamic optimal solution. It is not possible to change the system in a normal manufacturing system of real time. Furthermore, the allocation expression of BP and CFP in the steady-state is represented using stochastic theory. Finally, the numerical results are represented for theoretical verification.

2. **Production Systems in the Manufacturing Equipment Industry.** Figure 1 shows the flow of work for manufacturing an ordered product in-house and with a supply chain company for an order received from a customer. Figure 2 shows the image of an in-house organizational system for manufacturing ordered products, keeping delivery times and manufacturing according to specifications. The company decides whether to implement the ordered product manufacturing (Figure 1) with both BP and CFP, or with BP alone, or with CFP alone (Figure 2). This system is considered to be a "Make-to-order system with version control", which enables manufacturing after orders are received from clients, resulting in "volatility" according to its delivery date and lead time. In addition, there is volatility in the lead time, depending on the content of the make-to-order products (production equipment).

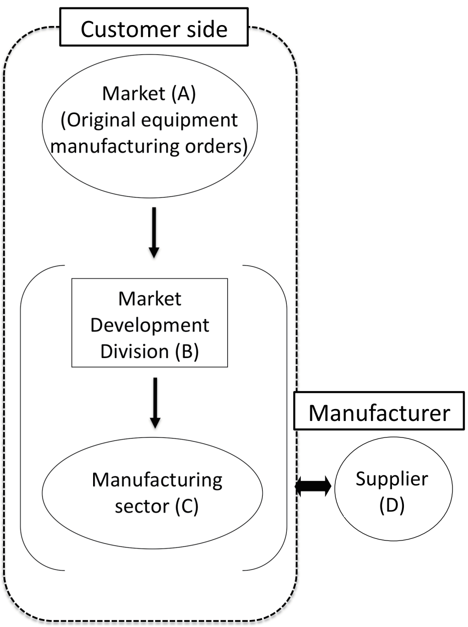

FIGURE 1. Business structure of company of research target

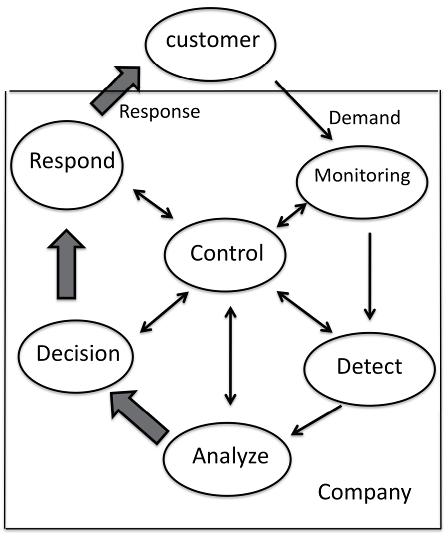

FIGURE 2. Decision-making process within the company

In Figure 1(A), the "Customer side" refers to an ordering company and "Supplier (D)" means the target company in this paper. The product manufacturer, which is the source of the ordered manufacturing equipment presents an order that takes account of the market price. In Figure 1(B), the market development department at the customers factory receives the order through the sale contract based on the predetermined strategy.

Figure 2 illustrates a company's decision-making process. The business monitors perceived demand trends. When a customer order is received, the perceived trend is analyzed.

Based on the analysis, the company is able to decide how to respond to the analyzed demand.

The two types of production methods which are BP and CFP, used in manufacturing equipment are briefly covered in this paper.

- BP – One skilled worker is in charge of all manufacturing processes. Sufficient time is required for the skilled worker, but it is usually difficult to occupy a certain process.
- CFP – This manufacturing process is employed in the production of control equipment. In this example, CFP consists of six stages. In each steps S1-S6 of the manufacturing process, material is being produced. The direction of the arrows represents the direction of CFP. In this process, production materials are supplied through the inlet and the end-product is shipped from the outlet. Figure 3 depicts a manufacturing process that is termed as CFP. For CFP, we make the following two assumptions.

Assumption 2.1. *The production structure is nonlinear.*

Assumption 2.2. *The production structure is a closed structure; that is, the production is driven by CFP.*

Assumption 2.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the throughput generation structure in a stochastic manufacturing process (hereafter called the manufacturing field). Because such a structure is at least dependent on the demand, it is considered to have a nonlinear structure.

Because the value of such a product depends on the throughput, its production structure is nonlinear. Therefore, Assumption 2.1 reflects the realistic production structure and is somewhat valid. Assumption 2.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption 2.2 is reasonable because new production starts from S1. Please refer to Appendix A. With respect to CFP, more information is provided in our report [11].

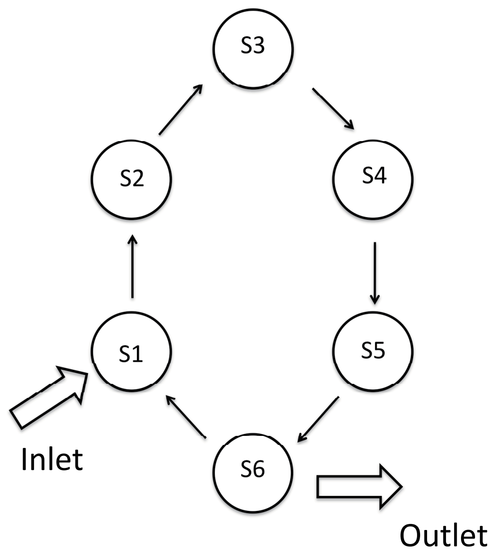

Figure 3. CFP

3. **Potential Energy and Production Cost in Production Process.** Non-productivity generates a static state in the production field. Transition to the dynamic state, modeled by the Hamilton-Jacobi equation, requires excitation energy, which increases the free energy of the system [12].

To retain a profitable business, products must be continually input to the static field. At the same time, sustained input of the order information is required. Figure 4 shows an overview of the production field concept.

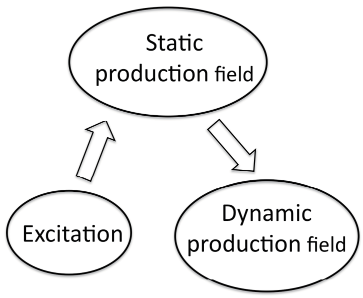

FIGURE 4. Overview of the production field concept

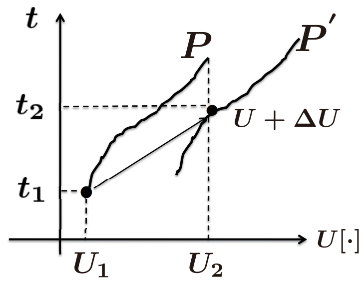

FIGURE 5. Transition from a lower-energy production process to the next process

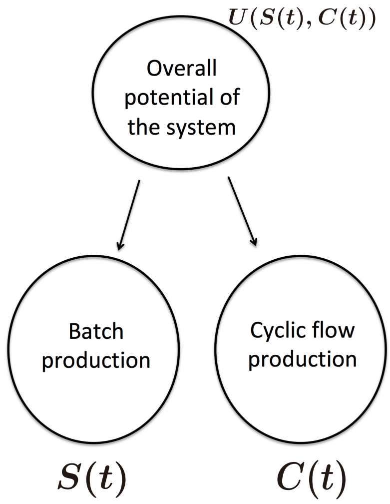

FIGURE 6. Overall potential of the system

The number of production units at each stage of a production unit i shifts over time. To function effectively, a production process requires a minimum number of personnel. This situation constitutes a shortest path problem. Production units can be considered to be physically located in mechanical fixtures. The production dynamics enable a company to profit from its business.

We consider that revenues are generated by the displacement of the potential in the production field. In other words, the entropy increase contributed by the production unit is another source of revenue. This is the principle of maximum entropy [12]. Figure 5 illustrates the transition from a lower-energy production process (energy state P) to the (higher-energy) next process (energy state P').

The potential is defined by considering the fluctuations of the internal state of the system. Therefore, the following is defined.

Definition 3.1. *Potential function $U(S(t), C(t))$*

$$U(S(t), C(t)) \equiv \mathcal{F}(S(t)) + \mathcal{G}(C(t)) + \mathcal{H}(dS(t), dC(t)) \tag{1}$$

where both of $\mathcal{F}(S(t))$ and $\mathcal{G}(C(t))$ are a production unit potential. $\mathcal{H}(dS(t), dC(t))$ is the potential of both $S(t)$ and $C(t)$ fluctuations. $S(t)$ and $C(t)$ are BP cost and CFP cost respectively.

Definition 3.2. *Marginal utility function for each cost variable*

$$\frac{\partial U[\cdot]}{\partial S(t)} : \text{Marginal utility function of BP} \tag{2}$$

$$\frac{\partial U[\cdot]}{\partial C(t)} : \text{Marginal utility function of CFP} \tag{3}$$

The following equation is obtained from Equation (1).

$$\frac{dS(t)}{dt} = \frac{\partial U[S(t), C(t)]}{\partial S(t)} = \frac{\delta \mathcal{F}(S)}{\delta S} + \frac{\delta \mathcal{H}(dS, dC)}{\delta S} \tag{4}$$

$$\frac{dC(t)}{dt} = \frac{\partial U[S(t), C(t)]}{\partial C(t)} = \frac{\delta \mathcal{G}(C)}{\delta C} + \frac{\delta \mathcal{H}(dS, dC)}{\delta C} \tag{5}$$

Since BP and CFP have no correlation with each other, it is assumed that the fluctuation is also linear as follows.

Assumption 3.1.

$$\mathcal{H}(dS, dC) \equiv \mathcal{H}^s(dS) + \mathcal{H}^c(dC) \tag{6}$$

$$\frac{\delta \mathcal{H}(dS, dC)}{\delta S} = \frac{\delta \mathcal{H}^s(dS)}{\delta S} = \frac{\delta \mathcal{H}^s(dS + \epsilon dg_s)}{\delta S} \tag{7}$$

where $g \in L_2(D)$ is a fluctuation function and D denotes a production field.

Moreover, $\mathcal{H}^s(dS + \epsilon dg_s)$ is as follows:

$$\mathcal{H}^s(dS + \epsilon dg) = \mathcal{H}^s(dS) + \epsilon \left\{ \frac{\partial \mathcal{H}^s(dS)}{\partial dS} dg \right\} + o\left((dS)^2\right) \tag{8}$$

where we ignore the derivative term twice $o\left((dS)^2\right)$.

From Gateau derivative, we obtain as follows:

$$d\mathcal{H}^s(d(S, g)) = \lim_{h \to 0} \frac{\mathcal{H}^s(dS + h) - \mathcal{H}^s(dS)}{h} = \frac{\partial \mathcal{H}^d(dS)}{\partial (dS)} dg(S) \tag{9}$$

Then, we replace Equation (9) as follows:

$$d\mathcal{F}^s \Big|_{S \in D} = \frac{\partial \mathcal{H}^s(dS)}{\partial (dS)} dg(S) \tag{10}$$

Equation (10) represents the fluctuation of the production function S.

Similarly, we can obtain for the production function C as follows:

$$d\mathcal{G}^s \Big|_{C \in D} = \frac{\partial \mathcal{H}^c(dC)}{\partial (dC)} dg(C) \tag{11}$$

Therefore, we define the free energy of the overall systems is derived as follows [12].

Definition 3.3. *Free energy of the overall systems*

$$\mathcal{L}(S, C) = \left[\mathcal{F}(S) + \frac{\delta \mathcal{H}^s(dS)}{\delta (dS)} \right] + \frac{K_s}{2} (\nabla S)^2 + \left[\mathcal{G}(C) + \frac{\delta \mathcal{H}^c(dC)}{\delta (dC)} \right] + \frac{K_c}{2} (\nabla C)^2 \tag{12}$$

where K_s and K_c are real parameters.

Here, $\delta\mathcal{G}(C)/\delta C$ represents the cost of production C.

4. **Optimal Problem.** In Section 3, the potential expression for the cost of both BP and CFP in the production system is derived. This is because the fluctuation of the internal state can be considered in the potential. The stochastic model of the overall production cost $W(t)$ is defined as follows.

Definition 4.1. *The stochastic model of the overall production cost* $W(t)$

$$dW(t) = \left[\rho^s \frac{\delta\mathcal{F}(S)}{\delta S} S + \rho^c \frac{\delta\mathcal{G}(C)}{\delta C} C + f(Q(S)) + g(Q(C)) + p^s q(t) - \kappa R(t)\right] \tag{13}$$

where $\delta\mathcal{F}(S)/\delta S$ *and* $\delta\mathcal{G}(C)/\delta C$ *are the marginal cost of* $S(t)$ *and marginal cost of* $C(t)$ *respectively.* ρ^s *and* ρ^c *are the parameters of marginal cost* $S(t)$ *and* $C(t)$ *respectively.* $Q(S)$ *and* $Q(C)$ *are a production cost for* S *and a production cost for* C *respectively.* $q(t)$ *is an outsourcing cost with external suppliers.* p^s *is the parameter of outsourcing cost.* $R(t)$ *is a profit function.* κ *is a parameter of profit function.*

Then, we make the following assumption. The reason is to make it an optimal problem regarding the production function $S(t)$. $C(t)$ is a production function, and there is no logical disadvantage even if it is considered as a function of the same type as $S(t)$.

Assumption 4.1.

$$C(t) \equiv k_s S(t) \tag{14}$$

$$g(Q(C)) \equiv k_g f(Q(S)) \tag{15}$$

where k_s *(*$0 < k_s < 1$*) and* k_g *(*$0 < k_g < 1$*) are a production ratio and a production cost ratio respectively.*

From Equations (14) and (15), we can replace it into the optimal problem on S as follows:

$$\begin{aligned} dW(t) = \max_{\{S\}} \Bigg[& \left\{\rho^s \frac{\delta\mathcal{F}(S)}{\delta S} + \rho^c \frac{\delta\mathcal{G}(C)}{\delta C} k_s\right\} S(t) + (1 + k_g) f(Q(S)) + p^s q(t) \\ & - \kappa R(t)\Bigg] dt \end{aligned} \tag{16}$$

Further, we replace $f(Q(S))$ as follows:

$$f(Q(S)) \equiv f_s \frac{\partial Q(S)}{\partial S} S \tag{17}$$

From Equation (17), Equation (16) is replaced as follows:

$$\begin{aligned} dW(t) = \Bigg[& \left\{\rho^s \frac{\delta\mathcal{F}(S)}{\delta S} + \rho^c \frac{\delta\mathcal{G}(k_s S)}{\delta k_s S} k_s\right\} S(t) + (1 + k_g) f_s \frac{\partial Q(S)}{\partial S} S + p^s q(t) \\ & - \kappa R(t)\Bigg] dt \end{aligned} \tag{18}$$

Here, we introduce the marginal cost of BP (C_f), the marginal cost of CFP (C_m) and the production cost for S. These variables are derived as follows:

$$C_f \equiv \rho^s \frac{\delta\mathcal{F}(S)}{\delta S} \tag{19}$$

$$C_m \equiv \rho^c \frac{\delta\mathcal{G}(k_s S)}{\delta k_s S} \tag{20}$$

$$C_g \equiv (1 + k_g) f_s \frac{\partial Q(S)}{\partial S} \tag{21}$$

$$\begin{aligned} dW(t) &= [(C_f + C_m)S + C_g S + p^s q(t) - \kappa R(t)]\, dt \\ &= [(C_f + C_m + C_g)S + p^s q(t) - \kappa R(t)]\, dt \end{aligned} \tag{22}$$

At this time, the cost ratio of S (BP) to the overall production cost W is defined as the following equation.

Definition 4.2. *Cost ratio of S (BP) to the overall production cost W*

$$\phi = \frac{S}{W} \tag{23}$$

From Equation (23), Equation (22) is derived as follows:

$$\begin{aligned} dW(t) &= [(C_f + C_m)S + C_g S + p^s q(t) - \kappa R(t)]\, dt \\ &= [\phi(C_f + C_m + C_g)W(t) + p^s q(t) - \kappa R(t)]\, dt \end{aligned} \tag{24}$$

Since $q(t)$ can be regarded as a function of S (BP), $q(t)$ is defined as follows.

Definition 4.3. *$q(t)$ is a linear function $h(S)$ of S (BP)*

$$q(t) = h(S) \tag{25}$$

Once again, the overall production cost model considering revenue and the model of overall cost only are as follows.

- Overall production cost model considering revenue

$$dW(t) = [\phi(C_f + C_m + C_g)S + p^s q(t)W(t) - \kappa R(t)]\, dt \tag{26}$$

- Overall cost only model

$$dW(t) = [\phi(C_f + C_m + C_g)S + p^s q(t)W(t)]\, dt \tag{27}$$

Then, we introduce the variables considering BP and CFP allocation ratio respectively.

- Allocation ratio ϕ_s of BP

$$\phi_s = \frac{S}{W} \tag{28}$$

- Allocation ratio ϕ_c of CFP

$$\phi_c = \frac{C}{W} \tag{29}$$

From Equations (28) and (29), we obtain as follows:

$$dW(t) = \left[\phi_s \left\{C_f + f_s \frac{\partial Q(S)}{\partial S}\right\} W(S) + \phi_c \left\{C_m + f_s \frac{\partial Q(C)}{\partial C}\right\} W(t) - \kappa R(t)\right] dt \tag{30}$$

where $\phi_s + \phi_c = 1$. ϕ_s and ϕ_c represent the allocation ratio of $S(t)$ (BP) and $C(t)$ (CFP) to $W(t)$.

A Winner process is added to treat Equation (31) in a strict sense as a stochastic differential equation model as follows:

$$\begin{aligned} dW(t) = &\left[\phi_s \left\{C_f + f_s \frac{\partial Q(S)}{\partial S}\right\} W(t) + \phi_c \left\{C_m + f_c \frac{\partial Q(C)}{\partial C}\right\} W(t) - \kappa R(t)\right] dt \\ &+ \sigma W(t) dB(t) \end{aligned} \tag{31}$$

where σ and $B(t)$ are a volatility and Winner process respectively.

We replace as in the following equation.

$$G_s = C_f + f_s \frac{\partial Q(S)}{\partial S} \tag{32}$$

$$G_c = C_m + f_c \frac{\partial Q(C)}{\partial C} \tag{33}$$

From Equations (32) and (33), we obtain as follows:

$$dW(t) = \left[\left\{ \phi_s G_s + \phi_c G_c \right\} W(t) - \kappa R(t) \right] dt + \sigma W(t) dB(t) \tag{34}$$

where $\phi_s + \phi_c = 1$.

$$\begin{aligned} dW(t) &= \left[[\phi_s . \phi_c] \begin{bmatrix} G_s \\ G_c \end{bmatrix} [W(t)] - \kappa R(t) \right] dt + \sigma W(t) dB(t) \\ &= \left[\sum_{n=1}^{2} \phi_n G_n W(t) - \kappa R(t) \right] dt + \sigma W(t) dB(t) \end{aligned} \tag{35}$$

where $\sum_{n=1}^{2} \phi_n = 1$. ϕ_1, ϕ_2, G_1 and G_2 are as follows

$$\phi_1 \equiv \phi_s, \quad \phi_2 \equiv \phi_c \tag{36}$$

$$G_1 \equiv G_s, \quad G_2 \equiv G_c \tag{37}$$

Equation (35) represents the stochastic allocation model of BP cost and CFP cost with respect to the overall production cost [13].

Here, the allocation coefficient ϕ_i is determined based on the stochastic allocation model of Equation (35).

Definition 4.4. *Cost functional $J[R(t), W(t), t]$*

$$J[R(t), W(t), t] = E\left[\int_0^T U[R(t)] dt + D(W(t), T) \right] \tag{38}$$

where $U[R(t)]$ represents the utility function.

Definition 4.5. *Evaluation function $V(W, t)$*

$$V(W, t) = \max_{R(\tau), \phi(\tau)} E\left[\int_0^T U(R(\tau)) d\tau + D(W(\tau), T) \middle| W(t) \right] \tag{39}$$

From Bellman's dynamic programming, we represent as follows:

$$\max_{R(\tau), \phi(\tau)} \{ U(R(t), t) + \mathcal{L}_W [V(W, t)] \} \tag{40}$$

where $\mathcal{L}_W$ satisfies the following equation.

$$\mathcal{L}_W = \frac{\partial V}{\partial t} + \left[\sum_{i=1}^{2} \phi_i G_i W(t) - R(t) \right] \frac{\partial V}{\partial W} + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \left[\sum_{i=1}^{2} \phi_i G_i W(t) \right]^2 \tag{41}$$

where the boundary condition is as follows:

$$V(W, T) = D(W(\tau)) \tag{42}$$

From the above description, Equation (40) is found the exact solution under the boundary condition Equation (42) based on the above constraint $\sum_{n=1}^{2} \phi_n = 1$.

Further, $U[R(t)]$ in Equation (38) is set as the following equation for simplicity.

$$U[R(t)] = \sqrt{R(t)} \tag{43}$$

Then, we substitute as follows.

$$f(\phi(t), \eta(t)) = V_t + V_W \cdot \sum_{i=1}^{2} [\phi_i G_i(t) - \eta(t)] + \frac{1}{2} V_{W \cdot W} W^2 \sum_{i=1}^{2} \phi_i^2 \sigma^2 + \sqrt{\eta(t)} \tag{44}$$

where $\eta(t) = \frac{R(t)}{W(t)}$ and also represents the rate of return. V_W and $V_{W \cdot W}$ represent the one-time partial derivative and two-time partial derivative respectively.

Therefore, Equation (40) is defined as follows.

Definition 4.6.

$$\max_{\eta(t),\phi(t)} \{U(\eta(t),t) + \mathcal{L}_W[V(W(t),t)]\} \tag{45}$$

We define the following Lagrange function to find the optimal solution.

Definition 4.7.

$$\mathcal{L}(\cdot) = f(\phi(t),\eta(t)) + \lambda \left[1 - \sum_{i=1}^{2} \phi_i(t)\right] \tag{46}$$

where λ is an undetermined multiplier.

The condition of the optimality is as follows under Lagrange's unknown multiplier method.

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \phi_i} = 0 \tag{47}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \eta(t)} = 0 \tag{48}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} = 1 - \sum_{i=1}^{2} \phi_i(t) = 0 \tag{49}$$

$i = 1$ and $i = 2$ represent BP and CFP respectively, and $f_i(\varphi_i)$ represents the throughput probability in Figure 7.

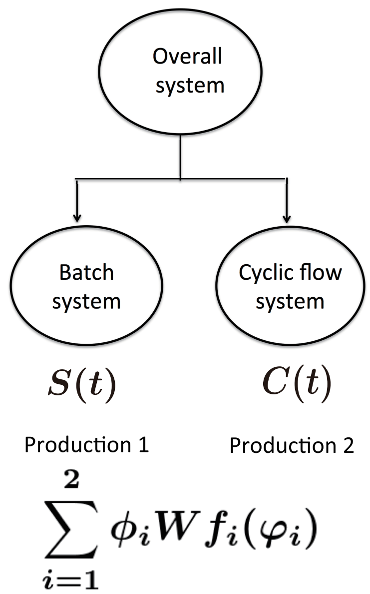

FIGURE 7. Steady-state process of BP and CFP

5. **Steady-State Production Allocation.** In Section 4, based on the potential obtained for both BP and CFP, the model expression of the whole production cost is represented. From the total production cost expression, the optimal allocation is determined by Lagrange multiplier method in order to obtain the optimal allocation using Bellman's dynamic programming. We extend to the steady-state system based on our thinking

described above. A real-time system change cannot be done in a dynamic production system. Therefore, the problem of production allocation in stationary systems is practically important.

We define the evaluation function for maximizing the stochastic throughput is defined to determine the allocation coefficient as follows [13].

Definition 5.1. *Evaluation function* $V(\theta)$

$$V(\theta) = \max_{\phi_i \in \theta} E\left[\sum_{i=1}^{2} \phi_i f_i(\varphi_i) W\right] = \max_{\phi_i \in \theta} E[\boldsymbol{f}(\theta) W] \tag{50}$$

where $\boldsymbol{f}$ *is as follows:*

$$\boldsymbol{f}(\theta) \equiv \{f_1(\varphi_1), f_2(\varphi_2)\} \tag{51}$$

We define the following Lagrangian function to maximize the evaluation function $V(\theta)$.

Definition 5.2. *Lagrangian function* $\mathcal{L}(\phi, \lambda)$

$$\mathcal{L}(\phi, \lambda) = \hat{V}(\boldsymbol{\mu}, \boldsymbol{\sigma}) + \lambda\left[1 - \sum_{i=1}^{2} \phi_i\right] \tag{52}$$

where $\hat{V}(\boldsymbol{\mu}, \boldsymbol{\sigma})$ *is as follows:*

$$\hat{V}(\boldsymbol{\mu}, \boldsymbol{\sigma}) = E[U(\boldsymbol{f})(\theta)] \tag{53}$$

From Equation (52), we obtain as follows:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \hat{V}_{\boldsymbol{\mu}} + 2\hat{V}_{\boldsymbol{\sigma}} + \frac{\partial}{\partial \phi_i}\left[\lambda\left\{1 - \sum_{i=1}^{2} \phi_i\right\}\right] = 0 \tag{54}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum_{i=1}^{2} \phi_i = 0 \tag{55}$$

$\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are represented as follows:

$$\mu_\phi = E[\boldsymbol{f}(\theta)] = E\left[\sum_{i=1}^{2} f_i(\varphi_i)\phi_i\right] = \boldsymbol{\mu} \tag{56}$$

$$\boldsymbol{\mu} = \left\{\overline{f_1(\varphi)}, \overline{f_2(\varphi)}\right\} = \{\mu_1\phi_1, \mu_2\phi_2\} \tag{57}$$

where $\bar{\bullet}$ denotes an average.

$$\sigma_\phi = V_{ar}\left[\sum_{i=1}^{2} f_i(\varphi_i)\phi_i\right] = \sum_{i,j=1}^{2} \phi_i\phi_j\sigma_{ij} = \boldsymbol{\sigma}^2 \tag{58}$$

Then, the dynamic mathematical models for reference are derived as follows:

$$df_1(t) = \mu_1 f_1(t)dt + \sigma_1 f_1(t)dZ(t) \tag{59}$$

$$df_2(t) = \mu_2 f_2(t)dt + \sigma_2 f_2(t)dZ(t) \tag{60}$$

where $Z(t)$ denotes a Wiener process.

From Equation (54), we obtain as follows:

$$\hat{V}_\mu \cdot \mu_i\phi_i + 2\hat{V}_{\sigma^2}\sigma_{ij}\phi_i - \lambda = 0 \tag{61}$$

where $\sigma_{ij} \neq 0$ $(i \neq j)$.

Now, assume the utility function $U[\boldsymbol{f}(\theta)]$ as follows [13].

Definition 5.3.

$$U[\boldsymbol{f}(\theta)] = -\exp(-\nu \boldsymbol{f}) \tag{62}$$

where $\nu > 0$.

Assuming that $\boldsymbol{f}(\theta)$ follows a normal distribution, the expectation value is as follows [13].

$$\hat{V}(\mu, \sigma) = E[U(\boldsymbol{f})] = \exp\left(-\nu\mu + \frac{1}{2}\nu^2\sigma^2\right) \tag{63}$$

From Equation (61), we obtain as follows:

$$\hat{V}_{\boldsymbol{\mu}} f_i(\varphi_i) + 2\hat{V}_{\sigma^2}\sigma_{ii}\phi_i - \lambda = 0 \tag{64}$$

Therefore, ϕ_i is derived as follows:

$$\phi_i = \frac{\lambda\sigma_{ii}^{-1} - \hat{V}_\mu f_i(\varphi_i)\sigma_{ii}^{-1}}{2\hat{V}_{\sigma^2}} \tag{65}$$

where $\sigma = 0$ $(i \neq j)$.

Moreover, λ is derived as follows:

$$\lambda = \frac{2\hat{V}_{\sigma^2}}{\sum\limits_{i=1}^{2}\sigma_{ii}^{-1}} + \frac{\hat{V}_\mu \sum\limits_{i=1}^{2} f_i(\varphi_i)\sigma_{ii}^{-1}}{\sum\limits_{i=1}^{2}\sigma_{ii}^{-1}} \tag{66}$$

Hence, ϕ_i is derived as follows [13].

$$\phi_i = \frac{\sigma_{ii}^{-1}}{\sum\limits_{i=1}^{2}\sigma_{ii}^{-1}}\left[1 - \frac{1}{\nu}\left\{\sum_{i=1}^{2}\mu_i\sigma_{ii}^{-1} - \mu_i\sum_{i=1}^{2}\sigma_{ii}^{-1}\right\}\right] \tag{67}$$

where the following replacements have been made.

$$\frac{\hat{V}_\mu}{2\hat{V}_{\sigma^2}} = \frac{1}{\nu} \tag{68}$$

6. **Numerical Examples.** In Section 5, the optimal work allocation of BP and CFP in steady-state is obtained. This is because, by obtaining the allocation in the steady-state, it can be used as the allocation suitable for the actual manufacturing operation. Therefore, this section seeks and verifies the optimal allocation based on numerical examples. In our past research using the risk sensitive control method previously, the synchronous system (when there is little variation among workers in each process) and the asynchronous system (when there is large variation among workers in the process), the synchronous method has 0.8, the asynchronous method reported the allocation of 0.2 [14]. Normally all orders can be handled in a synchronous method. However, our company is not only a skilled worker. In this paper, there are no reports of allocation of BP (0.2) and CFP (0.8) in the past. The production allocation was 0.8(CFP) : 0.2(BP). Therefore, the usefulness of the method of this paper was confirmed.

We introduce the following numerical results obtained by calculation using Equation (67).

Table 1 shows the example when the average and the volatility are set to different values. However, Table 2 shows the example in which the average and the volatility are set to almost the same value. As a matter of course, if the average and the volatility are set to exactly the same value, then $\phi_1 = 0.5$ and $\phi_2 = 0.5$. It can be seen in Table 3 that if the volatilities (σ_1 or σ_2) are the same, there are many allocations of systems with large

TABLE 1. Case-1

	μ_1	σ_1 $(=\sigma_{11})$	μ_2	σ_2 $(=\sigma_{22})$	ϕ_1	ϕ_2
$\nu=10$	0.9	0.1	0.7	0.3	0.8	0.2
$\nu=20$	0.9	0.1	0.7	0.3	0.77	0.23

TABLE 2. Case-2

	μ_1	σ_1 $(=\sigma_{11})$	μ_2	σ_2 $(=\sigma_{22})$	ϕ_1	ϕ_2
$\nu=10$	0.9	0.3	0.7	0.3	0.53	0.47
$\nu=20$	0.9	0.3	0.7	0.3	0.52	0.48

TABLE 3. Case-3

	μ_1	σ_1 $(=\sigma_{11})$	μ_2	σ_2 $(=\sigma_{22})$	ϕ_1	ϕ_2
$\nu=10$	0.9	0.1	0.7	0.3	0.8	0.2
$\nu=10$	0.8	0.1	0.7	0.2	0.7	0.3
$\nu=10$	0.7	0.1	0.7	0.25	0.71	0.37
$\nu=10$	0.4	0.3	0.6	0.2	0.36	0.64
$\nu=10$	0.9	0.3	0.7	0.1	0.3	0.7

trends (μ_1 or μ_2). On the other hand, when the trend is large but the volatility is large, there are many allocation of systems with small volatility.

From Equation (67), ϕ_1 and ϕ_2 are as follows:

$$\phi_1 = \frac{(\sigma_1)^{-1}}{(\sigma_1)^{-1}+(\sigma_2)^{-1}} \times \left\{1 - \frac{1}{\nu} \times \left\{\mu_1(\sigma_1)^{-1} + \mu_2(\sigma_2)^{-1}\right\} - \mu_1\left\{(\sigma_1)^{-1}+(\sigma_2)^{-1}\right\}\right\} \quad (69)$$

$$\phi_2 = \frac{(\sigma_2)^{-1}}{(\sigma_1)^{-1}+(\sigma_2)^{-1}} \times \left\{1 - \frac{1}{\nu} \times \left\{\mu_1(\sigma_1)^{-1} + \mu_2(\sigma_2)^{-1}\right\} - \mu_2\left\{(\sigma_1)^{-1}+(\sigma_2)^{-1}\right\}\right\} \quad (70)$$

7. **Conclusion.** There are cases where it is unavoidable to use BP method together to secure corporate profits. We presented the theoretical analysis on the kind of production system from the viewpoint of potential. Furthermore, we could present the numerical examples of production allocation. We were able to obtain numerical verification similar to the results of the theoretical methods we reported and the results of the risk sensitive methods. Therefore, we were able to confirm that our report was useful.

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Appendix A. **Analysis of the Test-run results.**

- (Test-run1): Because the throughput of each process (S1-S6) is asynchronous, the overall process throughput is asynchronous. In Table 5, we list the manufacturing time (min) of each process. In Table 6, we list the volatility in each process performed by the workers. Finally, Table 5 lists the target times. The theoretical throughput is obtained as $3 \times 199 + 2 \times 15 = 627$ (min). In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. In Figure 8, we plot the measurement data listed in Table 5, which represents the total working time of each worker (K1-K9). In Figure 9, we plot the data contained in Table 5, which represents the volatility of the working times.
- (Test-run2): Set to synchronously process the throughput. The target time listed in Table 7 is 500 (min), and the theoretical throughput (not including the synchronization idle time) is 400 (min). Table 8 presents the volatility of each working process (S1-S6) for each worker (K1-K9).
- (Test-run3): Introducing a preprocess stage. The process throughput is performed synchronously with the reclassification of the process. As shown in Table 9, the theoretical throughput (not including the synchronization idle time) is 400 (min). Table 10 presents the volatility of each working process (S1-S6) for each worker (K1-K9). On the basis of these results, the idle time must be set to 100 (min). Moreover, the theoretical target throughput $\left(T_s'\right)$ can be obtained using the "Synchronization with preprocess" method. This goal is as follows:

$$\begin{aligned} T_s &\sim 20 \times 6 \text{ (First cycle)} + 17 \times 6 \text{ (Second cycle)} + 20 \times 6 \text{ (Third cycle)} \\ &\quad + 20 \text{ (Previous process)} + 8 \text{ (Idle time)} \\ &\sim 370 \text{ (min)} \end{aligned} \tag{71}$$

The full synchronous throughput in one stage (20 min) is

$$T_s' = 3 \times 120 + 40 = 400 \text{ (min)} \tag{72}$$

Using the "Synchronization with preprocess" method, the throughput is reduced by approximately 10%. Therefore, we showed that our proposed "Synchronization with preprocess" method is realistic and can be applied in flow production systems. Below, we represent for a description of the "Synchronization with preprocess".

In Table 9, the working times of the workers K4, K7 show shorter than others. However, the working time shows around target time. Next, we manufactured one piece of equipment in three cycles. To maintain a throughput of six units/day, the production throughput must be as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25 \text{ (min)} \tag{73}$$

where the throughput of the preprocess is set to 20 (min). In Equation (73), the value 28 represents the throughput of the preprocess plus the idle time for synchronization. Similarly, the number of processes is 8 and the total number of processes is 9 (8 plus the preprocess). The value of 60 is obtained as 20 (min) × 3 (cycles).

In Table 4, Test-run3 indicates a best value for the throughput in the three types of theoretical working time. Test-run2 is ideal production method. However, because it is difficult for talented worker, Test-run3 is a realistic method.

Table 4. Correspondence between the table labels and the Test-run number

	Table number	Production process	Working time	Volatility
Test-run1	Table 5	Asynchronous process	627 (min)	0.29
Test-run2	Table 7	Synchronous process	500 (min)	0.06
Test-run3	Table 9	"Synchronization with preprocess" method	470 (min)	0.03

Table 5. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181
Deviation		27	39	54	30	29	36

Table 6. Volatility of Table 5

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

The results are as follows. Here, the trend coefficient, which is the actual number of pieces of equipment/the target number of equipment, represents a factor that indicates the degree of the number of pieces of manufacturing equipment.

Test-run1: 4.4 (pieces of equipment)/6 (pieces of equipment) = 0.73
Test-run2: 5.5 (pieces of equipment)/6 (pieces of equipment) = 0.92
Test-run3: 5.7 (pieces of equipment)/6 (pieces of equipment) = 0.95
Volatility data represent the average value of each Test-run.

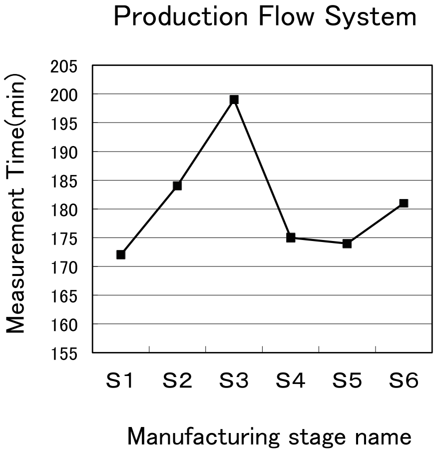

FIGURE 8. Total work time for each stage (S1-S6) in Table 5

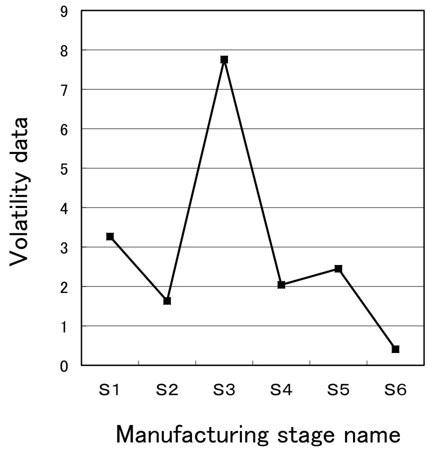

FIGURE 9. Volatility data for each stage (S1-S6) in Table 5

TABLE 7. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180
Deviation		12	16	2	3	2	0

TABLE 8. Volatility of Table 7

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 9. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	21	21	21	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	22	22	20	20	20	20
K9	20	25	25	25	20	20	20
Total	180	165	164	161	180	180	180
Deviation		−15	−16	-19	0	0	0

TABLE 10. Volatility of Table 9

K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.3	3	3	0	0	0
K5	1.3	1.3	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.3	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0