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# MEAN-SQUARE EXPONENTIAL STABILITY ANALYSIS AND $H_\infty$ PERFORMANCE OF IMPULSIVE STOCHASTIC SYSTEMS WITH TIME-VARYING UNCERTAIN PARAMETERS

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Abstract. *This paper deals with the problem of the mean-square exponential stability and $H_\infty$ performance of impulsive stochastic systems with time-varying uncertain parameters. First, by using the Lyapunov function approach and well-known vector inequalities, we develop criteria on the mean-square exponential stability of uncontrolled systems in the form of certain linear matrix inequalities (LMIs) for admissible uncertainty. Secondly, we give the memoryless state feedback controller and present sufficient conditions of robust stochastic stability of systems. Furthermore, we analyze the $H_\infty$ performance level of systems with time-varying uncertain parameters. Finally, we give a numerical example with simulation results to verify the validity of the obtained method.*

**Keywords:** Time-varying uncertain parameters, Mean-square exponential stability, $H_\infty$ performance, Impulsive stochastic system, Linear matrix inequalities

1. **Introduction.** Over the last few decades, many researchers have begun to pay increasing attention to performance analysis of dynamical systems with impulsive perturbations due to the important theoretical and practical significance of theoretical analysis of impulsive disturbed dynamic systems [1-6]. The impulse is an instantaneous change in the state of the system at some point. For instance, regular spraying of pesticides in agricultural pest control is a typical impulse phenomenon. On the one hand, impulsive perturbations will destroy the stability and reliability of the system, resulting in poor performance of the system [4]. On the other hand, for the original unstable system, a good control effect can be obtained by introducing proper impulse [5]. What is more, for the deterministic situation, many conclusions about the stability and control of the impulse disturbance system have been presented in [6].

Meanwhile, many real systems in information science, biology, physics, and engineering are stochastic dynamical systems. Many researchers have devoted efforts to expanding the results of deterministic systems where there are no stochastic perturbations to stochastic systems [7-13]. In [11], many innovative research results of stochastic systems theory were discussed in SSS, and many results on stochastic processes and stochastic systems are presented. Meanwhile, the stability of impulse systems was also extended to stochastic

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systems [12, 13]. In [12], the problem of mean-square exponential stability for stochastic systems with time-varying delay was solved by employing the formula for the variation of parameters and the Cauchy matrix. In [13], using Lyapunov-Razumikhin techniques, sufficient conditions for impulsive stabilization of stochastic differentia systems were established.

On the other hand, because parameter uncertainties are unavoidable in practical systems, many researchers have paid attention to stability analysis and control of impulsive stochastic uncertain systems [15-18]. For instance, in [16], exponential stability and $H_\infty$ performance of the uncertain impulsive stochastic system were investigated by constructing a controller. In [17], by applying Lyapunov-Razumikhin techniques, some criteria of robust mean-square exponential stability for stochastic systems with parameter uncertainties were developed. Unlike the constraint conditions of uncertain parameters in [17], in [18], a sufficient condition of robust stochastic stability of impulsive stochastic systems with time-varying uncertainties was developed. Stochastic stability and $H_\infty$ performance of the system were ensured by designing a filter.

Motivated by the above discussion, this paper focuses on the problem of mean-square exponential stability and $H_\infty$ performance of a class of impulsive stochastic systems with time-varying uncertain parameters. We utilize the Lyapunov function approach and well-known vector inequalities to establish the novel criteria of mean-square exponential stability of the uncontrolled system with time-varying uncertain parameters. Then, we propose the memoryless state feedback controller to make sure that the system is robustly stochastically stable. Furthermore, some criteria are given to guarantee that the system can achieve a prescribed $H_\infty$ performance level. At last, a numerical example is given to verify the validness of our research.

The remaining of the paper is arranged as follows. In Section 2, the considered problems are formulated, and some relevant definitions and lemmas are reviewed. The mean-square exponential stability of the system with time-varying uncertainty is discussed in Section 3, which is followed by a discussion of robust stochastic stability of the discussed systems in Section 4. In Section 5, the $H_\infty$ performance of the proposed systems is analyzed. Section 6 provides an example that demonstrates our theoretical results. Section 7 presents some conclusions.

Notations: In this note, $\mathbb{R}^n$ denotes the $n$-dimensions Euclidean space; $\mathbb{R}^{m\times n}$ is a set of real $m \times n$ matrices; $X > Y$ (respectively $X \geq Y$), where $X$ and $Y$ are the symmetric matrices, means that $X - Y$ is a positive definite matrix (respectively, semi-definite matrix); $Y^{-1}$ and $Y^T$ denote the inverse and transpose of $Y$; $\pounds_2[0,\infty)$ is the space of square-integrable vector functions on $[0,\infty)$; $\|\cdot\|_2$ represents the $\pounds_2[0,\infty)$ norm over $[0,\infty)$; $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ stand for minimum and maximum eigenvalues of a symmetric matrix $\lambda$; $He(X)$ refers to $X + X^T$; $E\{\cdot\}$ denotes the mathematical expectation.

## 2. **Problem Statement and Preliminaries.**

We will examine the stability of the following system:

$$\Sigma_1 = \begin{cases} dx(t) = [(A + \triangle A(s))x(t) + (B + \triangle B(\mu))u(t) + B_0 v(t)]dt \\ \qquad\qquad + [Dx(t) + Hu(t) + D_0 v(t)]dw(t), \quad t \in (t_k, t_{k+1}], \\ \Delta x(t_k) = Fx(t), \quad t = t_k, \\ z(t) = Cx(t) + Gv(t), \\ x(t_0^+) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $v(t) \in \mathbb{R}^q$ is the continuous disturbance which pertains to $\pounds_2[0,\infty)$, $z(t) \in \mathbb{R}^p$ is the controlled output,

$w(t)$ is zero-mean real scalar Wiener process satisfying that $E\{dw(t)\}$ is zero and

$$E\left\{dw(t)^2\right\} = dt. \tag{2}$$

$$\Delta x(t) = x\left(t^+\right) - x\left(t^-\right), \quad \lim_{h\to 0^+} x(t-h) = x\left(t^-\right), \quad \lim_{h\to 0^+} x(t+h) = x\left(t^+\right). \tag{3}$$

$\{t_k,\ k = 0, 1, 2, \dots\}$ are the impulsive time instants and satisfy: $0 \le t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots$, and $\lim_{k\to\infty} t_k = +\infty$. Under normal circumstances, it can be supposed that $x(t_k) = x(t_k^-) = lim_{h\to 0^+} x(t_k - h)$. $A$, $B$, $B_0$, $C$, $G$, $D$, $H$, $D_0$ and $F$ are known real constant matrices. $\Delta A$ and $\Delta B$ represent time-varying uncertain parameters which satisfy

$$|\Delta A| < A_1, \quad |\Delta B| < B_1, \tag{4}$$

where $A_1$, $B_1$ are known nonnegative real constant matrices. The notation $|\Delta| < \overline{\Delta}$, means $|e_{ij}| \le \overline{e}_{ij}$, where $e_{ij}$ $(\overline{e}_{ij})$ is the elements at the corresponding position in the matrix $\Delta$ $\left(\overline{\Delta}\right)$, respectively. Furthermore, the parameters of uncertainties satisfy $s \in \Lambda \subset \mathbb{R}^{n_1}$, $\mu \in \Xi \subset \mathbb{R}^{m_1}$, where $\Lambda$, $\Xi$ refer to bounded compact set.

**Definition 2.1.** [5] *The equilibrium $x^* = 0$ of the system $\Sigma_1$ with $u(t) = 0$ and $v(t) = 0$ is said to be mean-square exponential stable, if there exist constants $m > 0$, $\gamma > 0$ such that*

$$\mathbf{E}\{\|x(t)\|\} \le m\mathbf{E}\,\|x(t_0)\|\, e^{-\lambda(t-t_0)}, \quad \forall t \ge t_0.$$

**Definition 2.2.** [5] *Given a scalar $\gamma > 0$, the system $\Sigma_1$ is said to robustly stochastically stable (RSS) and has the $H_\infty$ performance $\gamma$, if it is RSS in zero initial conditions, for all $v \in \pounds_2[0,\infty)$ and all admissible uncertainties $\Delta A$, $\Delta B$, such that*

$$\|z(t)\| < \gamma\|v(t)\|.$$

**Lemma 2.1.** [18] *Let $\mathfrak{A}$, $\mathfrak{D}$ be real matrices of appropriate dimensions, then we have: for any scalar $\varepsilon > 0$ and vectors $a, b \in \mathbb{R}^n$,*

$$2a^T\mathfrak{D}\mathfrak{A}b \le \varepsilon^{-1}a^T\mathfrak{D}\mathfrak{D}^Ta + \varepsilon b^T\mathfrak{A}^T\mathfrak{A}b.$$

**Lemma 2.2.** [18] *Let $n \times m$ matrix $\Delta A$ satisfying $|\Delta A| < A_1$, and then we have*

$$\Omega(A_1) \ge \Delta A\Delta A^T, \quad \Gamma(A_1) \ge \Delta A^T\Delta A,$$

*where*

$$\Omega(A_1) = \begin{cases} \left\|A_1A_1^T\right\| I_{n\times n}, & \left\|A_1A_1^T\right\| I \le n\cdot diag\left(A_1A_1^T\right), \\ n\cdot diag\left(A_1A_1^T\right), & otherwise, \end{cases}$$

$$\Gamma(A_1) = \begin{cases} \left\|A_1^TA_1\right\| I_{m\times m}, & \left\|A_1^TA_1\right\| I \le m\cdot diag\left(A_1^TA_1\right), \\ m\cdot diag\left(A_1^TA_1\right), & otherwise, \end{cases}$$

*where $diag(R) = diag(r_{11}, r_{22}, \dots, r_{nn})$, $R = (r_{ij})$ is an $n \times n$ symmetric matrix.*

**Lemma 2.3.** [19] *For given symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, the following three conditions are equivalent*

*1) $S < 0$.*
*2) $S_{11} < 0$, $S_{22} - S_{12}^TS_{11}^{-1}S_{12} < 0$.*
*3) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.*

**Remark 2.1.** *The system $\Sigma_1$ is said to be RSS if the system $\Sigma_1$ with $v = 0$ is mean-square exponential stable for all admissible time-varying uncertain parameters $\Delta A$ and $\Delta B$.*

3. **Mean-Square Exponential Stability.** First considering the situation of $u(t) = 0$ and $v(t) = 0$ in system $\Sigma_1$, then $\Sigma_1$ becomes the following uncontrolled system:

$$\Sigma_2 = \begin{cases} dx(t) = [(A + \Delta A(s))x(t)]dt + Dx(t)dw(t), \quad t \in (t_k, t_{k+1}], \\ \Delta x(t_k) = Fx(t), \quad t = t_k, \\ z(t) = Cx(t), \\ x(t_0^+) = x_0. \end{cases} \tag{5}$$

**Theorem 3.1.** *Given scalar $\lambda_k$ and $\alpha > 0$, and supposing there exist $\varepsilon_1 > 0$, a matrix $X > 0$ such that the following matrix inequalities hold*

$$\begin{bmatrix} \Theta_0 & XD^T & X(\Gamma(A_1))^{\frac{1}{2}} \\ DX & -X & 0 \\ (\Gamma(A_1))^{\frac{1}{2}}X & 0 & -\varepsilon_1 I \end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} -\lambda_k X & (X + FX)^T \\ X + FX & -X \end{bmatrix} \le 0, \tag{7}$$

*where*

$$\Theta_0 = He(AX) + \varepsilon_1 I + 2\alpha X, \quad 0 < -\lambda_k \le e^{\alpha(t_k - t_{k-1})},$$

*then, the system $\Sigma_2$ is mean-square exponential stable.*

**Proof:** We choose an appropriate Lyapunov functional candidate as:

$$V(x) = x^T Px. \tag{8}$$

Adopting the Itô formula to (5), it is noted that

$$\begin{aligned} dV(x) &= LV(x)dt + 2x^T(t)PDx(t)dw \\ &= 2x^T(t)P(A + \Delta A)x(t) + x^T(t)D^T PDx(t) + 2x^T(t)PDx(t)dw(t). \end{aligned} \tag{9}$$

Using Lemma 2.1 and Lemma 2.2, it can be seen that

$$2x^T P\Delta Ax \le x^T \left[\varepsilon_1 PP + \varepsilon_1^{-1}\Delta A^T \Delta A\right] x \le x^T \left[\varepsilon_1 PP + \varepsilon_1^{-1}\Gamma(A_1)\right] x. \tag{10}$$

From (9) and (10), it can be confirmed that

$$LV(x) \le x^T \Phi x, \tag{11}$$

where

$$\Phi = \left[He(PA) + \varepsilon_1 PP + \varepsilon_1^{-1}\Gamma(A_1) + D^T PD\right].$$

Let $P = X^{-1}$. Left- and right-multiplying by $diag[P, I, I]$ matrix Inequality (6), which combined Lemma 2.3 yields

$$\begin{bmatrix} \Theta_1 & D^T & (\Gamma(A_1))^{\frac{1}{2}} \\ D & -P^{-1} & 0 \\ (\Gamma(A_1))^{\frac{1}{2}} & 0 & -\varepsilon_1 I \end{bmatrix} < 0, \tag{12}$$

where

$$\Theta_1 = He(PA) + \varepsilon_1 PP + 2\alpha P, \quad X = P^{-1}.$$

Based on Lemma 2.3, (12) is equivalent to

$$\Phi + 2\alpha P < 0. \tag{13}$$

It follows from (11) and (13) that

$$LV(x) < -2\alpha x^T Px = -2\alpha V(x). \tag{14}$$

Therefore,

$$dV(x) < -2\alpha V(x)dt + 2x^T P(D + \Delta D)xdw. \tag{15}$$

Then, adopting the formula of integration by parts, we can deduce that

$$\begin{aligned} d\left(e^{2\alpha t}V(x)\right) &< 2\alpha e^{\alpha t}V(x)dt + e^{2\alpha t}dV(x) \\ &< e^{2\alpha t}\left[2\alpha V(x)dt - 2\alpha V(x)dt + 2x^T P(D + \Delta D)xdw\right] \\ &= 2e^{2\alpha t}x^T P(D + \Delta D)xdw. \end{aligned} \tag{16}$$

From $t_{k-1}$ to $t$, integrating both sides of Inequality (16) and taking expectations yields

$$E\{V(t)\} \le e^{-2\alpha(t-t_{k-1})}E\left\{V(t_{k-1})\right\}. \tag{17}$$

Hence, when $t = t_k^+$, in view of (7), it gives

$$(I + F)^T P(I + F) - \lambda_k P \le 0.$$

$$E\left\{V\left(t_k^+\right)\right\} - \lambda_k E\{V(t_k)\} = E\left\{x^T(t_k)\left((I + F)^T P(I + F) - \lambda_k P\right)x(t_k)\right\} \le 0. \tag{18}$$

That is

$$E\left\{V(t_k^+)\right\} \le \lambda_k E\{V(t_k)\}. \tag{19}$$

From (17), we can deduce that for any $t \in (t_0, t_1]$

$$E\{V(t)\} \le E\left\{V(t_0^+)\right\}e^{-2\alpha(t-t_0)}. \tag{20}$$

When $t = t_1$, (20) is equivalent to

$$E\{V(t_1)\} \le e^{-2\alpha(t_1-t_0)}E\left\{V\left(t_0^+\right)\right\}. \tag{21}$$

Therefore, for $t \in (t_1, t_2]$, combining (19) and (21) yields

$$E\left\{V\left(t_1^+\right)\right\} \le \lambda_1 e^{-2\alpha(t_1-t_0)}E\left\{V\left(t_0^+\right)\right\}. \tag{22}$$

For any $t > 0$ and $t \in (t_{k-1}, t_k]$, applying mathematical induction, one obtains

$$E\{V(t)\} \le \left(\prod_{i=1}^{k-1}\lambda_i\right)E\left\{V\left(t_0^+\right)\right\}e^{-2\alpha(t-t_0)}. \tag{23}$$

Considering the condition $0 < \lambda_k \le e^{\alpha(t_k-t_{k-1})}$, we can obtain that

$$\begin{aligned} E\{V(t)\} &\le \left(\prod_{i=1}^{k-1}\lambda_i\right)e^{-\alpha(t_k-t_0)}E\left\{V\left(t_0^+\right)\right\}e^{-\alpha(t-t_0)}e^{-\alpha(t-t_{k-1})} \\ &\le E\left\{V\left(t_0^+\right)\right\}e^{-\alpha(t-t_0)} \\ &= e^{\alpha(t_0)}E\left\{V\left(t_0^+\right)\right\}e^{-\alpha t}. \end{aligned} \tag{24}$$

Let $m = \sqrt{e^{\alpha t_0}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} > 0$, $\gamma = \alpha/2 > 0$. It follows that

$$E\{\|x(t)\|\} \le me^{-\gamma t}E\{\|x(t_0)\|\}. \tag{25}$$

This ends the proof.

4. **Robust Stochastic Stability.** Now, in consideration of $v(t) = 0$ and uncertainties, we will address the problem that the system $\Sigma_1$ is RSS. We construct a memoryless state feedback controller which presents with formula $u(t) = Kx(t)$, and the corresponding system becomes

$$\Sigma_3 = \begin{cases} dx(t) = [A + \Delta A(s) + (B + \Delta B(\mu))K]x(t)dt + [(D + HK)x(t)]dw(t), \\ \Delta x(t_k) = Fx(t), \quad t = t_k, \\ x(t_0^+) = x_0. \end{cases} \tag{26}$$

**Theorem 4.1.** *Given scalars $\lambda_k$, $\alpha > 0$, if there are $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ and matrices $X > 0$, $Y$ such that the following inequalities hold:*

$$\begin{bmatrix} \Theta_2 & Y^T H^T + X D^T & X(\Gamma(A_1))^{\frac{1}{2}} & Y^T(\Gamma(B_1))^{\frac{1}{2}} \\ DX + HY & -X & 0 & 0 \\ (\Gamma(A_1))^{\frac{1}{2}} X & 0 & -\varepsilon_2 I & 0 \\ (\Gamma(B_1))^{\frac{1}{2}} Y & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0, \tag{27}$$

$$\begin{bmatrix} -\lambda_k X & (X + FX)^T \\ X + FX & -X \end{bmatrix} \leq 0, \tag{28}$$

*where*

$$\Theta_2 = He(AX + BY) + (\varepsilon_2 + \varepsilon_3)I + 2\alpha X, \quad 0 < \lambda_k \leq e^{\alpha(t_k - t_{k-1})},$$

*then, the system $\Sigma_3$ is RSS with controller $u(t) = Kx(t)$, and $K = YX^{-1}$.*

**Proof:** For simplicity and convenience, we define

$$\Xi = D + HK. \tag{29}$$

Employing Itô formula to the system $\Sigma_3$, we can get that

$$dV(x) = LV(x)dt + 2x^T P \Xi x dw, \tag{30}$$

where $V(x)$ is mentioned in (8) and

$$LV(x) = 2x^T P[A + \Delta A + (B + \Delta B)K]x + x^T \Xi^T P \Xi x. \tag{31}$$

Using Lemma 2.2, one obtains

$$2x^T P \Delta A x \leq x^T \left[\varepsilon_2 PP + \varepsilon_2^{-1} \Delta A^T \Delta A\right] x \leq x^T \left[\varepsilon_2 PP + \varepsilon_2^{-1} \Gamma(A_1)\right] x, \tag{32}$$

and

$$2x^T P \Delta B K x \leq x^T \left[\varepsilon_3 PP + \varepsilon_3^{-1} K^T \Delta B^T \Delta B K\right] x \leq x^T \left[\varepsilon_3 PP + \varepsilon_3^{-1} K^T \Gamma(B_1) K\right] x. \tag{33}$$

Thus, it follows from (31)-(33) that

$$LV(x) < x^T[\Pi_1 + \Psi_1]x, \tag{34}$$

where

$$\Pi_1 = He(P(A + BK)) + (\varepsilon_2 + \varepsilon_3)PP,$$
$$\Psi_1 = \varepsilon_2^{-1}\Gamma(A_1) + \varepsilon_3^{-1} K^T \Gamma(B_1) K + \Xi^T P \Xi.$$

Left- and right-multiplying both side of (27) by $[P, I, I, I]$ and combining Lemma 2.3, where $X = P^{-1}$, the following inequality holds

$$\begin{bmatrix} \Theta_3 & K^T H^T + D^T & (\Gamma(A_1))^{\frac{1}{2}} & K^T(\Gamma(B_1))^{\frac{1}{2}} \\ D + HK & -P^{-1} & 0 & 0 \\ (\Gamma(A_1))^{\frac{1}{2}} & 0 & -\varepsilon_2 I & 0 \\ (\Gamma(B_1))^{\frac{1}{2}} K & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0, \tag{35}$$

where

$$\Theta_3 = He(A + BK) + (\varepsilon_2 + \varepsilon_3) PP + 2\alpha P.$$

Combining (29) and (35), applying Lemma 2.3, we can obtain that

$$\Pi_1 + \Psi_1 \leq -2\alpha P. \tag{36}$$

From (34) and (36), we have

$$LV(x) < -2\alpha x^T P x = -2\alpha V(x). \tag{37}$$

Combining (30) and (37), it gives

$$dV(x) < -2\alpha V(x)dt + 2x^T P\Xi x dw. \tag{38}$$

Similar to the proof of Theorem 3.1. We can get that

$$E\{\|x(t)\|\} \leq mE\{\|x(t_0)\|\}e^{-\gamma t}, \tag{39}$$

where

$$m = \sqrt{e^{\alpha t_0}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} > 0, \quad \gamma = \alpha/2 > 0.$$

The proof settles the issue of mean-square stability of the system $\Sigma_3$, which also indicates the system $\Sigma_1$ is RSS.

## 5. $H_\infty$ Performance Analysis.

This section is devoted to studying $H_\infty$ performance level of the system $\Sigma_1$.

**Theorem 5.1.** *Given $\gamma > 0$, $\alpha > 0$, $\lambda_k$, if there are $\varepsilon_4 > 0$, $\varepsilon_5 > 0$, a matrix $Y$ and $X > 0$, such that*

$$\begin{bmatrix} \Theta_4 & B_0^T & X(\Gamma(A_1))^{\frac{1}{2}} & Y^T(\Gamma(B_1))^{\frac{1}{2}} & XC^T & (DX+HY)^T \\ B_0 & -\gamma^2 I & 0 & 0 & G^T & D_0^T \\ (\Gamma(A_1))^{\frac{1}{2}}X & 0 & -\varepsilon_4 I & 0 & 0 & 0 \\ (\Gamma(B_1))^{\frac{1}{2}}Y & 0 & 0 & -\varepsilon_5 I & 0 & 0 \\ CX & G & 0 & 0 & -I & 0 \\ DX+HY & D_0 & 0 & 0 & 0 & -X \end{bmatrix} < 0, \tag{40}$$

$$\begin{bmatrix} -\lambda_k X & (X+FX)^T \\ X+FX & -X \end{bmatrix} < 0, \tag{41}$$

*where*

$$\Theta_4 = He(AX+BY) + (\varepsilon_4+\varepsilon_5)I + 2\alpha X, \quad 0 < \lambda_k \leq e^{\alpha(t_k - t_{k-1})},$$

*then, the system $\Sigma_1$ has $H_\infty$ performance level $\gamma$ with controller $u(t) = Kx(t)$, and $K = YX^{-1}$.*

**Proof:** Set

$$\Xi = D + HK. \tag{42}$$

Applying Itô formula to the system $\Sigma_1$, we can derive that

$$\begin{aligned} dV(x) &= LV(x)dt + 2x^T P[\Xi x + D_0 v]d\omega \\ &= \big[2x^T P(A + \Delta A + (B+\Delta B)K)x + 2x^T PB_0 v \\ &\quad + (\Xi x + D_0 v)^T P(\Xi x + D_0 v)\big]\, dt + 2x^T P(\Xi x + D_0 v)d\omega. \end{aligned} \tag{43}$$

Similar to the derivation of Inequality (34), for any $t \in [ih, (i+1)h)$, we have that

$$\mathrm{d}V(x(t)) \leq \left[x(t)^{\mathrm{T}} v(t)^{\mathrm{T}}\right] \Phi_1 \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \mathrm{d}t + 2x^T P[\Xi x + D_0 v]d\omega, \tag{44}$$

where

$$\Phi_1 = \begin{bmatrix} \Pi_2 & PB_0^T \\ * & 0 \end{bmatrix} + \begin{bmatrix} \Xi^T \\ D_0^T \end{bmatrix} P \begin{bmatrix} \Xi & D_0 \end{bmatrix},$$

$$\Pi_2 = He(P(A+BK)) + (\varepsilon_4+\varepsilon_5)PP + \varepsilon_4^{-1}\Gamma(A_1) + \varepsilon_5^{-1}K^T\Gamma(B_1)K.$$

Left- and right-multiplying both sides of (40) by $diag[P, I, I, I, I, I]$ and combining Lemma 2.3, where $X = P^{-1}$, it is obtained that

$$\begin{bmatrix} \Theta_5 & PB_0^T & (\Gamma(A_1))^{\frac{1}{2}} & K^T(\Gamma(B_1))^{\frac{1}{2}} & C^T & (D+HK)^T \\ B_0P & -\gamma^2 I & 0 & 0 & G^T & D_0^T \\ (\Gamma(A_1))^{\frac{1}{2}} & 0 & -\varepsilon_4 I & 0 & 0 & 0 \\ (\Gamma(B_1))^{\frac{1}{2}}K & 0 & 0 & -\varepsilon_5 I & 0 & 0 \\ C & G & 0 & 0 & -I & 0 \\ D+HK & D_0 & 0 & 0 & 0 & -P^{-1} \end{bmatrix} < 0, \tag{45}$$

where

$$\Theta_5 = He(P(A+BK)) + (\varepsilon_4+\varepsilon_5)PP + 2\alpha P.$$

Applying Itô formula, we can derive

$$E\{V(x(t),t)\} = E\left\{\int_0^t dV(x(\varsigma))\right\} = E\left\{\int_0^t LV(x(\varsigma),\varsigma)d\varsigma\right\}. \tag{46}$$

So, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, let

$$\begin{aligned} J(t) &= E\left\{\int_0^t \left[z(\varsigma)^T z(\varsigma) - \gamma^2 v(\varsigma)^T v(\varsigma)\right] d\varsigma\right\} \\ &= E\left\{\int_0^t \left[z(\varsigma)^T z(\varsigma) - \gamma^2 v(\varsigma)^T v(\varsigma) + LV(x(\varsigma),\varsigma)\right] d\varsigma\right\} - E\{V(x(t),t)\} \\ &\le E\left\{\int_0^t \left[z(\varsigma)^T z(\varsigma) - \gamma^2 v(\varsigma)^T v(\varsigma) + LV(x(\varsigma),\varsigma)\right] d\varsigma\right\}. \end{aligned} \tag{47}$$

Observe

$$z(\varsigma)^T z(\varsigma) - \gamma^2 v(\varsigma)^T v(\varsigma) + LV(x(\varsigma),\varsigma) \le \begin{bmatrix} x^T(\varsigma) & v^T(\varsigma) \end{bmatrix} \Pi \begin{bmatrix} x(\varsigma) \\ v(\varsigma) \end{bmatrix}, \tag{48}$$

where

$$\Pi = \begin{bmatrix} \Pi_3 & PB_0 \\ B_0^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \Xi^T \\ D_0^T \end{bmatrix} P \begin{bmatrix} \Xi & D_0 \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}} \\ G^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & G \end{bmatrix},$$

$$\Pi_3 = He(P(A+BK)) + (\varepsilon_4+\varepsilon_5)PP + \varepsilon_4^{-1}\Gamma(A_1) + \varepsilon_5^{-1}K^T\Gamma(B_1)K.$$

Using Lemma 2.3 and considering Equation (42), (45) is equal to $\Pi < 0$. By means of (47) and (48), we have

$$\begin{aligned} J(t) \le\ & E\left\{\int_{0^+}^{t_1} \begin{bmatrix} x^T(\varsigma) & v^T(\varsigma) \end{bmatrix} \Pi \begin{bmatrix} x(\varsigma) \\ v(\varsigma) \end{bmatrix} ds\right\} \\ & + E\left\{\int_{t_1^+}^{t_2} \begin{bmatrix} x^T(\varsigma) & v^T(\varsigma) \end{bmatrix} \Pi \begin{bmatrix} x(\varsigma) \\ v(\varsigma) \end{bmatrix} ds\right\} \\ & + \cdots + E\left\{\int_{t_{k-1}^+}^{t_k} \begin{bmatrix} x^T(\varsigma) & v^T(\varsigma) \end{bmatrix} \Pi \begin{bmatrix} x(\varsigma) \\ v(\varsigma) \end{bmatrix} ds\right\} < 0. \end{aligned} \tag{49}$$

When $t = t_k^+$,

$$\beta(t) = LV(x(t)) + 2\alpha V(x(t)) + z(t)^T z(t) - \gamma^2 v(t)^T v(t). \tag{50}$$

By (45), we uncover $\beta(t) < 0$. In view of the proof of the above two theorems, we can derive

$$LV(x) \le -2\alpha V(x) - \Theta, \tag{51}$$

where $\Theta = \|z\|^2 - \gamma^2\|v\|^2$, and

$$\begin{aligned} d\big(e^{2\alpha t}V(x(t))\big) &= 2\alpha e^{2\alpha t}V(x(t))dt + e^{2\alpha t}dV(x(t)) \\ &< e^{2\alpha t}\big[2\alpha V(x(t))dt - 2\alpha V(x(t))dt - \Theta(t)dt \\ &\quad + 2x^T(t)P(\Xi x(t) + D_0 v(t))d\omega(t)\big] \\ &= -e^{2\alpha t}[\Theta(t)dt - 2x^T(t)P(\Xi x(t) + D_0 v(t))d\omega(t)]. \end{aligned} \tag{52}$$

Integrating both sides of Formula (52) simultaneously from $t_{k-1}$ to $t$, it gives

$$\begin{aligned} E\{V(t)\} &\le E\{V(t_{k-1})\}e^{-2\alpha(t-t_{k-1})} - E\left\{\int_{t_{k-1}}^{t} \Theta(\varsigma)d\varsigma\right\} \\ &\le \left(\prod_{i=1}^{k-1}\lambda_i\right) E\left\{V(t_0^+)\right\} e^{-2\alpha(t-t_0)} - E\left\{\int_{t_{k-1}}^{t} \Theta(\varsigma)d\varsigma\right\}. \end{aligned} \tag{53}$$

When $t_0 = 0$, $x(t_0) = 0$, aiming at Inequality (41), we can confirm

$$V\left(t_0^+\right) = x^T\left(t_0^+\right)Px\left(t_0^+\right) = x^T(t_0)(I+F)^T P(I+F)x(t_0). \tag{54}$$

On account of $E\{V(t)\} > 0$, $e^{-2\alpha(t-\varsigma)} > 0$, from (53), we deduce that

$$E\left\{\int_{t_{k-1}}^{t} e^{-2\alpha(t-\varsigma)}\Theta(\varsigma)d\varsigma\right\} < 0, \tag{55}$$

which means that

$$\Theta(t) = \|z(t)\|^2 - \gamma^2\|v(t)\|^2 < 0.$$

This completes the proof.

6. **A Numerical Example.** In this part, an example is given to demonstrate the usefulness of the theoretical results.

**Example 6.1.** *Consider the uncertain impulsive stochastic system (1) with the following parameters:*

$$\begin{aligned} &A = \begin{bmatrix} -2 & 1.2 \\ 0.8 & -2 \end{bmatrix},\ B = \begin{bmatrix} -1 & 2 \\ 0.4 & 3 \end{bmatrix},\ B_0 = \begin{bmatrix} 2 & 0 \\ 0.2 & -1 \end{bmatrix},\ D = \begin{bmatrix} 1.2 & 0 \\ -2 & 0.5 \end{bmatrix}, \\ &H = \begin{bmatrix} 0.5 & 0 \\ -1 & 2 \end{bmatrix},\ D_0 = \begin{bmatrix} 2 & 0.4 \\ -1 & -4 \end{bmatrix},\ F = \begin{bmatrix} 1.9 & 0 \\ 0 & 1.9 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 1.2 & 0.1 \end{bmatrix}, \\ &G = \begin{bmatrix} 0.1 & 1 \\ 0.2 & -0.1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 5 \\ 0.02 & 10 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0,2 \\ 1.2 & 1 \end{bmatrix}. \end{aligned}$$

*Choose $\gamma = 1.2$, $\alpha = 1$, the the impulsive time interval $t_{k+1} - t_k = 0.5$. The solutions of LMIs (28) and (29) are*

$$\varepsilon_4 = 0.9558,\ \varepsilon_5 = 0.4691,\ X = \begin{bmatrix} 0.1278 & -0.1755 \\ -0.1755 & 0.5569 \end{bmatrix},\ Y = \begin{bmatrix} 0.1564 & -0.8262 \\ -0.1380 & -1.1560 \end{bmatrix}.$$

*Thus, the memoryless state feedback controller is taken as*

$$u(t) = \begin{bmatrix} -1.4349 & -1.9359 \\ -6.9307 & -4.2604 \end{bmatrix} x(t).$$

*The simulation results are given in Figures 1 and 2. From Figure 2, it can be seen that $E|x(t)|^2$ converges to 0 rapidly.*

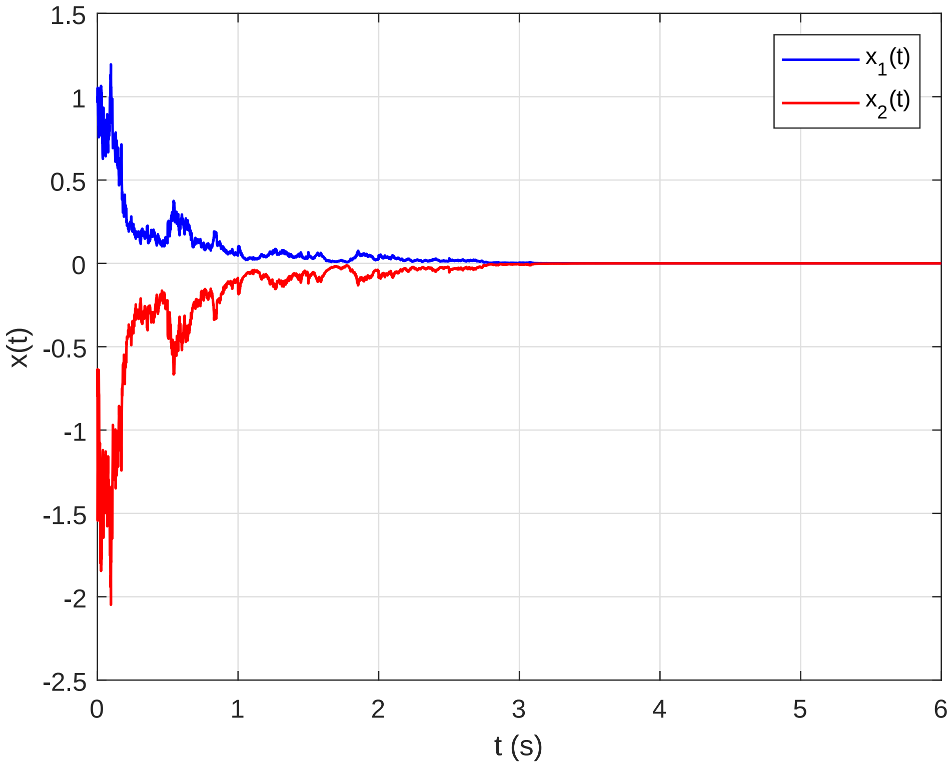


FIGURE 1. State trajectory of the uncertain impulsive stochastic system (1) in Example 6.1

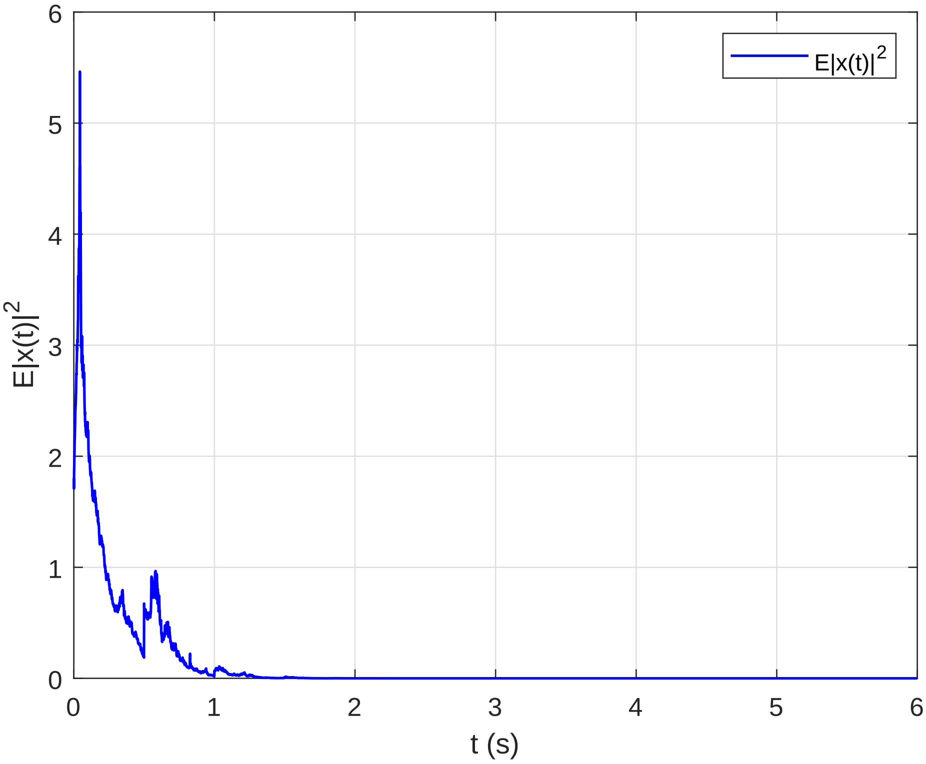


FIGURE 2. The mean square of the solution to Example 6.1

7. **Conclusions.** The problem of the mean-square exponential stability and $H_\infty$ performance for impulsive stochastic systems with time-varying uncertain parameters has been addressed by designing a memoryless state feedback controller. Using the Lyapunov function approach and well-known vector inequalities, the sufficient conditions of mean-square exponential stability and robust stochastic stability have been given in terms of linear matrix inequalities. Then, we have designed a memoryless feedback controller, which leads to having a prescribed $H_\infty$ performance level of the system with uncertainties. The validity of the proposed method is illustrated by numerical simulation. The further work directions may extend to stability analysis of linear time-delay systems with parameter uncertainties and Semi-Markovian jump systems with parameter uncertainties.

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