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STATE SPACE DESIGN METHOD FOR UNKNOWN INPUT OBSERVERS

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Abstract. *The unknown input observer has been employed to estimate the state variable of the plant in the presence of unknown input. Initially, the unknown input observer is examined by Kudva, Viswanadham and Ramakrishna. Since then, several papers have investigated designing unknown input observers, and have revealed that the unknown input observer for the plant $(A, B, C, 0)$ can be designed if and only if the following two conditions hold true: (c1)* $\operatorname{rank} CB = \operatorname{rank} B$ *and (c2) the plant $(A, B, C, 0)$ has no invariant zero in the closed right half plane. As those conditions are often restrictive, various studies have been dedicated to designing unknown input observers, even if the above two conditions are not satisfied. In this paper, we newly consider designing unknown input observers for non-minimum phase plants by focusing on the intended bandwidth of the control system. The proposed design methods are derived based on the parametrization of all state observers, and do not require the restrictive conditions (c1) and (c2). Numerical examples are provided to examine the effectiveness of the proposed design methods.*
Keywords: Unknown input observer, Non-minimum phase plant, Parametrization of all state observers

1. **Intoroduction.** State observers are employed to estimate the unavailable state variable of the controlled plant. The state observer theory is initially established by Luenberger [1, 2, 3]. Subsequently, the parametrization of all state observers [4] and of all linear functional observers [5] is derived. The design methods in [1, 2, 3, 4, 5] require the access to the control input for estimating the state variable. In contrast, in this paper, we address design methods of unknown input observers, which are state observers independent of the control input.

In some cases such as estimation of an IC engine torque [6] and velocity and angle of planar gantry crane [7], both the state variable and the control input are unavailable, and state observers are required to estimate the state variable using only the measured output. Such a state observer is called the unknown input observer. That is, the unknown input observer has been used to estimate the state variable of the plant in the presence of unknown input. In addition, the unknown input observer is applied to the systems

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with nonlinearities or time-varying parameters [25, 26]. Initially, the unknown input observer is examined by Kudva et al. [8]. Since then, several papers have been published to design unknown input observers [9, 10, 11]. According to these papers, the unknown input observer for the plant $(A, B, C, 0)$ can be designed if and only if the following two conditions hold true: (c1) rank CB = rank B and (c2) the plant $(A, B, C, 0)$ has no invariant zero in the closed right half plane. The first condition (c1) implies that the number of outputs should be greater than or equal to that of inputs. The second condition (c2) means that the controlled plant is of non-minimum phase. Since the conditions (c1) and (c2) are rather restrictive, a number of authors have considered designing unknown input observers by relaxing the conditions (c1) and (c2). [12, 13, 14, 15] considered the problem of designing unknown input observers without requiring the first condition (c1). Unfortunately, the design methods in [12, 13, 14, 15] cannot be applied if the second condition (c2) fails. In contrast, [16, 17] tackled approximately lifting both of the first and second conditions (c1) and (c2) via the minimal polynomial bases approach and eigenstructure assignment approach, respectively. Those approaches are highly algebraic and it is difficult to intuitively tune the input-output characteristics of the resulting control systems. [24] proposes to augment the controlled plant with a low-pass filter so that the augmented controlled plant satisfies the conditions (c1) and (c2). The design method in [24] requires to increase the number of sensors for measuring the overall output of the augmented plant, and hence from a cost-aware point of view it is not readily employed when the original plant is given.

In this paper, we propose alternative design methods of unknown input observers for non-minimum phase plants, such that they are handily applicable when the intended bandwidth of the control system is specified. The proposed design methods do not require neither of the conditions (c1) and (c2) nor plant augmentation [24]. The idea behind that is as follows: If the controlled plant is intended to function in the lower-frequency range, the control input mainly contains low-frequency-range signal components, that is, the control input decays in the higher frequency range. Note that, if we design a state observer discarding the high-frequency-range signal components of the control input, then the resulting state observer works as an unknown input observer. In order to embody this idea, we utilize the parametrization of all state observers [4, 5] and stable left filtered inverses [20, 21] as underlying techniques. It is shown that the stable left filtered inverses [20, 21] enable to determine the Youla parameter in the state observer parametrization so that the resulting observer works as an unknown input observer. Furthermore, as a complement to the proposed design methods, we describe that the resulting unknown input observers can be employed for constructing output feedback control systems if it is combined with the H_∞ state feedback control [22].

This paper is organized as follows. In Section 2, the problem considered in this paper is formulated. In Section 3, we derive the unknown input observer design methods. In Section 4, we construct an output feedback control by employing the proposed unknown input observer. In Section 5, the features of the resulting control systems are illustrated through numerical examples. In Section 6, the present contributions are summarized.

2. Problem Formulation.

Consider the plant written by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^m$ is the measured output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$. The transfer function from the control input $u(s)$ to the measured output $y(s)$ in (1) is denoted by

$$y(s) = G(s)u(s), \tag{2}$$

where

$$G(s) = C(sI - A)^{-1}B \in R^{m\times p}(s), \tag{3}$$

where $R(s)$ denotes the set of real-rational transfer functions. It is assumed that (A, B) is stabilizable, (C, A) is detectable, and $G(s)$ is of full row normal rank:

$$\text{rank}\, G(s) = p, \tag{4}$$

and has no invariant zero on the imaginary axis. Furthermore, this paper focuses on the situation that the frequency component range of the control input $u(j\omega)$ is limited to $0 \le \omega \le \omega_{\max}$, where $\omega_{\max}$ specifies the maximum frequency component of the control input $u(j\omega)$. We note that $G(s)$ is allowed to have some invariant zeros in the open right half plane, that is, $G(s)$ is allowed to be of non-minimum phase.

When the state variable $x(t)$ in (1) is not available, as is the case in many practical control problems, we employ state observers, which estimate the state variable $x(t)$ in (1) utilizing the available information on $y(t)$ and $u(t)$. The general form of state estimates based on the available information on $y(t)$ and $u(t)$ is given as follows (Figure 1):

$$\xi(s) = F_1(s)y(s) + F_2(s)u(s), \tag{5}$$

where $\xi(t) \in \mathbb{R}^n$ is the estimate of the state variable $x(t)$. The transfer functions $F_1(s) \in R^{n\times m}(s)$ and $F_2(s) \in R^{n\times p}(s)$ in (5) are required to satisfy the condition

$$\lim_{t\to\infty} (x(t) - \xi(t)) = 0. \tag{6}$$

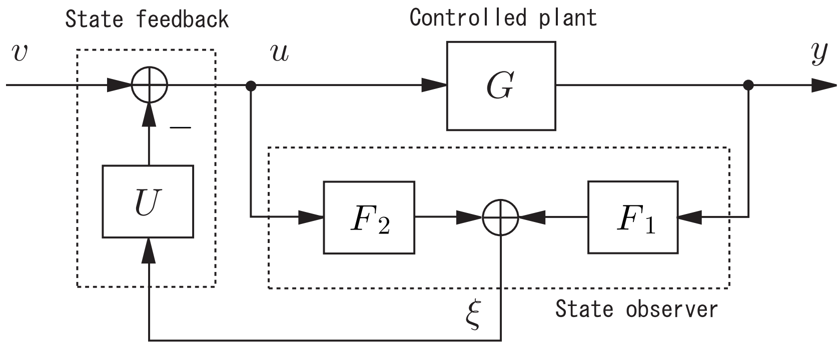

Figure 1. Configuration of control system

In some cases, not only the state variable $x(t)$ but also the control input $u(t)$ are unavailable and the state observer (5) depends only on the measured output $y(t)$. Such a state observer is called an unknown input (state) observer. The purpose of this paper is to propose alternative design methods of unknown input observers for the non-minimum phase plant (1).

3. **Design Methods of Unknown Input Observers.** In this section, we describe the unknown input observer design methods. According to [4, 5], the parametrization of all state observers in (5) for the plant $G(s)$ in (1) is written by

$$F_1(s) = (sI - A + BU)^{-1}BX(s) + Q(s)\tilde{D}(s) \tag{7}$$

and

$$F_2(s) = (sI - A + BU)^{-1}BY(s) - Q(s)\tilde{N}(s), \tag{8}$$

where $U \in \mathbb{R}^{p\times n}$ makes $A - BU$ have no eigenvalue in the closed right half plane. Furthermore, $\tilde{N}(s) \in RH_\infty^{m\times p}$ and $\tilde{D}(s) \in RH_\infty^{m\times m}$ are coprime factors of $G(s)$ on RH_∞ (i.e., the set of stable real-rational functions) satisfying

$$G(s) = \tilde{D}^{-1}(s)\tilde{N}(s) = N(s)D^{-1}(s), \tag{9}$$

where $X(s) \in RH_\infty^{p\times m}$ and $Y(s) \in RH_\infty^{p\times p}$ are functions satisfying

$$\begin{aligned}\begin{bmatrix} Y(s) & X(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix}\begin{bmatrix} D(s) & -\tilde{X}(s) \\ N(s) & \tilde{Y}(s) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} D(s) & -\tilde{X}(s) \\ N(s) & \tilde{Y}(s) \end{bmatrix}\begin{bmatrix} Y(s) & X(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix}\end{aligned} \tag{10}$$

and $Q(s)$ is an arbitrary function in $RH_\infty^{n\times m}$.

From (8), it is observed that if there exists $Q(s) \in RH_\infty^{n\times m}$ satisfying

$$(sI - A + BU)^{-1}BY(s) - Q(s)\tilde{N}(s) = 0, \tag{11}$$

we can obtain an unknown input observer. However, in order to find $Q(s) \in RH_\infty^{n\times m}$ satisfying (11), $G(s)$ must be of minimum-phase.

In order to design unknown input observers for non-minimum phase plants, we adopt the following idea. If we design $Q(s) \in RH_\infty^{n\times m}$ such that

$$Q(j\omega)\tilde{N}(j\omega) \simeq (j\omega I - A + BU)^{-1}BY(j\omega) \ \ (0 \le \forall\, \omega \le \omega_{\max}), \tag{12}$$

then

$$F_2(j\omega) \simeq 0 \ \ (0 \le \forall\, \omega \le \omega_{\max}) \tag{13}$$

holds true. Together with (13), the assumption that the frequency component range of the control input $u(t)$ is limited to $0 \le \omega \le \omega_{\max}$ implies

$$\bar{u}(t) = \mathcal{L}^{-1}\{F_2(s)u(s)\} \simeq 0, \tag{14}$$

where $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transformation. Therefore, when $Q(s)$ is settled to satisfy (12), the state estimate $\xi(s)$ in (5) reduces to

$$\xi(s) = F_1(s)y(s) \tag{15}$$

with $F_1(s)$ defined by (7) working as an unknown input observer.

Hence in the rest of this section, we consider designing $Q(s) \in RH_\infty^{n\times m}$ which satisfies (12). Specifically, we propose to settle $Q(s)$ so that the following condition is satisfied:

$$Q(s)\tilde{N}(s) = (sI - A + BU)^{-1}BY(s)G_K(s)Q_l(s), \tag{16}$$

where $G_K(s) \in RH_\infty^{p\times p}$ is an inner part of the transfer function $\tilde{N}(s)$ with $G_K(0) = I$. Furthermore, $Q_l(s)$ is given as the diagonal matrix with $\dfrac{1}{(1+sT_1)^{\alpha_1}}$ on its i-th diagonal entry:

$$Q_l(s) = \mathrm{diag}\left\{ \frac{1}{(1+sT_1)^{\alpha_1}} \ \cdots \ \frac{1}{(1+sT_p)^{\alpha_p}} \right\}, \tag{17}$$

where α_i $(i = 1, \dots, p)$ are positive integers chosen to make $Q(s)$ proper and T_i $(i = 1, \dots, p)$ are positive real numbers chosen to satisfy the condition

$$I - G_K(j\omega)\mathrm{diag}\left\{ \frac{1}{(1+j\omega T_1)^{\alpha_1}} \ \cdots \ \frac{1}{(1+j\omega T_p)^{\alpha_p}} \right\} \simeq 0 \ \ (0 \le \forall\, \omega \le \omega_{\max}). \tag{18}$$

The following identity confirms that $Q(s)$ settled by (16) satisfies (12):

$$\begin{aligned}&(j\omega I - A + BU)^{-1}BY(j\omega) - Q(j\omega)\tilde{N}(j\omega) \\ &= (j\omega I - A + BU)^{-1}BY(j\omega)\end{aligned}$$

$$\left[I - G_K(j\omega)\mathrm{diag}\left\{ \begin{array}{ccc} \frac{1}{(1+j\omega T_1)^{\alpha_1}} & \cdots & \frac{1}{(1+j\omega T_p)^{\alpha_p}} \end{array} \right\} \right]. \tag{19}$$

Next, we provide state-space design methods of $Q(s) \in RH_\infty^{n\times m}$ satisfying (16). Before proceeding, let the state space realization of $\tilde{N}(s)$ in (10) be given by

$$\tilde{N}(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array} \right], \tag{20}$$

where generally speaking, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ represents the transfer function $C(sI-A)^{-1}B+D$. Then we assume that it holds that

$$\mathrm{rank}\ \Phi = p, \tag{21}$$

where the matrix Φ is constructed from the parameters of the state-space realization of $\tilde{N}(s)$ as follows:

$$\Phi = \left[\begin{array}{c} \tilde{B}_1^{\mathrm{T}} \left(\tilde{A}^{\mathrm{T}}\right)^{\alpha_1-1} \tilde{C}^{\mathrm{T}} \\ \vdots \\ \tilde{B}_p^{\mathrm{T}} \left(\tilde{A}^{\mathrm{T}}\right)^{\alpha_p-1} \tilde{C}^{\mathrm{T}} \end{array} \right], \tag{22}$$

$$\tilde{B} = \left[\begin{array}{ccc} \tilde{B}_1 & \cdots & \tilde{B}_p \end{array} \right] \ \left(\tilde{B}_i \in \mathbb{R}^n \, (i=1,\ldots,p)\right) \tag{23}$$

and

$$\alpha_i = \min\left(j | \tilde{B}_i^{\mathrm{T}} \left(\tilde{A}^{\mathrm{T}}\right)^{j-1} \tilde{C}^{\mathrm{T}} \neq 0; j=1,\ldots,n \right) \ \ (i=1,\ldots,p). \tag{24}$$

We note that the assumption (21) means that $G(s)$ can be decoupled using static feedback control, and hence does not impose severe restriction. Under the assumption (21), we below propose (Method 1) and (Method 2) to determine $Q(s) \in RH_\infty^{n\times m}$ satisfying (16).

(Method 1) This method is based on the result in [20] and determines $Q(s) \in RH_\infty^{n\times m}$ as follows:

$$Q(s) = (sI - A + BU)^{-1} BY(s)\hat{G}(s), \tag{25}$$

where

$$\hat{G}(s) = \left[\begin{array}{c|c} \tilde{A} + K\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C} & K\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}} \\ \hline \Gamma^{-1}\left(E^{-\frac{1}{2}}\right)^{\mathrm{T}} \bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C} & \Gamma^{-1}\left(E^{-\frac{1}{2}}\right)^{\mathrm{T}} \bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}} \end{array} \right], \tag{26}$$

$$\Phi\hat{\Phi} = I_p, \tag{27}$$

$$X = \mathrm{diag}\left\{ \begin{array}{ccc} \beta_{1\alpha_1} & \cdots & \beta_{p\alpha_p} \end{array} \right\}, \tag{28}$$

$$\beta_{ij} = \alpha_i C_j (T_i)^{-j} \ \ (i=1,\ldots,p;\ j=1,\ldots,\alpha_i), \tag{29}$$

and $\bar{D}_l \in \mathbb{R}^{p\times p}$ is an arbitrary constant nonsingular matrix satisfying

$$E = \bar{D}_l^{-1}\left(\bar{D}_l^{-1}\right)^{\mathrm{T}}. \tag{30}$$

Furthermore, the auxiliary feedback gain K and scaling matrix Γ are defined by

$$K = -\Psi X^{-1}\bar{D}_l - P\left(\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}\right)^{\mathrm{T}}\left(E^{-1}\right)^{\mathrm{T}}, \tag{31}$$

$$\Gamma = -\left(E^{-\frac{1}{2}}\right)^{\mathrm{T}} \bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}\left(\tilde{A} + K\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}\right)^{-1}\left(\Psi X^{-1} + K\bar{D}_l^{-1}\right) + \left(E^{-\frac{1}{2}}\right)^{\mathrm{T}}\bar{D}_l^{-1}, \tag{32}$$

$$\Psi = \left[\ \tilde{A}^{\alpha_1}\tilde{B}_1 + \cdots + \beta_{1\alpha_1}\tilde{B}_1 \quad \ldots \quad \tilde{A}^{\alpha_p}\tilde{B}_p + \cdots + \beta_{p\alpha_p}\tilde{B}_p\ \right], \tag{33}$$

where $P = P^{\mathrm{T}} \geq 0$ is the unique solution of the Riccati equation

$$\begin{aligned} &P\left(\tilde{A}^{\mathrm{T}} - \tilde{C}^{\mathrm{T}}\hat{\Phi}\Psi^{\mathrm{T}}\right) + \left(\tilde{A}^{\mathrm{T}} - \tilde{C}^{\mathrm{T}}\hat{\Phi}\Psi^{\mathrm{T}}\right)^{\mathrm{T}} P \\ &- P\left(\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}\right)^{\mathrm{T}} E^{-1}\left(\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}\right) P = 0 \end{aligned} \tag{34}$$

to make $\tilde{A} + K\bar{D}_l^{-1}X\hat{\Phi}^{\mathrm{T}}\tilde{C}$ have no eigenvalue in the closed right half plane.

(Method 2) This method is based on the result in [21] and determines $Q(s) \in RH_\infty^{n\times m}$ as follows:

$$Q(s) = (sI - A + BU)^{-1}BY(s)G_K(s)G_0(s), \tag{35}$$

where

$$G_0(s) = \left[\begin{array}{c|c} \tilde{A} - \Psi\hat{\Phi}^{\mathrm{T}}\tilde{C} & -\Psi\hat{\Phi}^{\mathrm{T}} \\ \hline X\hat{\Phi}^{\mathrm{T}}\tilde{C} & X\hat{\Phi}^{\mathrm{T}} \end{array}\right] = \begin{bmatrix} G_{01}(s) \\ \vdots \\ G_{0p}(s) \end{bmatrix} \quad \left(G_{0i}(s) \in R^{1\times m}(s)\ (i = 1, \ldots, p)\right), \tag{36}$$

$\hat{\Phi}$, X, β_{ij} and Ψ are given by (27), (28), (29) and (33), respectively. In addition, $G_K(s)$ is designed as follows: Let the minimal realization of $G_{0i}(s)$ $(i = 1, \ldots, p)$ be

$$G_{0i}(s) = \left[\begin{array}{c|c} A_{0i} & B_{0i} \\ \hline C_{0i} & D_{0i} \end{array}\right] \quad (i = 1, \ldots, p). \tag{37}$$

Then, from this realization, $G_K(s)$ is obtained by

$$\begin{aligned} G_K(s) &= \mathrm{diag}\left\{ \frac{1}{1 + C_{01}(sI - A_{01})^{-1}K_1} \quad \cdots \quad \frac{1}{1 + C_{0p}(sI - A_{0p})^{-1}K_p} \right\} \\ &= \left[\begin{array}{cc|cc} A_{01} - K_1C_{01} & 0 & K_1 & 0 \\ & \ddots & & \ddots \\ 0 & A_{0p} - K_pC_{0p} & 0 & K_p \\ \hline -C_{01} & 0 & 1 & 0 \\ & \ddots & & \ddots \\ 0 & -C_{0p} & 0 & 1 \end{array}\right], \end{aligned} \tag{38}$$

where

$$K_i = P_iC_{0i}^{\mathrm{T}} \quad (i = 1, \ldots, p) \tag{39}$$

and $P_i \geq 0$ $(i = 1, \ldots, p)$ is the unique stabilizing solution of the Riccati equation

$$P_iA_{0i}^{\mathrm{T}} + A_{0i}P_i - P_iC_{0i}^{\mathrm{T}}C_{0i}P_i = 0 \quad (i = 1, \ldots, p). \tag{40}$$

The key point common in (Method 1) and (Method 2) is that the Youla parameter $Q(s)$ includes a stable left filtered inverse of $\tilde{N}(s)$. In (Method 1), $\hat{G}(s)$ is the stable left filtered inverse, and yields the inner function $G_K(s)$ for (16), which is not necessarily diagonal. In (Method 2), $G_K(s)G_0(s)$ is the stable left filtered inverse, and yields the inner function $G_K(s)$ for (16), which has the diagonal structure.

4. **Output Feedback Controller Design.** In accordance with the proposed unknown input observer design methods, this section describes how to construct the output feedback control system in Figure 1.

Consider the output feedback control

$$u(t) = -U\xi(t) + v(t) \tag{41}$$

for the controlled plant $G(s)$, where $\xi(t)$ is the state estimate of the state variable $x(t)$ and $v(t) \in \mathbb{R}^p$ is an external input exerted on the control system. A method of designing the state feedback gain U and state estimate $\xi(t)$ for the output feedback control (41) is summarized as follows:

1) Specify the frequency component range $0 \leq \omega \leq \omega_{\max}$ from the supposed bandwidth of the external input $v(t)$.
2) Using the design method of H_∞ state feedback controllers in [22], fix U in (41) so that the maximal singular value of the transfer function from $v(j\omega)$ to $u(j\omega)$ is made negligible outside the frequency component range $0 \leq \omega \leq \omega_{\max}$.
3) Using (Method 1) or (Method 2) in Section 3, design the unknown input observer (5) which produces the state estimate $\xi(t)$ used for (41).

5. **Numerical Example.** In this section, we design the output feedback control in Section 4 for two sample cases, and examine the features of the proposed unknown input observer design methods.

5.1. **Numerical example 1.** Consider employing (Method 1) in Section 3 to design the output feedback control (41) for the controlled plant $G(s)$ written by

$$\left\{ \begin{aligned} \dot{x}(t) &= \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & -30 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 4 & 0 & 5 \end{bmatrix} x(t) \end{aligned} \right. . \tag{42}$$

The above controlled plant is of non-minimum phase, since it has invariant zeros at $(10, 0)$ and $(20, 0)$.

It is supposed that the external input $v(t)$ in (41) and initial state $x(0)$ are given by

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \sin(0.1t) \\ 2\sin(0.1t) \end{bmatrix} \tag{43}$$

and

$$x(0) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^{\mathrm{T}}, \tag{44}$$

respectively. Referring to the angular frequency of the external input $v(t)$, we specify the frequency component range by $\omega_{\max} = 0.1$.

Using the method in [18], $\tilde{N}(s)$ satisfying (9) is obtained as

$$\tilde{N}(s) = \left[\begin{array}{cccc|cc} -10 & 0 & 0 & 0 & 1 & 0 \\ 0 & -20 & 0 & 0 & 0 & 1 \\ 0 & 0 & -30 & 0 & 1 & 0 \\ 0 & 0 & 0 & -30 & 0 & 1 \\ \hline 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 0 & 0 \end{array} \right]. \tag{45}$$

The matrix Φ in (22), constructed from the state-space representation (45), satisfies the condition

$$\operatorname{rank} \Phi = 2, \tag{46}$$

as Φ in (22) is given by

$$\Phi = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}, \tag{47}$$

with $\alpha_1 = 1$ and $\alpha_2 = 1$. By (46), $\hat{\Phi}$ satisfying (27) is obtained as

$$\hat{\Phi} = \begin{bmatrix} 0.1667 & 0 \\ 0 & 0.111 \end{bmatrix}. \tag{48}$$

We choose the time constants in (17) as $T_1 = 0.001$, $T_2 = 0.002$ so that the condition (18) is satisfied in the frequency component range $0 \leq \omega \leq \omega_{\max} = 0.1$. Setting $\bar{D}_l = I$, together with (28), (29), (30), (31), (33) and (34), we have

$$\begin{cases} \beta_{11} = 1000 \\ \beta_{21} = 500 \end{cases}, \tag{49}$$

$$X = \begin{bmatrix} 1000 & 0 \\ 0 & 500 \end{bmatrix}, \tag{50}$$

$$E = I, \tag{51}$$

$$\Psi = \begin{bmatrix} 990 & 0 \\ 0 & 480 \\ 970 & 0 \\ 0 & 470 \end{bmatrix}, \tag{52}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

and

$$K = \begin{bmatrix} -0.990 & 0 \\ 0 & -0.960 \\ -0.970 & 0 \\ 0 & -0.940 \end{bmatrix}. \tag{54}$$

Substituting the above parameters into (25), $Q(s)$ is obtained. Consequently, the unknown input observer (5) reduces to (15) in the intended bandwidth of the control system.

The state estimation error

$$e(t) = x(t) - \xi(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} - \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \xi_4(t) \end{bmatrix} \tag{55}$$

evolves over time as depicted in Figure 2, where the solid, dotted, alternate long/short dash, broken lines correspond with $x_1(t)-\xi_1(t)$, $x_2(t)-\xi_2(t)$, $x_3(t)-\xi_3(t)$ and $x_4(t)-\xi_4(t)$, respectively. It is observed that the state variable $x(t)$ is effectively estimated by the unknown input observer designed using (Method 1).

5.2. **Numerical example 2.** Consider employing (Method 1) in Section 3 to design the output feedback control (41) for the controlled plant $G(s)$ written by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & -11 & -2 & 12 \\ 1 & -16 & -1 & 17 \end{bmatrix} x(t) \end{cases}. \tag{56}$$

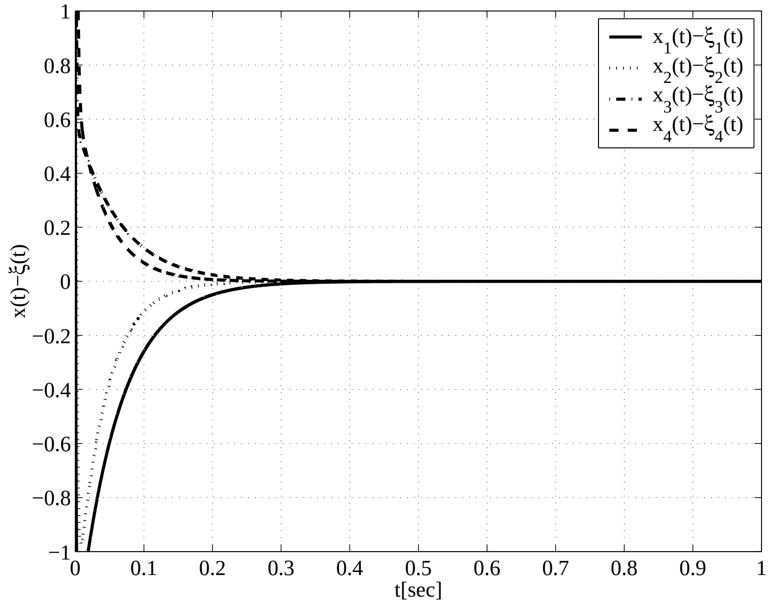

FIGURE 2. Time response of state estimation error $x(t) - \xi(t)$

The above controlled plant is of non-minimum phase, since it has an invariant zero at $(20, 0)$.

Using the method in [18], $\tilde{N}(s)$ satisfying (9) is obtained as

$$\tilde{N}(s) = \left[\begin{array}{cccc|cc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ \hline 2 & -11 & -2 & 12 & 0 & 0 \\ 1 & -16 & -1 & 17 & 0 & 0 \end{array}\right]. \tag{57}$$

The matrix Φ in (22), constructed from the state-space representation (57), satisfies the condition

$$\operatorname{rank} \Phi = 2, \tag{58}$$

as Φ in (22) is given by

$$\Phi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \tag{59}$$

with

$$\alpha_1 = 2 \tag{60}$$

and

$$\alpha_2 = 1. \tag{61}$$

By (58), $\hat{\Phi}$ satisfying (27) is given by

$$\hat{\Phi} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \tag{62}$$

Setting

$$T_1 = 0.001, \tag{63}$$

$$T_2 = 0.002, \tag{64}$$

$$\bar{D}_l = I, \tag{65}$$

together with (28), (29), (30), (31), (33) and (34), we have

$$\left\{ \begin{array}{l} \beta_{11} = 2000 \\ \beta_{12} = 1000000 \\ \beta_{21} = 500 \end{array} \right. , \tag{66}$$

$$X = \begin{bmatrix} 1000000 & 0 \\ 0 & 500 \end{bmatrix}, \tag{67}$$

$$E = I, \tag{68}$$

$$\Psi = \begin{bmatrix} 998001 & 0 \\ 0 & 499 \\ 996004 & 0 \\ 0 & 498 \end{bmatrix}, \tag{69}$$

$$P = \begin{bmatrix} 0.0869 & -0.0174 & 0.0827 & -0.0166 \\ -0.0174 & 0.0035 & -0.0166 & 0.0033 \\ 0.0827 & -0.0166 & 0.0788 & -0.0158 \\ -0.0166 & 0.0033 & -0.0158 & 0.0032 \end{bmatrix} \tag{70}$$

and

$$K = \begin{bmatrix} 0.8298 & -0.3657 \\ -0.3657 & -0.9248 \\ 0.7452 & -0.3484 \\ -0.3484 & -0.9263 \end{bmatrix}. \tag{71}$$

Substituting the above parameters into (25), $Q(s)$ is obtained. Consequently, the unknown input observer (5) reduces to (15) in the intended bandwidth of the control system.

When the external input $v(t)$ and initial state $x(0)$ are supplied as the same with (43) and (44), respectively, the state estimation error $x(t) - \xi(t)$ evolves over time as depicted in Figure 3, where the solid, dotted, alternate long/short dash, broken lines correspond with $x_1(t) - \xi_1(t)$, $x_2(t) - \xi_2(t)$, $x_3(t) - \xi_3(t)$ and $x_4(t) - \xi_4(t)$, respectively.

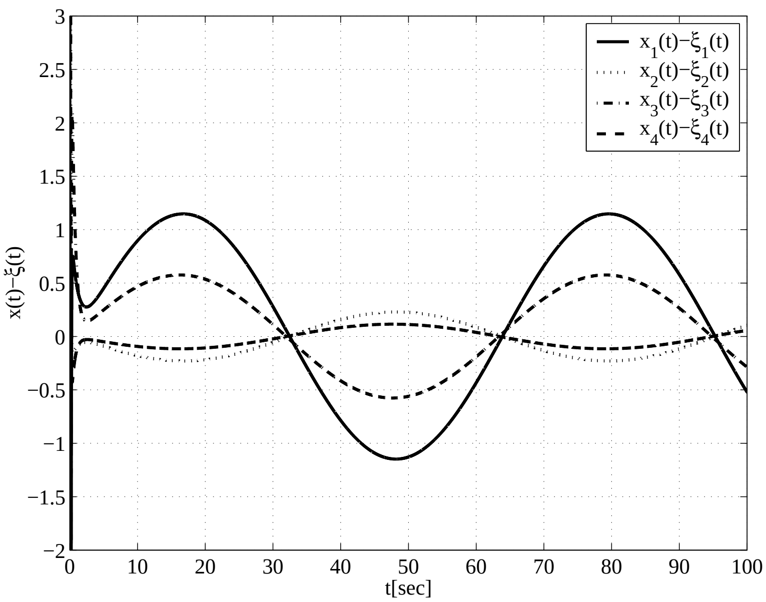

FIGURE 3. Time response of state estimation error $x(t) - \xi(t)$

Figure 3 shows that the state variable $x(t)$ is not fully estimated by the unknown input observer designed using (Method 1). The reason why (Method 1) failed is that $G_K(s)$ in (16) is not a diagonal inner function. Next, to circumvent this problem, we will design the unknown input observer according to (Method 2).

Let $\tilde{N}(s)$, Φ, α_i $(i = 1, 2)$, $\hat{\Phi}$, T_i $(i = 1, 2)$, $\bar{D}_l$, β_{ij} $(i = 1, 2;\ j = 1, \ldots, \alpha_i)$, X, E and Ψ be the same with (57), (59), (60), (61), (62), (63), (64), (65), (66), (67), (68) and (69), respectively. Using these parameters, $G_0(s)$ is determined by (36). By obtaining the minimal realization of $G_{0i}(s)$ $(i = 1, 2)$ and calculating P_i $(i = 1, 2)$, we have

$$K_1 = \begin{bmatrix} -1.9010 & 0.3803 & -1.8109 & 0.3623 \end{bmatrix}^{\mathrm{T}} \tag{72}$$

and

$$K_2 = \begin{bmatrix} 9.5011 & -1.9010 & 9.0511 & -1.8109 \end{bmatrix}^{\mathrm{T}}. \tag{73}$$

Using above parameters, $G_K(s)$ in (38) is obtained as

$$G_K(s) = \left[\begin{array}{cc|cc} -20.0000 & 0 & 2.6775 & 0 \\ 0 & -20.0000 & 0 & -2.6255 \\ \hline 14.9393 & 0 & -1 & 0 \\ 0 & -15.2355 & 0 & -1 \end{array}\right] = \begin{bmatrix} \dfrac{-s+20}{s+20} & 0 \\ 0 & \dfrac{-s+20}{s+20} \end{bmatrix}. \tag{74}$$

It is verified that $G_K(s)$ in (74) is a diagonal inner function. In Figure 4, we depict the state estimation error resulting from the unknown input observer designed using (Method 2), and confirm that the state variable $x(t)$ is effectively estimated.

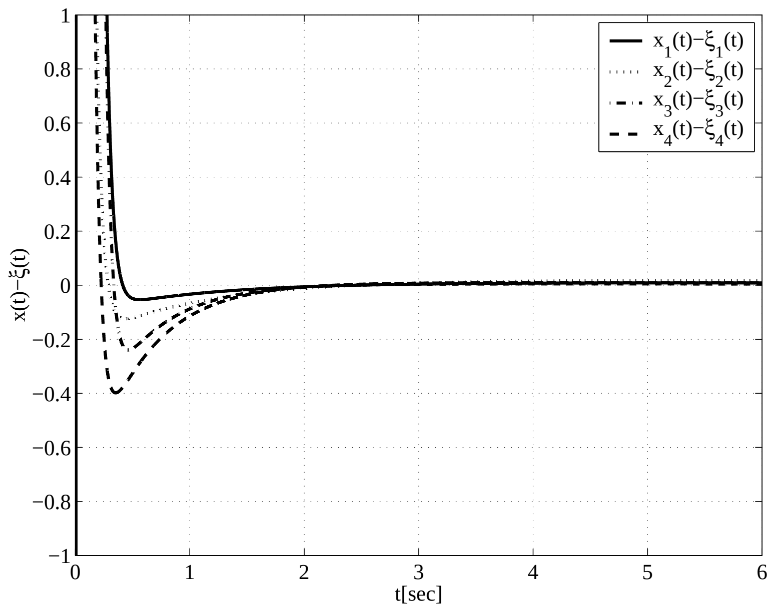

FIGURE 4. Time response of state estimation error $x(t) - \xi(t)$

6. **Conclusion.** In this paper, we proposed alternative design methods of unknown input observers for the non-minimum phase plant (1) by focusing on the intended bandwidth of the control system. The proposed design methods start from the parametrization of all state observers (5), (7), (8), and determine the free-parameter $Q(s)$ by utilizing the techniques of the stable left filtered inverses [20, 21]. The stable left filtered inverses in [20] and [21] led to the two methods (Method 1) and (Method 2), respectively. In Sections 4 and 5, it is also described that the proposed unknown input observer can be employed for

constructing output feedback control systems. Two sample cases were considered in order to illustrate the features of (Method 1) and (Method 2). It was observed that (Method 2) enabled to estimate the state variable effectively even in the case (Method 1) failed. In the recent authors' work [27], the underlying technique of the stable left filtered inverses [20, 21] is extended to a class of nonlinear systems. Hence a future subject of research is to enhance the proposed design methods of unknown input observers to the extent of handling the nonlinear systems directly.

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