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ROBUST H_∞ FILTERING OF UNCERTAIN TWO-DIMENSIONAL DISCRETE SYSTEMS WITH DELAYS

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Abstract. *This work is concerned with the robust H_∞ filtering of two-dimensional (2-D) discrete systems with state delays described by the Roesser state space model with uncertain parameters of the polytopic type problem. By using the parameter-dependent Lyapunov-Krasovski functional approach and by introducing some slack matrix variables, new sufficient conditions for the H_∞ performance analysis are developed in terms of linear matrix inequalities (LMIs). Based on those results, both types of parameter-dependent and independent filters are designed by solving a convex optimization problem. Finally, two numerical examples are introduced to illustrate the effectiveness, reduced conservatism and potential of the developed theoretical results.*

Keywords: 2-D discrete systems, Delays, Roesser model, Polytopic uncertainties, H_∞ performance, H_∞ filter

1. **Introduction.** The two-dimensional (2-D) systems have widely attracted interest with their theoretical importance such as linear repetitive control [1] and iterative learning control [2, 3] and with a practical significance in image processing and process control [4].

It has been well known that time-delays are inevitable in practical systems, especially in the 2-D class due to the finite speed of information processing and data transmission among various parts of the system. The time-delay often degrades the system performance and even causes the system instability. Therefore, the highlighting of stability of time-delay systems plays an important role in applied models and has been greatly studied in control theory and signal processing fields [5].

The H_∞ technique introduced in [6] has attracted the attention of many researchers, for example, [7, 8, 9, 10]. It is a technique well known in literature to minimize the impact of perturbations on systems. Over the past decades, considerable attention has been devoted to the problem of state estimation. When a priori information on the external noises is not incisively known, although the famed Kalman filtering provides an optimal state estimation approach in the sense of error variance, it has been acknowledged that the traditional Kalman filter is considerably sensitive to system parameters [5] but not robust enough against large uncertainties. Therefore, many attempts have been established toward other more robust filtering arrangements, among which, H_∞ filtering is the most concerned one [11, 12, 13], in which the input signal is assumed to be energy bounded and the prime aim is to minimize the H_∞ norm of the filtering error system.

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In this paper, by constructing a Lyapunov-Krasovski functional [14] on a 2-D system described by Rosser model [15], a delay-dependant H_∞ performance analysis is established for error systems by retaining some useful terms from the difference of Lyapunov-Krasovski functions. As a result, the H_∞ filter is designed in terms of linear matrix inequalities (LMIs).

Many useful results on the problem of H_∞ filtering for various dynamic 2-D systems have appeared [16]. In contrast, the problem of H_∞ filtering for uncertain 2-D discrete systems described by the Roesser model with delays has not been fully investigated yet and deserves more attention, the increase in computational complexity due to the addition of more decision variables in our case constant delays, motivates the present study.

This paper is adjusted to six sections. In Section 2, the problem under study is formulated. In Section 3, new criterion is obtained in terms of LMI, which ensures the H_∞ performance of the 2-D discrete system described by the Rosser model and the filtering design is thus established in Section 4. Numerical examples are given to highlight the results in Section 5. Finally, some conclusions are provided in Section 6.

Notations. Throughout the paper, $\mathbb{R}^p$ denotes the p-dimensional real Euclidean space, and $\mathbb{R}^{p\times q}$ denotes the set of all $p \times q$ matrices. 0 and I represent zero matrix and identity matrix respectively. $diag\{\dots\}$ denotes a block-diagonal matrix in symmetric block matrices or long matrix expressions. X^T stand for the transpose of the matrix X. $Q > 0$ ($Q < 0$) means that Q is real symmetric and positive (negative) definite matrix. The notation $||x||$ stands for the Euclidean norm of the vector x.

2. **Problem Formulation.** Consider the 2-D systems with polytopic uncertainties and constant delays described as such:

$$\begin{aligned}
\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A(\alpha)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_d(\alpha)\begin{bmatrix} x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix} \\
&\quad + B(\alpha)w(i,j) \\
y(i,j) &= C(\alpha)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D(\alpha)w(i,j) \\
z(i,j) &= E(\alpha)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + F(\alpha)w(i,j)
\end{aligned} \tag{1}$$

where $x^v(i,j) \in \mathbb{R}^{n_v}$, $x^h(i,j) \in \mathbb{R}^{n_h}$ are vertical and horizontal state vectors, respectively, $w(i,j) \in \mathbb{R}^q$ disturbance input, $z(i,j) \in \mathbb{R}^s$ signal to be estimated, $y(i,j) \in \mathbb{R}^m$ is the measured output, d_1 and d_2 are positive integers representing constant delays along horizontal and vertical directions, respectively. The matrices

$$A(\alpha) = \begin{bmatrix} A_{11}(\alpha) & A_{12}(\alpha) \\ A_{21}(\alpha) & A_{22}(\alpha) \end{bmatrix}, \quad B(\alpha) = \begin{bmatrix} B_1(\alpha) \\ B_2(\alpha) \end{bmatrix}, \quad E(\alpha) = \begin{bmatrix} E_1(\alpha) & E_2(\alpha) \end{bmatrix}$$

$$A_d(\alpha) = \begin{bmatrix} A_{d11}(\alpha) & A_{d12}(\alpha) \\ A_{d21}(\alpha) & A_{d22}(\alpha) \end{bmatrix}, \quad C(\alpha) = \begin{bmatrix} C_1(\alpha) & C_2(\alpha) \end{bmatrix}$$

$F(\alpha)$ and $D(\alpha)$ are presumed to belong to a known convex bounded polyhedral domain **D** described as:

$$\begin{aligned}
\Omega(\alpha) \triangleq \big[\ & A_{11}(\alpha),\ A_{12}(\alpha),\ A_{21}(\alpha),\ A_{22}(\alpha),\ A_{d11}(\alpha),\ A_{d12}(\alpha),\ A_{d21}(\alpha),\ A_{d22}(\alpha), \\
& B_1(\alpha),\ B_2(\alpha),\ C_1(\alpha),\ C_2(\alpha),\ D(\alpha),\ E_1(\alpha),\ E_2(\alpha),\ F(\alpha)\ \big] \in \mathbf{D}
\end{aligned}$$

where

$$\mathbf{D} \triangleq \left[\Omega(\alpha) \backslash \Omega(\alpha) = \sum_{n=1}^{N} \alpha_n \Omega_n ; \sum_{n=1}^{N} \alpha_n = 1, \alpha_n \geq 0 \right] \tag{2}$$

with

$$\Omega_n \triangleq [\ A_{11n},\ A_{12n},\ A_{21n},\ A_{22n},\ A_{d11n},\ A_{d12n},\ A_{d21n},\ A_{d22n},\ B_{1n},\ B_{2n},\ C_{1n},\ C_{2n},\ D_n,\ E_{1n},\ E_{2n},\ F_n\ ]$$

denoting the n-th vortex of the polyhedral domain.

The purpose of this work is to estimate the signal $z(i,j)$ with ensured H_∞ performance. To achieve that the following full-order 2-D discrete filter will be observed:

$$\begin{aligned} \begin{bmatrix} \tilde{x}^h(i+1,j) \\ \tilde{x}^v(i,j+1) \end{bmatrix} &= A_f \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + B_f y(i,j) \\ \tilde{z}(i,j) &= C_f \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + D_f y(i,j) \\ \tilde{x}^h(i,j) &= \tilde{x}^v(i,j) = 0, \quad i,j = 1,2,\ldots \end{aligned} \tag{3}$$

The matrices A_f, B_f and C_f have the following form:

$$A_f = \begin{bmatrix} A_{f11} & A_{f12} \\ A_{f21} & A_{f22} \end{bmatrix}, \quad B_f = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix}, \quad C_f = \begin{bmatrix} C_{f1} & C_{f2} \end{bmatrix}$$

From (1) and (3), the filtering error system is obtained as follows:

$$\begin{aligned} \begin{bmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{bmatrix} = \hat{A}(\alpha) \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} + \hat{A}_d(\alpha) \begin{bmatrix} \hat{x}^h(i-d_1,j) \\ \hat{x}^v(i,j-d_2) \end{bmatrix} \\ + \hat{B}(\alpha) w(i,j) \end{aligned} \tag{4}$$

$$e(i,j) = \hat{C}(\alpha) \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} + \hat{D}(\alpha) w(i,j) \tag{5}$$

where

$$\begin{aligned} e(i,j) &= z(i,j) - \tilde{z}(i,j) \\ \hat{x}^h(i,j) &= \begin{bmatrix} x^h(i,j) \\ \tilde{x}^h(i,j) \end{bmatrix} \\ \hat{x}^v(i,j) &= \begin{bmatrix} x^v(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} \end{aligned}$$

and

$$\hat{A}(\alpha) = \begin{bmatrix} \hat{A}_{11}(\alpha) & \hat{A}_{12}(\alpha) \\ \hat{A}_{21}(\alpha) & \hat{A}_{22}(\alpha) \end{bmatrix} = \begin{bmatrix} A_{11}(\alpha) & 0 & A_{12}(\alpha) & 0 \\ B_{f1}C_1(\alpha) & A_{f11} & B_{f1}C_2(\alpha) & A_{f12} \\ A_{21}(\alpha) & 0 & A_{22}(\alpha) & 0 \\ B_{f2}C_1(\alpha) & A_{f21} & B_{f2}C_2(\alpha) & A_{f22} \end{bmatrix}$$

$$\hat{A}_d(\alpha) = \begin{bmatrix} \hat{A}_{d11}(\alpha) & \hat{A}_{d12}(\alpha) \\ \hat{A}_{d21}(\alpha) & \hat{A}_{d22}(\alpha) \end{bmatrix} = \begin{bmatrix} A_{d11}(\alpha) & 0 & A_{d12}(\alpha) & 0 \\ 0 & 0 & 0 & 0 \\ A_{d21}(\alpha) & 0 & A_{d22}(\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\hat{B}(\alpha) &= \begin{bmatrix} B_1(\alpha) \\ B_{f1}D(\alpha) \\ B_2(\alpha) \\ B_{f2}D(\alpha) \end{bmatrix} \\
\hat{C}(\alpha) &= \begin{bmatrix} \hat{C}_1(\alpha) & \hat{C}_2(\alpha) \end{bmatrix} \\
\hat{C}_1(\alpha) &= \begin{bmatrix} E_1(\alpha) - D_f C_1(\alpha) & -C_{f1} \end{bmatrix} \\
\hat{C}_2(\alpha) &= \begin{bmatrix} E_2(\alpha) - D_f C_2(\alpha) & -C_{f2} \end{bmatrix} \\
\hat{D}(\alpha) &= F(\alpha) - D_f D(\alpha)
\end{aligned}$$

To get the main results of this paper, the following remark and lemma are needed.

Lemma 2.1. [17] *For given symmetric matrices*

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$$

where S_{11} and S_{22} are square matrices, the following conditions are equivalent

1) $S < 0$;

2) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;

3) $S_{22} < 0$, $S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0$.

Remark 2.1. *The H_∞ filtering problem is important due to its theoretical and practical value in control engineering and signal processing. In this work, we consider the problem of robust H_∞ filtering for a class of uncertain 2-D discrete systems with polytopic uncertainties and constant delays. The model used in this paper can find many applications, for example, the thermal processes in chemical reactors* [18] *and the stationary random field in image processing* [19].

3. Main Results.

In this section, we solve the robust H_∞ filtering problem.

3.1. H_∞ performance analysis.

Theorem 3.1. *Consider the uncertain 2-D system with delays (1) and given a positive scalar γ, an admissible full-order filter of the form (4) assuring a prescribed H_∞ performance and the robust stability of the filtering error system exists if there exist matrices $G^h(\alpha) > 0$ and $G^v(\alpha) > 0$, L, J, $Z_1 > 0$, $Z_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, N_1, N_2, X_h, X_v such that the following LMIs are feasible*

$$E(\alpha) = \begin{bmatrix} E_1(\alpha) & \sqrt{d_1}\phi_h^T Z_1 & \sqrt{d_2}\phi_v^T Z_2 & E_2^T(\alpha) & E_3^T(\alpha) & E_4^T(\alpha) \\ * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & -Z_2 & 0 & 0 & 0 \\ * & * & * & -G^h(\alpha) & 0 & 0 \\ * & * & * & * & -G^v(\alpha) & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \qquad (6)$$

$$\psi_1 = \begin{bmatrix} X_h & N_1 \\ * & Z_1 \end{bmatrix} \geq 0$$

$$\psi_2 = \begin{bmatrix} X_v & N_2 \\ * & Z_2 \end{bmatrix} \geq 0$$

with

$$E_1(\alpha) = \begin{bmatrix} \phi_{h_1} - sym(L) & 0 & \left(\frac{L}{2}\right)^T + G^{hT}(\alpha) & 0 & \phi_{h2} & 0 & 0 \\ * & \phi_{v_1} - sym(J) & 0 & \left(\frac{J}{2}\right)^T + G^{vT}(\alpha) & 0 & \phi_{v2} & 0 \\ * & * & -2G^h(\alpha) & 0 & 0 & 0 & 0 \\ * & * & * & -2G^v(\alpha) & 0 & 0 & 0 \\ * & * & * & * & \phi_{h3} & 0 & 0 \\ * & * & * & * & * & \phi_{v3} & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$\begin{aligned} &sym(L) = L + L^T \\ &sym(J) = J + J^T \\ &\phi_{h1} = E^T \left(N_{11} + N_{11}^T + Q_1 + d_1 \times X_{h11}\right) E \\ &\phi_{h2} = E^T \left(-N_{11} + N_{21}^T + d_1 \times X_{h12}\right) \\ &\phi_{h3} = -N_{21} - N_{21}^T - Q_1 + d_1 \times X_{h22} \\ &\phi_{v1} = E^T \left(N_{12} + N_{12}^T + Q_2 + d_2 \times X_{v11}\right) E \\ &\phi_{v2} = E^T \left(-N_{12} + N_{22}^T + d_2 \times X_{v12}\right) \\ &\phi_{v3} = -N_{22} - N_{22}^T - Q_2 + d_2 \times X_{v22} \\ &E = [I \quad 0] \end{aligned}$$

and

$$E_2^T(\alpha) = \begin{bmatrix} \hat{A}_{11}^T(\alpha)L^T \\ \hat{A}_{12}^T(\alpha)L^T \\ 0 \\ 0 \\ \hat{A}_{d11}^T(\alpha)L^T \\ \hat{A}_{d12}^T(\alpha)L^T \\ \hat{B}_1^T(\alpha)L^T \end{bmatrix}, \quad E_3(\alpha)^T = \begin{bmatrix} \hat{A}_{21}^T(\alpha)J^T \\ \hat{A}_{22}^T(\alpha)J^T \\ 0 \\ 0 \\ \hat{A}_{d21}^T(\alpha)J^T \\ \hat{A}_{d22}^T(\alpha)J^T \\ \hat{B}_2^T(\alpha)J^T \end{bmatrix}$$

$$E_4(\alpha)^T = \begin{bmatrix} \hat{C}_1^T(\alpha) \\ \hat{C}_2^T(\alpha) \\ 0 \\ 0 \\ 0 \\ 0 \\ \hat{D}^T(\alpha) \end{bmatrix}, \quad \phi_h^T = \begin{bmatrix} E^T \left(A_{11}^T(\alpha) - I\right) \\ E^T A_{12}^T(\alpha) \\ 0 \\ 0 \\ A_{d11}^T(\alpha) \\ A_{d12}^T(\alpha) \\ B_1^T(\alpha) \end{bmatrix}$$

$$\phi_v^T = \begin{bmatrix} E^T A_{21}^T(\alpha) \\ E^T \left(A_{22}^T(\alpha) - I\right) \\ 0 \\ 0 \\ A_{d21}^T(\alpha) \\ A_{d22}^T(\alpha) \\ B_2^T(\alpha) \end{bmatrix}$$

Proof: Suppose the parameter-dependent matrices $G^v(\alpha)$ and $G^h(\alpha)$ have the following form:

$$\begin{aligned} G^h(\alpha) &= \sum_{n=1}^{N} \alpha_n G_n^h, \quad G_n^h > 0; \\ G^v(\alpha) &= \sum_{n=1}^{N} \alpha_n G_n^v, \quad G_n^v > 0; \quad n = 1, \dots, N \end{aligned}$$

The Lyapunov-Krasovski function of our system is

$$V(i,j) = V^h(i,j) + V^v(i,j)$$

where

$$\begin{aligned} V^h(i,j) &= V_1^h(i,j) + V_2^h(i,j) + V_3^h(i,j) \\ V^v(i,j) &= V_1^v(i,j) + V_2^v(i,j) + V_3^v(i,j) \end{aligned}$$

and

$$\begin{aligned} V_1^h(i,j) &= \hat{x}^{hT}(i,j) L^T G^h(\alpha)^{-1} L \hat{x}^h(i,j) \\ V_1^v(i,j) &= \hat{x}^{vT}(i,j) J^T G^v(\alpha)^{-1} J \hat{x}^v(i,j) \\ V_2^h(i,j) &= \sum_{l=i-d_1}^{i-1} x^{hT}(l,j) Q_1 x^h(l,j) \\ V_2^v(i,j) &= \sum_{l=j-d_2}^{j-1} x^{vT}(i,l) Q_2 x^v(i,l) \\ V_3^h(i,j) &= \sum_{\theta=1-d_1}^{0} \sum_{l=i-1+\theta}^{i-1} \rho_h^T(l,j) Z_1 \rho_h(l,j) \\ V_3^v(i,j) &= \sum_{\theta=1-d_2}^{0} \sum_{l=j-1+\theta}^{j-1} \rho_v^T(i,l) Z_2 \rho_v(i,l) \end{aligned}$$

$Q_1, Q_2, Z_1, Z_2 > 0$ matrices with appropriate dimensions.

Let us find the increment of the function $V(i,j)$.

$$\begin{aligned} \Delta V(i,j) &= \left[\Delta V_1^h(i,j) + \Delta V_2^h(i,j) + \Delta V_3^h(i,j)\right] \\ &\quad + \left[\Delta V_1^v(i,j) + \Delta V_2^v(i,j) + \Delta V_3^v(i,j)\right] \\ \Delta V_1^h(i,j) &= \hat{x}^{hT}(i+1,j) L^T G^h(\alpha)^{-1} L \hat{x}^h(i+1,j) \\ &\quad - \hat{x}^{hT}(i,j) L^T G^h(\alpha)^{-1} L \hat{x}^h(i,j) \end{aligned}$$

and

$$\begin{aligned} \Delta V_2^h(i,j) &= \sum_{l=i-d_1+1}^{i-1+1} x^{hT}(l,j) Q_1 x^h(l,j) - V_2^h(i,j) \\ &= \hat{x}^{hT}(i,j) E^T Q_1 E \hat{x}^h(i,j) - x^{hT}(i-d_1,j) Q_1 x^h(i-d_1,j) \end{aligned}$$

and

$$\Delta V_3^h(i,j) = \sum_{\theta=1-d_1}^{0} \sum_{l=i-1+\theta+1}^{i-1+1} \rho_h^T(l,j) Z_1 \rho_h(l,j) - V_3^h(i,j)$$

$$\Delta V_3^h(i,j) = d_1 \rho_h^T(i,j) Z_1 \rho_h(i,j) - \sum_{l=i-d_1}^{i-1} \rho_h^T(i,l) Z_1 \rho_h(i,l)$$

same for the vertical direction:

$$\begin{aligned}
\Delta V_1^v(i,j) &= \hat{x}^{vT}(i,j+1)J^T G^v(\alpha)^{-1} J\hat{x}^v(i,j+1) \\
&\quad - \hat{x}^{vT}(i,j)J^T G^v(\alpha)^{-1} J\hat{x}^v(i,j) \\
\Delta V_2^v(i,j) &= \hat{x}^{vT}(i,j)E^T Q_2 E\hat{x}^v(i,j) - x^{vT}(i,j-d_2)Q_2 x^v(i,j-d_2) \\
\Delta V_3^v(i,j) &= d_2\rho_v^T(i,j)Z_2\rho_v(i,j) - \sum_{l=j-d_2}^{j-1} \rho_v^T(i,l)Z_2\rho_v(i,l)
\end{aligned}$$

Denote

$$\begin{aligned}
\rho_h(i,j) &= x^h(i+1,j) - x^h(i,j) \\
\rho_h(i,j) &= \phi_h \xi_{sys}(i,j)
\end{aligned}$$

with

$$\begin{aligned}
\xi_{sys}(i,j) = \big[&\ \hat{x}^{hT}(i,j) \quad \hat{x}^{vT}(i,j) \quad \hat{x}^{hT}(i,j)L^T G_h^{-T}(\alpha) \quad \hat{x}^{vT}(i,j)J^T G_v^{-T}(\alpha) \\
&\ x^{hT}(i-d_1,j) \quad x^{vT}(i,j-d_2) \quad w^T(i,j)\ \big]^T
\end{aligned}$$

and

$$\phi_h = \left[\ (A_{11}(\alpha)-I)E \quad A_{12}(\alpha)E \quad 0 \quad 0 \quad A_{d11}(\alpha) \quad A_{d12}(\alpha) \quad B_1(\alpha) \ \right]$$

the same goes for the vertical direction:

$$\rho_v(i,j) = \phi_v \xi_{sys}(i,j)$$

with

$$\phi_v = \left[\ A_{21}E \quad (A_{12}(\alpha)-I)E \quad 0 \quad 0 \quad A_{d21}(\alpha) \quad A_{d22}(\alpha) \quad B_2(\alpha) \ \right]$$

The addition and substraction of the same terms in the next equations are true:

$$\begin{aligned}
\eta_1 &= 2\big(\hat{x}^{hT}(i,j)L^T G_h^{-1}(\alpha)L\hat{x}^h(i,j) + \hat{x}^{hT}(i,j)G_h(\alpha)G_h^{-1}(\alpha)L\hat{x}^h(i,j) \\
&\quad - \hat{x}^{hT}(i,j)L^T G_h^{-1}(\alpha)G_h(\alpha)G_h^{-1}(\alpha)L\hat{x}^h(i,j) - \hat{x}^{hT}(i,j)L\hat{x}^h(i,j)\big) \\
&= 0 \\
\eta_2 &= 2\big(\hat{x}^{vT}(i,j)J^T G_v^{-1}(\alpha)J\hat{x}^v(i,j) + \hat{x}^{vT}(i,j)G_v(\alpha)G_v^{-1}(\alpha)J\hat{x}^v(i,j) \\
&\quad - \hat{x}^{vT}(i,j)J^T G_v^{-1}(\alpha)G_v(\alpha)G_v^{-1}(\alpha)J\hat{x}^v(i,j) - \hat{x}^{vT}(i,j)J\hat{x}^v(i,j)\big) \\
&= 0 \\
&\sum_{l=i-d_1}^{i-1} \rho^h(l,j) = \sum_{l=i-d_1}^{i-1} x^h(l+1,j) - \sum_{l=i-d_1}^{i-1} x^h(l,j) \\
&0 = x^h(i,j) - x^h(i-d_1,j) - \sum_{l=i-d_1}^{i-1} \rho^h(l,j)
\end{aligned}$$

the same for the vertical direction:

$$0 = x^v(i,j) - x^v(i,j-d_2) - \sum_{l=j-d_2}^{j-1} \rho^v(i,l)$$

we have

$$\begin{aligned}
0 &= 2\left[\hat{x}^{hT}(i,j)E^T N_{11} + x^{hT}(i-d_1,j)N_{21}\right] \\
&\quad \times \left[E\hat{x}^h(i,j) - x^h(i-d_1,j) - \sum_{l=i-d_1}^{i-1} \rho^h(l,j)\right] \\
0 &= 2\left[\hat{x}^{vT}(i,j)E^T N_{12} + x^{vT}(i,j-d_2)N_{22}\right]
\end{aligned}$$

$$\times \left[E\hat{x}^v(i,j) - x^v(i,j-d_2) - \sum_{l=j-d_2}^{j-1} \rho^v(i,l) \right]$$

for $N_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}$ and $N_2 = \begin{bmatrix} N_{12} \\ N_{22} \end{bmatrix}$ with appropriate dimensions.

For $X_h = X_h^T > 0$, $X_v = X_v^T > 0$ the following equations are true:

$$0 = d_1 \xi_1^T(i,j) X_h \xi_1(i,j) - \sum_{l=i-d_1}^{i-1} \xi_1^T(i,j) X_h \xi_1(i,j)$$

$$0 = d_2 \xi_2^T(i,j) X_v \xi_2(i,j) - \sum_{l=j-d_2}^{j-1} \xi_2^T(i,j) X_v \xi_2(i,j)$$

with

$$\xi_1(i,j) = \begin{bmatrix} x^{hT}(i,j) & x^{hT}(i-d_1,j) \end{bmatrix}^T$$
$$\xi_2(i,j) = \begin{bmatrix} x^{vT}(i,j) & x^{vT}(i,j-d_2) \end{bmatrix}^T$$

and

$$X_h = \begin{bmatrix} X_{h11} & X_{h12} \\ * & X_{h22} \end{bmatrix}$$
$$X_v = \begin{bmatrix} X_{v11} & X_{v12} \\ * & X_{v22} \end{bmatrix}$$

If we add $\Delta V(i,j)$ to all previous equations we will find that

$$\begin{aligned} &\Delta V(i,j) + e^T(i,j)e(i,j) - \gamma^2 w^T(i,j)w(i,j) \\ &< \xi_{sys}^T(i,j) \left[\phi + \phi_1^T G^h(\alpha)^{-1}\phi_1 + \phi_2^T G^v(\alpha)^{-1}\phi_2 + \phi_3^T\phi_3 + \phi_h^T d_1 Z_1 \phi_h \right. \\ &\left. + \phi_v^T d_2 Z_2 \phi_v \right] \xi_{sys}(i,j) - \sum_{l=i-d_1}^{i-1} \xi_h^T(l,j)\psi_1\xi_h(l,j) \\ &- \sum_{l=j-d_2}^{j-1} \xi_v^T(i,l)\psi_2\xi_v(i,l) \end{aligned}$$

with

$$\xi_h(l,j) = \begin{bmatrix} \xi_1^T(i,j) & \rho_h^T(l,j) \end{bmatrix}^T$$
$$\xi_v(i,l) = \begin{bmatrix} \xi_2^T(i,j) & \rho_v^T(i,l) \end{bmatrix}^T$$
$$\psi_1 = \begin{bmatrix} X_h & N_1 \\ * & Z_1 \end{bmatrix} \geq 0$$
$$\psi_2 = \begin{bmatrix} X_v & N_2 \\ * & Z_2 \end{bmatrix} \geq 0$$

with

$$\phi = E_1(\alpha), \quad \phi_1 = E_2(\alpha), \quad \phi_2 = E_3(\alpha), \quad \phi_3 = E_4(\alpha)$$

thus, if $\psi_i \geq 0$, $i = 1, 2$, and

$$\xi_{sys}^T(i,j) \left[\phi + \phi_1^T G^h(\alpha)^{-1}\phi_1 + \phi_2^T G^v(\alpha)^{-1}\phi_2 + \phi_3^T\phi_3 + \phi_h^T d_1 Z_1 \phi_h + \phi_v^T d_2 Z_2 \phi_v \right] \xi_{sys}(i,j) < 0$$

then

$$\Delta V(i,j) + e^T(i,j)e(i,j) - \gamma^2 w^T(i,j)w(i,j) < 0$$

which mean: $\Delta V(i,j) < 0$, which ensures stability under zero-conditions for all nonzero $w(i,j) \in L_2\{[0,\infty),[0,\infty)\}$. This completes the proof.

4. $\boldsymbol{H_\infty}$ Filter Design.

4.1. Filter design.

Theorem 4.1. *Taking into consideration the uncertain 2-D system with constant delays (1) and given a positive scalar γ, an admissible full-order filter of the form (4) assuring an established H_∞ performance and the robust stability of the filtering error system exists if there exist matrices $G_n^h > 0$ and $G_n^v > 0$ with $n \in 1,\dots,N$, L, J, A_{f11}, A_{f12}, A_{f21}, A_{f22}, B_{f1}, B_{f2}, C_{f1}, C_{f2} and D_f such that the following LMI is feasible for $n \in 1,\dots,N$*

$$LMI = \begin{bmatrix}
\phi_{h1} - sym(L) & 0 & \left(\frac{L}{2}\right)^T + G_n^{hT} & 0 & \phi_{h2} & 0 & 0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \\
* & \phi_{v1} - sym(J) & 0 & \left(\frac{J}{2}\right)^T + G_n^{vT} & 0 & \phi_{v2} & 0 & \pi_6 & \pi_7 & \pi_8 & \pi_9 & \pi_{10} \\
* & * & -2G_n^h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -2G_n^v & 0 & 0 & 0 & \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & 0 \\
* & * & * & * & \phi_{h3} & 0 & 0 & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & 0 \\
* & * & * & * & * & \phi_{v3} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & \pi_{19} & \pi_{20} & \pi_{21} & \pi_{22} & \pi_{23} \\
* & * & * & * & * & * & * & -Z_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -Z_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -G_n^h & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -G_n^v & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0$$

with

$$G_n^h = \begin{bmatrix} G_{1n}^h & G_{2n}^h \\ * & G_{3n}^h \end{bmatrix}, \quad G_n^v = \begin{bmatrix} G_{1n}^v & G_{2n}^v \\ * & G_{3n}^v \end{bmatrix}$$

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{12} \end{bmatrix}, \quad J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{12} \end{bmatrix}$$

$$\psi_1 = \begin{bmatrix} X_{h11} & X_{h12} & N_{11} \\ * & X_{h22} & N_{21} \\ * & * & Z_1 \end{bmatrix} \geq 0, \quad \psi_2 = \begin{bmatrix} X_{v11} & X_{v12} & N_{12} \\ * & X_{v22} & N_{22} \\ * & * & Z_2 \end{bmatrix} \geq 0$$

where

$$\pi_1 = \begin{bmatrix} \sqrt{d_1}A_{11}^T Z_1 - \sqrt{d_1}Z_1 \\ 0 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \sqrt{d_2}A_{21}^T Z_2 \\ 0 \end{bmatrix}$$

$$\pi_3 = \begin{bmatrix} A_{11n}^T L_{11}^T + C_{1n}^T B_{f1}^T & A_{11n}^T L_{21}^T + C_{1n}^T B_{f1}^T \\ Af_{11}^T & Af_{11}^T \end{bmatrix}$$

$$\pi_4 = \begin{bmatrix} A_{21n}^T J_{11}^T + C_{1n}^T B_{f2}^T & A_{21n}^T J_{21}^T + C_{1n}^T B_{f2}^T \\ A_{f21}^T & A_{f21}^T \end{bmatrix}$$

$$\pi_5 = \begin{bmatrix} E_{1n}^T - C_{1n}^T D_f^T \\ -C_{f1}^T \end{bmatrix}, \quad \pi_6 = \begin{bmatrix} \sqrt{d_1}A_{12}^T Z_1 \\ 0 \end{bmatrix}$$

$$\pi_7 = \begin{bmatrix} \sqrt{d_2}A_{22}^T Z_2 - \sqrt{d_2}Z_2 \\ 0 \end{bmatrix}$$

$$\pi_8 = \begin{bmatrix} A_{12n}^T L_{11}^T + C_{2n}^T B_{f1}^T & A_{12n}^T L_{21}^T + C_{2n}^T B_{f1}^T \\ A_{f11}^T & A_{f11}^T \end{bmatrix}$$

$$\pi_9 = \begin{bmatrix} A_{22n}^T J_{11}^T + C_{2n}^T B_{f2}^T & A_{22n}^T J_{21}^T + C_{2n}^T B_{f2}^T \\ A_{f11}^T & A_{f11}^T \end{bmatrix}$$

$$\pi_{10} = \begin{bmatrix} E_{2n}^T - C_{2n}^T D_f^T \\ -C_{f2}^T \end{bmatrix}, \quad \pi_{11} = \begin{bmatrix} \sqrt{d_1}A_{d11}^T Z_1 \\ 0 \end{bmatrix}$$

$$\pi_{12} = \begin{bmatrix} \sqrt{d_2}A_{d21}^T Z_2 \\ 0 \end{bmatrix}, \quad \pi_{13} = \begin{bmatrix} A_{d11}^T L_{11}^T \\ 0 \end{bmatrix}$$

$$\pi_{14} = \begin{bmatrix} A_{d11}^T L_{21}^T \\ 0 \end{bmatrix}, \quad \pi_{15} = \begin{bmatrix} A_{d21}^T J_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{16} = \begin{bmatrix} A_{d21}^T J_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{17} = \begin{bmatrix} \sqrt{d_1}A_{d12}^T Z_1 \\ 0 \end{bmatrix}, \quad \pi_{18} = \begin{bmatrix} \sqrt{d_2}A_{d22}^T Z_2 \\ 0 \end{bmatrix}$$

$$\pi_{19} = \begin{bmatrix} A_{d12}^T L_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{20} = \begin{bmatrix} A_{d12}^T L_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{21} = \begin{bmatrix} A_{d22}^T J_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{22} = \begin{bmatrix} A_{d22}^T J_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{23} = \begin{bmatrix} \sqrt{d_1}B_1^T Z_1 \end{bmatrix}$$

An appropriate filter in the form of (3) is given:

$$\begin{bmatrix} \bar{A}_{f11} & \bar{A}_{f12} & \bar{B}_{f1} \\ \bar{A}_{f21} & \bar{A}_{f22} & \bar{B}_{f2} \\ \bar{C}_{f1} & \bar{C}_{f2} & \bar{D}_f \end{bmatrix} = \begin{bmatrix} L_{12}^{-1} & 0 & 0 \\ 0 & J_{12}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{f11} & A_{f12} & B_{f1} \\ A_{f21} & A_{f22} & B_{f2} \\ C_{f1} & C_{f2} & D_f \end{bmatrix}$$

Proof: Let

$$l = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}, \quad j = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}$$

$$\bar{G}^h(\alpha) = \begin{bmatrix} \bar{G}_1^h(\alpha) & \bar{G}_2^h(\alpha) \\ * & \bar{G}_3^h(\alpha) \end{bmatrix}, \quad \bar{G}^v(\alpha) = \begin{bmatrix} \bar{G}_1^v(\alpha) & \bar{G}_2^v(\alpha) \\ * & \bar{G}_3^v(\alpha) \end{bmatrix}$$

Suppose that l_{12}, l_{22} and j_{12}, j_{22} are nonsingular, and determining the following matrices:

$$K^h = diag\left\{I, l_{22}^{-1} l_{12}\right\}$$

$$K^v = diag\left\{I, j_{22}^{-1} j_{12}\right\}$$

$$K = diag\left\{K^h, K^v, K^h, K^v, K^h, K^v, I, K^h, K^v, K^h, K^v, I\right\}$$

Pre- and post-multiply $E(\alpha)$ in (6) by K^T and K respectively, and consider the following variables change:

$$\begin{aligned} K^{hT}\bar{G}^h(\alpha)K^h &= G^h(\alpha) = \begin{bmatrix} G_1^h(\alpha) & G_2^h(\alpha) \\ * & G_3^h(\alpha) \end{bmatrix} \\ &= \sum_{n=1}^{N} \alpha_n \begin{bmatrix} G_{1n}^h & G_{2n}^h \\ * & G_{3n}^h \end{bmatrix} \\ K^{vT}\bar{G}^v(\alpha)K^v &= G^v(\alpha) = \begin{bmatrix} G_1^v(\alpha) & G_2^v(\alpha) \\ * & G_3^v(\alpha) \end{bmatrix} \\ &= \sum_{n=1}^{N} \alpha_n \begin{bmatrix} G_{1n}^v & G_{2n}^v \\ * & G_{3n}^v \end{bmatrix} \end{aligned}$$

$$K^{hT} l K^h = L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{12} \end{bmatrix}, \quad K^{vT} j K^v = J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{12} \end{bmatrix}$$

and

$$\begin{bmatrix} A_{f11} & A_{f12} \\ A_{f21} & A_{f22} \end{bmatrix} = \begin{bmatrix} l_{12} & 0 \\ 0 & j_{12} \end{bmatrix} \begin{bmatrix} \bar{A}_{f11} & \bar{A}_{f12} \\ \bar{A}_{f21} & \bar{A}_{f22} \end{bmatrix} \begin{bmatrix} l_{22}^{-1} l_{12} & 0 \\ 0 & j_{22}^{-1} j_{12} \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix} = \begin{bmatrix} l_{12} & 0 \\ 0 & j_{12} \end{bmatrix} \begin{bmatrix} \bar{B}_{f1} \\ \bar{B}_{f2} \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} C_{f1} & C_{f2} \end{bmatrix} = \begin{bmatrix} \bar{C}_{f1} & \bar{C}_{f2} \end{bmatrix} \begin{bmatrix} l_{22}^{-1} l_{12} & 0 \\ 0 & j_{22}^{-1} j_{12} \end{bmatrix} \tag{9}$$

Then, we assure that the filtering error system (3) is robustly asymptotically stable with an H_∞ disturbance attenuation level γ if the following inequality holds

$$S(\alpha) = K^T E(\alpha) K = \sum_{1}^{N} \alpha_n S_n < 0$$

This completes the proof.

4.2. **Parameter-dependant H_∞ filter design.** In this subsection we will try the design of robust parameter-dependant H_∞ filter where the parameter matrices and the parameter α jointly vary. We consider a parameter-dependant full-order filter in the following form:

$$\begin{aligned} \begin{bmatrix} \tilde{x}^h(i+1,j) \\ \tilde{x}^v(i,j+1) \end{bmatrix} &= A_f(\alpha) \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + B_f(\alpha) y(i,j) \\ \tilde{z}(i,j) = C_f(\alpha)(\alpha) &\begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + D_f(\alpha) y(i,j) \\ \tilde{x}^h(i,j) = \tilde{x}^v(i,j) &= 0, \quad i,j = 1,2,\ldots \end{aligned} \tag{10}$$

Theorem 4.2. *Considering the uncertain 2-D system with constant delays (1) and given a positive scalar γ, the precedent parameter-dependant filter exists, such as assuring a prescribed H_∞ performance and the robust stability of the filtering error system if there exist matrices $G_n^h > 0$ and $G_n^v > 0$ with $n \in 1,\ldots,N$, L, J, A_{f11n}, A_{f12n}, A_{f21n}, A_{f22n}, B_{f1n}, B_{f2n}, C_{f1n}, C_{f2n} and D_f such that the following LMI is feasible for $n \in 1,\ldots,N$*

$$\begin{bmatrix}
\phi_{h1}-sym(L) & 0 & \left(\frac{L}{2}\right)^T+G_n^{hT} & 0 & \phi_{h2} & 0 & 0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \\
* & \phi_{v1}-sym(J) & 0 & \left(\frac{J}{2}\right)^T+G_n^{vT} & 0 & \phi_{v2} & 0 & \pi_6 & \pi_7 & \pi_8 & \pi_9 & \pi_{10} \\
* & * & -2G_n^h(\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -2G_n^v(\alpha) & 0 & 0 & 0 & \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & 0 \\
* & * & * & * & \phi_{h3} & 0 & 0 & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & 0 \\
* & * & * & * & * & \phi_{v3} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & \pi_{19} & \pi_{20} & \pi_{21} & \pi_{22} & \pi_{23} \\
* & * & * & * & * & * & * & -Z_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -Z_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -G_n^h & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -G_n^v & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0$$

with

$$\pi_1 = \begin{bmatrix} \sqrt{d_1}A_{11n}^T Z_1 - \sqrt{d_1}Z_1 \\ 0 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \sqrt{d_2}A_{21n}^T Z_2 \\ 0 \end{bmatrix}$$

$$\pi_3 = \begin{bmatrix} A_{11n}^T L_{11}^T + C_{1n}^T B_{f1n}^T & A_{11n}^T L_{21}^T + C_{1n}^T B_{f1n}^T \\ A_{f11n}^T & A_{f11n}^T \end{bmatrix}$$

$$\pi_4 = \begin{bmatrix} A_{21n}^T J_{11}^T + C_{1n}^T B_{f2n}^T & A_{21n}^T J_{21}^T + C_{1n}^T B_{f2n}^T \\ A_{f21n}^T & A_{f21n}^T \end{bmatrix}$$

$$\pi_5 = \begin{bmatrix} E_{1n}^T - C_{1n}^T D_{fn}^T \\ -C_{f1n}^T \end{bmatrix}, \quad \pi_6 = \begin{bmatrix} \sqrt{d_1}A_{12n}^T Z_1 \\ 0 \end{bmatrix}$$

$$\pi_7 = \begin{bmatrix} \sqrt{d_2}A_{22n}^T Z_2 - \sqrt{d_2}Z_2 \\ 0 \end{bmatrix}$$

$$\pi_8 = \begin{bmatrix} A_{12n}^T L_{11}^T + C_{2n}^T B_{f1n}^T & A_{12n}^T L_{21}^T + C_{2n}^T B_{f1n}^T \\ A_{f11n}^T & A_{f11n}^T \end{bmatrix}$$

$$\pi_9 = \begin{bmatrix} A_{22n}^T J_{11}^T + C_{2n}^T B_{f2n}^T & A_{22n}^T J_{21}^T + C_{2n}^T B_{f2n}^T \\ A_{f11n}^T & A_{f11n}^T \end{bmatrix}$$

$$\pi_{10} = \begin{bmatrix} E_{2n}^T - C_{2n}^T D_{fn}^T \\ -C_{f2n}^T \end{bmatrix}, \quad \pi_{11} = \begin{bmatrix} \sqrt{d_1}A_{d11n}^T Z_1 \\ 0 \end{bmatrix}$$

$$\pi_{12} = \begin{bmatrix} \sqrt{d_2}A_{d21n}^T Z_2 \\ 0 \end{bmatrix}, \quad \pi_{13} = \begin{bmatrix} A_{d11n}^T L_{11}^T \\ 0 \end{bmatrix}$$

$$\pi_{14} = \begin{bmatrix} A_{d11n}^T L_{21}^T \\ 0 \end{bmatrix}, \quad \pi_{15} = \begin{bmatrix} A_{d21n}^T J_{11}^T \\ 0 \end{bmatrix}$$

$$\pi_{16} = \begin{bmatrix} A_{d21n}^T J_{21}^T \\ 0 \end{bmatrix}, \quad \pi_{17} = \begin{bmatrix} \sqrt{d_1} A_{d12n}^T Z_1 \\ 0 \end{bmatrix}$$

$$\pi_{18} = \begin{bmatrix} \sqrt{d_2} A_{d22n}^T Z_2 \\ 0 \end{bmatrix}, \quad \pi_{19} = \begin{bmatrix} A_{d12n}^T L_{11}^T \\ 0 \end{bmatrix}$$

$$\pi_{20} = \begin{bmatrix} A_{d12n}^T L_{21}^T \\ 0 \end{bmatrix}, \quad \pi_{21} = \begin{bmatrix} A_{d22n}^T J_{11}^T \\ 0 \end{bmatrix}$$

$$\pi_{22} = \begin{bmatrix} A_{d22n}^T J_{21}^T \\ 0 \end{bmatrix}, \quad \pi_{23} = \begin{bmatrix} \sqrt{d_1} B_{1n}^T Z_1 \end{bmatrix}$$

Furthermore, a suitable filter in the form of (3) is given by

$$\begin{bmatrix} \bar{A}_{f11n} & \bar{A}_{f12n} & \bar{B}_{f1n} \\ \bar{A}_{f21n} & \bar{A}_{f22n} & \bar{B}_{f2n} \\ \bar{C}_{f1n} & \bar{C}_{f2n} & \bar{D}_{fn} \end{bmatrix} = \begin{bmatrix} L_{12}^{-1} & 0 & 0 \\ 0 & J_{12}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{f11n} & A_{f12n} & B_{f1n} \\ A_{f21n} & A_{f22n} & B_{f2n} \\ C_{f1n} & C_{f2n} & D_{fn} \end{bmatrix}$$

Proof: By considering the uncertain 2-D discrete system in (1) and the parameter-dependent filter (3), similar to the proof of Theorem 4.1, the corresponding filtering error system is robustly asymptotically stable and has a prescribed H_∞ disturbance attenuation performance level γ, if the following inequality holds

$$\left[\begin{array}{cccccccccccc} \phi_{h1} - sym(L) & 0 & \left(\frac{L}{2}\right)^T + G^{hT}(\alpha) & 0 & \phi_{h2} & 0 & 0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \\ * & \phi_{v1} - sym(J) & 0 & \left(\frac{J}{2}\right)^T + G^{vT}(\alpha) & 0 & \phi_{v2} & 0 & \pi_6 & \pi_7 & \pi_8 & \pi_9 & \pi_{10} \\ * & * & -2G^h(\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -2G^v(\alpha) & 0 & 0 & 0 & \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & 0 \\ * & * & * & * & \phi_{h3} & 0 & 0 & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & 0 \\ * & * & * & * & * & \phi_{v3} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & \pi_{19} & \pi_{20} & \pi_{21} & \pi_{22} & \pi_{23} \\ * & * & * & * & * & * & * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -G_h & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -G_v & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -I \end{array}\right] < 0$$

with

$$\pi_1 = \begin{bmatrix} \sqrt{d_1}A_{11}^T(\alpha)Z_1 - \sqrt{d_1}Z_1 \\ 0 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \sqrt{d_2}A_{21}^T(\alpha)Z_2 \\ 0 \end{bmatrix}$$

$$\pi_3 = \begin{bmatrix} A_{11}^T(\alpha)L_{11}^T + C_1^T(\alpha)B_{f1}^T(\alpha) & A_{11}^T(\alpha)L_{21}^T + C_1^T(\alpha)B_{f1}^T(\alpha) \\ A_{f11}^T(\alpha) & A_{f11}^T(\alpha) \end{bmatrix}$$

$$\pi_4 = \begin{bmatrix} A_{21}^T(\alpha)J_{11}^T + C_1^T(\alpha)B_{f2}^T(\alpha) & A_{21}^T(\alpha)J_{21}^T + C_1^T(\alpha)B_{f2}^T(\alpha) \\ A_{f21}^T(\alpha) & A_{f21}^T(\alpha) \end{bmatrix}$$

$$\pi_5 = \begin{bmatrix} E_1^T(\alpha) - C_1^T(\alpha)D_f^T(\alpha) \\ -C_{f1}^T(\alpha) \end{bmatrix}, \quad \pi_6 = \begin{bmatrix} \sqrt{d_1}A_{12}^T(\alpha)Z_1 \\ 0 \end{bmatrix}$$

$$\pi_7 = \begin{bmatrix} \sqrt{d_2}A_{22}^T(\alpha)Z_2 - \sqrt{d_2}Z_2 \\ 0 \end{bmatrix}$$

$$\pi_8 = \begin{bmatrix} A_{12}^T(\alpha)L_{11}^T + C_2^T(\alpha)B_{f1}^T(\alpha) & A_{12}^T(\alpha)L_{21}^T + C_2^T(\alpha)B_{f1}^T(\alpha) \\ A_{f11}^T & A_{f11}^T \end{bmatrix}$$

$$\pi_9 = \begin{bmatrix} A_{22}^T(\alpha)J_{11}^T + C_2^T(\alpha)B_{f2}^T(\alpha) & A_{22}^T(\alpha)J_{21}^T + C_2^T(\alpha)B_{f2}^T(\alpha) \\ A_{f11}^T(\alpha) & A_{f11}^T(\alpha) \end{bmatrix}$$

$$\pi_{10} = \begin{bmatrix} E_2^T(\alpha) - C_2^T(\alpha)D_f^T(\alpha) \\ -C_{f2}^T(\alpha) \end{bmatrix}$$

$$\pi_{11} = \begin{bmatrix} \sqrt{d_1}A_{d11}^T(\alpha)Z_1 \\ 0 \end{bmatrix}, \quad \pi_{12} = \begin{bmatrix} \sqrt{d_2}A_{d21}^T(\alpha)Z_2 \\ 0 \end{bmatrix}$$

$$\pi_{13} = \begin{bmatrix} A_{d11}^T(\alpha)L_{11}^T \\ 0 \end{bmatrix} \quad \pi_{14} = \begin{bmatrix} A_{d11}^T(\alpha)L_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{15} = \begin{bmatrix} A_{d21}^T(\alpha)J_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{16} = \begin{bmatrix} A_{d21}^T(\alpha)J_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{17} = \begin{bmatrix} \sqrt{d_1}A_{d12}^T(\alpha)Z_1 \\ 0 \end{bmatrix}, \quad \pi_{18} = \begin{bmatrix} \sqrt{d_2}A_{d22}^T(\alpha)Z_2 \\ 0 \end{bmatrix}$$

$$\pi_{19} = \begin{bmatrix} A_{d12}^T(\alpha)L_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{20} = \begin{bmatrix} A_{d12}^T(\alpha)L_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{21} = \begin{bmatrix} A_{d22}^T(\alpha)J_{11}^T \\ 0 \end{bmatrix}, \quad \pi_{22} = \begin{bmatrix} A_{d22}^T(\alpha)J_{21}^T \\ 0 \end{bmatrix}$$

$$\pi_{23} = \begin{bmatrix} \sqrt{d_1}B_1^T(\alpha)Z_1 \end{bmatrix}$$

Remark 4.1. *Compared to 1-D systems, the analyses of 2-D systems are not easy due to their complex structures for which the dynamics depend on two independent variables. Theorem 3.1 gives a sufficient condition for H_∞ performance of uncertain 2-D discrete systems described by the Roesser model. Note that if system (1) reduces to a 1-D system with polytopic uncertainty, Theorem 3.1 coincides with H_∞ performance for 1-D systems. Thus, Theorem 3.1 can be viewed as an extension of existing results on the H_∞ performance and filtering for 1-D systems to the 2-D case.*

Remark 4.2. *The study of the uncertain 2-D discrete systems has shown a powerful ability to represent plenty of real systems. In the numerical examples section, we will consider a real process (thermal processes) to clarify the importance of our results.*

5. **Numerical Example.** In this section, two numerical examples are considered to show the less conservatism and the efficiency of the proposed approaches. Example 5.1 will be presented to consider the H_∞ filtering problem of a system without uncertainty. And Example 5.2 will be provided to consider the H_∞ filtering problem of the thermal processes in the presence of uncertainty.

Example 5.1. *Let us consider a Roesser state system with the following matrices:*

$$A = \begin{bmatrix} 0 & 1 \\ 0.25 & 0.75 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -0.003 & -0.004 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.05 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$E = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Given $d_1 = 1$, $d_2 = 1$ by solving the LMIs, the minimun H_∞ norm bound for this example is $\gamma = 0.07$ and filter matrices are as follows:

$$\begin{bmatrix} A_{f11} & A_{f12} & B_{f1} \\ A_{f21} & A_{f22} & B_{f2} \\ C_{f1} & C_{f2} & D_f \end{bmatrix} = \begin{bmatrix} -0.0017 & 0.0113 & -0.0032 \\ 0.0049 & 0.0030 & -0.009 \\ 0.8187 & -0.0091 & 0.9978 \end{bmatrix}$$

The transfer function of the filtering system is shown as Figure 1.

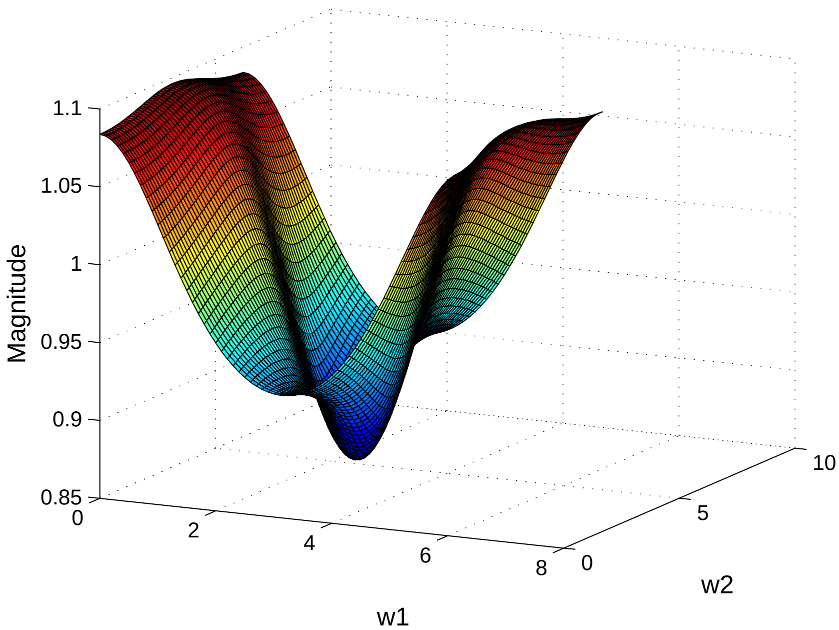

FIGURE 1. Transfer function

As shown in the example upward the solution of the method is feasible with mild results.

Example 5.2. *Consider the thermal processes* [20, 21] *in chemical reactors, heat exchangers, and pipe furnaces shown in Figure 2, which can be expressed in the partial differential equation with uncertainty and time delays:*

$$\begin{aligned}\frac{\partial T(x,t)}{\partial x} = &-\frac{\partial T(x,t)}{\partial t} - a_0(\sigma+1)\partial T(x,t) - a_1(\sigma+1)\partial T(x-x_d,t) \\ &- a_2(\sigma+1)\partial T(x,t-\tau) + b(\sigma+1)u(x,t) \end{aligned} \qquad (11)$$

where $T(x,t)$ is usually the temperature at x (space) $\in [0, x_f]$ and t (time) $\in [0, \infty]$, $u(x,t)$ is a given force function, a_0, a_1 and a_2, b are real coefficients, σ is unknown real time invariant parameter satisfying $|\sigma| \le \bar{\sigma}$, τ and x_d representing the time delay and the space delay respectively.

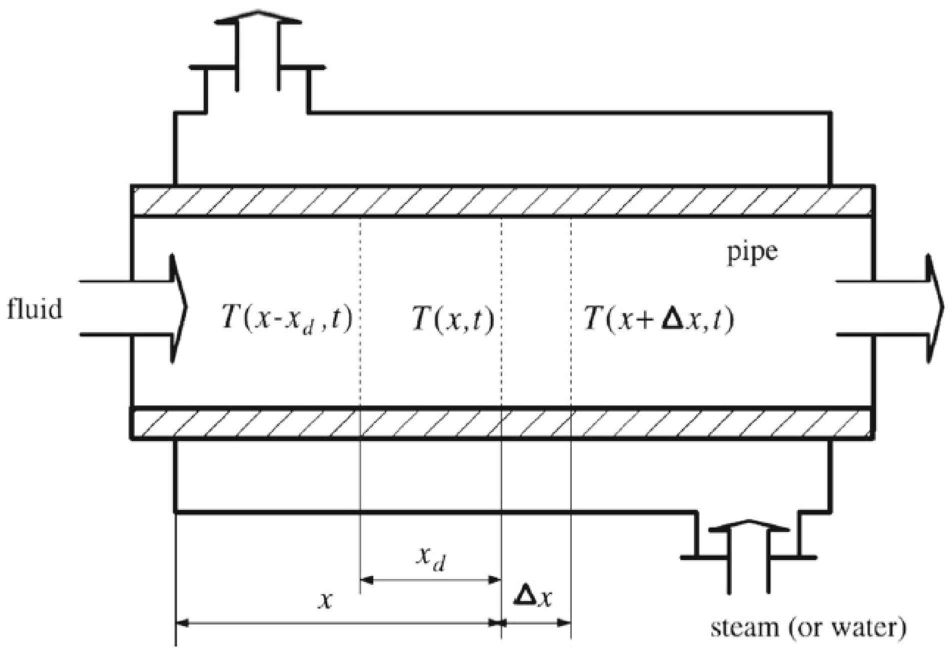

FIGURE 2. Exchanger

Denote $x^h(i,j) = T(i-1,j)$, $x^v(i,j) = T(i,j)$, $d_1 = int(x_d/\Delta x)$ and $d_2 = int(\tau/\Delta t+1)$ (where $int(.)$ is the integer function) and define

$$c(\alpha) = \sigma = \alpha_1\bar{\sigma} - \alpha_2\bar{\sigma}$$

It is easy to verify that Equation (11) can be converted into the Roesser model (1) with parameter matrices:

$$A(\alpha) = \begin{bmatrix} 0 & 1 \\ \dfrac{\Delta t}{\Delta x} & 1 - \dfrac{\Delta t}{\Delta x} - a_0(1+c(\alpha))\Delta t \end{bmatrix}$$

$$Ad(\alpha) = \begin{bmatrix} 0 & 0 \\ -a_1(1+c(\alpha))\Delta t & -a_2(1+c(\alpha))\Delta t \end{bmatrix}$$

Let $\Delta t = 0.2$, $\Delta x = 0.4$, $a_0 = 1$, $a_1 = 0.3$, $a_2 = 0.4$, $b = 1$. By considering the problem of H_∞ disturbance attenuation, the thermal process is modeled in the form (1) with

$$B(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C(\alpha) = \begin{bmatrix} 0.05 & 1 \end{bmatrix}, \quad D(\alpha) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$E(\alpha) = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Now, assume that $\bar{\sigma} = 0.9$, the system can be modeled as a polytope with two vertices:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.48 \end{bmatrix}$$

$$A_{d1} = \begin{bmatrix} 0 & 0 \\ -0.114 & -0.152 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0 \\ -0.006 & -0.008 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.05 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$E = \begin{bmatrix} 0 & 2.5 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Applying Theorem 4.2, we can obtain the minimum attenuation level $\gamma_{\min} = 1.5109$ *with* $d_1 = d_2 = 3$.

The filter matrices for the first vertex are as follows:

$$\begin{bmatrix} A_{f11} & A_{f12} & B_{f1} \\ A_{f21} & A_{f22} & B_{f2} \\ C_{f1} & C_{f2} & D_f \end{bmatrix} = \begin{bmatrix} -0.0084 & 0.0648 & -0.3718 \\ 0.3420 & 0.0053 & -0.1861 \\ 0.0792 & -0.1253 & 2.2333 \end{bmatrix}$$

and for the second vertex

$$\begin{bmatrix} A_{f11} & A_{f12} & B_{f1} \\ A_{f21} & A_{f22} & B_{f2} \\ C_{f1} & C_{f2} & D_f \end{bmatrix} = \begin{bmatrix} 0.0023 & 0.0295 & -0.2978 \\ 0.1042 & 0.0126 & -0.1904 \\ 0.0204 & -0.1270 & 2.2633 \end{bmatrix}$$

Based on Theorem 4.2, the actual H_∞ *performances obtained at the vertices is shown in Figure 3, which is below the guaranteed value 1.5109.*

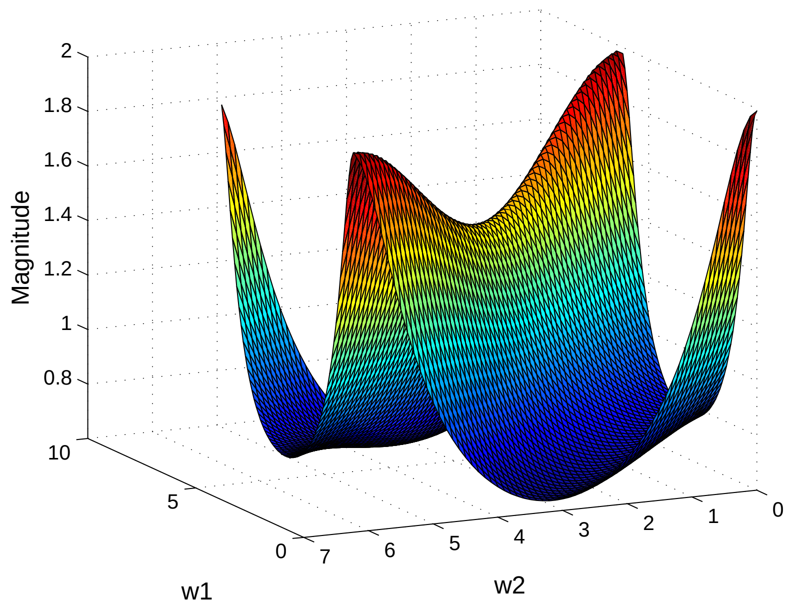

Figure 3. Transfer function example 2

6. **Conclusion.** The problem of robust H_∞ filtering has been considered in this paper for 2-D state delay systems described by the Roesser model with polytopic parameter uncertainty. The robust filters have been designed in terms of a feasible LMI, which guarantees the filtering error system to be asymptotically stable with a prescribed H_∞ performance for all admissible parameter uncertainties. An optimal filter design problem is also provided by optimizing the filtering performances. Numerical examples are given to illustrate the effectiveness of the proposed results.

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