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ELASTIC LIMIT ANALYSIS OF COMPOSED THICK-WALL CYLINDER CONSIDERING DIFFERENCE IN TENSION-COMPRESSION STRENGTH

Wengang Ao[1,2,*], Zongmin Liu[1], Huiyan Zhang[1] and Yonggang Huang[2]

[1]National Research Base of Intelligent Manufacturing Service
[2]Chongqing Municipal Key Laboratory of Mechanism Design and Control for Manufacturing Equipment
Chongqing Technology and Business University
No. 19, Xuefu Avenue, Nan'an District, Chongqing 400067, P. R. China
*Corresponding author: aowg@ctbu.edu.cn

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Abstract. *Based on the unified strength theory and the principle of equal strength design, considering the influence of difference in tension-compression strength and intermediate principal stress, the elastic limit analysis and structural parameter calculation of two- and three-layer composite thick-wall cylinder are carried out, and the calculation expressions of ultimate internal pressure, stratified radius and interference amount are obtained respectively. Through the analysis of examples, we can get the influence of difference in tension-compression strength and intermediate principal stress effect on the structural strength of composed thick-wall cylinder: when considering difference in tension-compression strength of materials, the structural strength of composed thick-wall cylinder increases; the value of intermediate principal stress coefficient increases gradually from 0 to 1. This research is universal and can provide reference for the strength analysis and structural design of cold extrusion combining die, high pressure vessel and barrel.*

Keywords: Unified strength theory, Composed thick-wall cylinder, Stratified radius, Interference amount

1. **Introduction.** The thick-wall cylinder is a common important component, such as pressure vessel, high pressure pipeline, composite support, die, chemical equipment, barrel, and many scholars have analyzed the strength and structural parameters of the thick-wall cylinder [1-9]. If the inner radius of the thick-wall cylinder is determined, to improve the ultimate bearing capacity, there are many limitations in only increasing the wall thickness or selecting higher strength materials, so the composed thick-wall cylinder can be adopted to make full use of the strength potential of each layer of thick-wall cylinder material by adjusting stress distribution and improve the bearing capacity of the composed thick-wall cylinder structure [10]. Meanwhile, such components are often made of hard alloy, high strength steel, composite materials, aluminum alloy, concrete and other materials with difference in tension-compression strength [17-19]. However, the traditional strength theory such as Tresca and Mises can only be used for materials with the same tension-compression strength. In the analysis for the strength and structural parameters of the combination die that is carried out on the basis of the unified strength theory [11,12], the influences of the difference in tension-compression strength and the intermediate principal stress influence coefficient of the materials on the structural ultimate bearing capacity can be considered simultaneously. Considering the effect of intermediate

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principal stress, the different tension-compression strength and modulus of materials, the work in [3] proposed the unified solutions of elastic limit internal pressure, plastic limit internal pressure and shakedown limit internal pressure of thick-wall cylinder based on the unified strength theory. Considering the influence of intermediate principal stress and the different tension-compression strength, the basic solutions of optimum stratified radius, sleeve pressure and sleeve interference were investigated based on the unified strength theory [6], and the unified solution of elastic limit internal pressure and plastic limit internal pressure of composed thick-wall cylinder were obtained.

Furthermore, this paper will be based on the unified strength theory and the principle of equal strength design [13-16], considering the material pressure of the opposite sex, and the intermediate principal stress effect and practical factors such as stress state, the inner cylinder is derived for brittle material or material toughness about two or three layers of thick wall cylinder elastic limit bearing capacity of structure and optimal parameters such as delamination radius, interference quantity, give full play to the strength of composite potential, and reduce the overall size of the structure.

2. **Analysis for Yield Conditions of Each Layer of Composed Cylinder Based on the Unified Strength Theory.** The combination die, composite support, high pressure vessel, barrel and other structures can be regarded as composed thick-wall cylinder for strength analysis, and the inner layer or other layers of cylinder materials of such structures may be hard alloy, high-strength alloy steel, cast iron, aluminum alloy, composite material, concrete and other materials with obvious difference in tension-compression strength. The yield condition can be determined by using the unified strength theory [11,12]. The mathematical expression of the unified strength theory [6,7] in the principal stress space is

$$f = \sigma_1 - \frac{\alpha}{1+b}\left(b\sigma_2 + \sigma_3\right) = \sigma_t, \text{ if } \sigma_2 \le \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{1}$$

$$f' = \frac{1}{1+b}\left(\sigma_1 + b\sigma_2\right) - \alpha\sigma_3 = \sigma_t, \text{ if } \sigma_2 \ge \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{2}$$

where σ_1, σ_2 and σ_3 are the first, second and third principal stresses respectively, the parameter $\alpha = \sigma_t/\sigma_c \le 1$ is the ratio of the ultimate tensile strength σ_t to the ultimate compression strength σ_c of the material, the parameter $b = \frac{(1+\alpha)\tau_b - \sigma_t}{\sigma_t - \tau_b} \le 1$ is the intermediate stress influence coefficient, and τ_b is shear strength of the material.

2.1. **Principal stress analysis of thick-wall cylinder.** According to Lamé Formula, three-direction principal stresses (circumferential stress σ_θ, radial stress σ_ρ, axial stress σ_Z) of the thick-wall cylinder can be expressed as

$$\sigma_\rho = \frac{R^2 r^2 (p_2 - p_1)}{(R^2 - r^2)\rho^2} + \frac{r^2 p_1 - R^2 p_2}{R^2 - r^2} \tag{3}$$

$$\sigma_\theta = -\frac{R^2 r^2 (p_2 - p_1)}{(R^2 - r^2)\rho^2} + \frac{r^2 p_1 - R^2 p_2}{R^2 - r^2} \tag{4}$$

$$\sigma_Z = m(\sigma_\theta + \sigma_\rho) \tag{5}$$

where p_1 is the inner wall pressure of the thick-wall cylinder, and p_2 is the outer wall pressure of the thick-wall cylinder. For the thick-wall cylinder subjected to uniform internal and external pressure, $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_Z$, $\sigma_3 = \sigma_\rho$. In Formula (5), σ_Z is the intermediate principal stress (second principal stress), and the value needs to consider the stress state of the structure: when the structure can be simplified as the plane stress state, $m = 0$; when the structure can be simplified as the plane strain state, $m = \nu$

(elastic state); in structural plastic area, based on the condition of the structural volume remaining unchanged, $m = 0.5$.

2.2. **Failure condition of composed thick-wall cylinder.** In Formula (5), according to $\sigma_Z = m(\sigma_\theta + \sigma_\rho) \leq 1/2(\sigma_\theta + \sigma_\rho)$, it can be derived that

$$\sigma_Z = m(\sigma_\theta + \sigma_\rho) \leq \frac{\sigma_\theta + \alpha\sigma_\rho}{1+\alpha} \tag{6}$$

According to Formula (1) and Formula (6), the yield condition of each layer of the composed cylinder is

$$\frac{1+b-mb\alpha}{1+b}\sigma_\theta - \frac{mb\alpha+\alpha}{1+b}\sigma_\rho = \sigma_S \tag{7}$$

If the inner layer of the composed cylinder is made of hard alloy, stone and other fragile materials, it can bear greater compressive stress, while the tensile stress bearing capacity is very small. Therefore, it is designed so that the circumferential stress at the inner wall is not greater than zero, and the failure condition is

$$\sigma_\theta = 0 \tag{8}$$

When the inner radius of the thick-wall cylinder is fixed, it is not enough to increase the strength only by increasing the wall thickness and selecting the materials with higher strength. However, it is an effective scheme to adjust the stress distribution of the composed thick-wall cylinder structure to improve the structural strength. The optimum design of the composed cylinder is to make the equivalent stress of each layer of composed cylinder reach the allowable stress of material at the same time by adjusting the stratified radius and interference amount, so as to maximize the strength potential of each layer of the composed cylinder. In this paper, the strength analysis of two-layer and three-layer composed thick-wall cylinder structure is carried out. In order to simplify the derivation process expression, the following parameters are set:

$$\begin{aligned}
&A_1 = 1+b-mb\alpha_1, \quad B_1 = 1+b+\alpha_1, \quad C_1 = 1+b-2mb\alpha_1-\alpha_1 \\
&A_2 = 1+b-mb\alpha_2, \quad B_2 = 1+b+\alpha_2, \quad C_2 = 1+b-2mb\alpha_2-\alpha_2 \\
&B_3 = 1+b+\alpha_3, \quad C_3 = 1+b-2mb\alpha_3-\alpha_3
\end{aligned}$$

where α_1, α_2, α_3 are the tension-compression strength ratios of three-layer composed thick-wall cylinder material from inside to outside, b is the intermediate principal stress influence coefficient, and m is the parameter reflecting the structural stress state.

3. **Elastic Limit Analysis and Structural Parameter Calculation of Two-Layer Composed Thick-Wall Cylinder.** Under the action of internal pressure, the inner walls of the inner and outer layers of the composed thick-wall cylinder fail at the same time, and the strength potentials of the two-layer cylinder materials can be brought into full play, that is, the composed thick-wall cylinder with the optimum structure in theory. The stratified radius, elastic limit internal pressure, interference amount, etc., are derived accordingly.

3.1. **Calculation of elastic limit and stratified radius when the inner layer of cylinder is made of ductile material.** If the inner layer of cylinder is made of ductile material, the inner walls of the inner and outer layers of cylinder are required to yield, obtaining

$$\frac{1+b-mb\alpha_1}{1+b}\left[\frac{(r_1^2+r^2)\,P_{e1} - 2r_1^2 q}{(r_1^2-r^2)}\right] + \frac{mb\alpha_1+\alpha_1}{1+b}P_{e1} = \sigma_{S1} \tag{9}$$

$$\frac{1+b-mb\alpha_2}{1+b}\frac{R^2+r_1^2}{R^2-r_1^2}q+\frac{mb\alpha_2+\alpha_2}{1+b}q=\sigma_{S2} \tag{10}$$

In the above two formulae, R, r are the inner radius and outer radius of the composed cylinder, respectively, and r_1 is the stratified radius. The compressive stress of the inner walls of the inner and outer layers of cylinder can be derived as follows from Formulae (9) and (10) when the composed thick-wall cylinder yields

$$P_{e1}=\frac{2A_1\left(1+b\right)\left(R^2-r_1^2\right)r_1^2}{\left(B_1r_1^2+C_1r^2\right)\left(B_2R^2+C_2r_1^2\right)}\sigma_{S2}+\frac{\left(1+b\right)\left(r_1^2-r^2\right)}{\left(B_1r_1^2+C_1r^2\right)}\sigma_{S1} \tag{11}$$

$$q=\frac{\left(1+b\right)\left(R^2-r_1^2\right)}{B_2R^2+C_2r_1^2}\sigma_{S2} \tag{12}$$

According to Formula (11), the internal pressure P_{e1} on the inner wall of the composed cylinder is the function of the stratified radius r_1; when $\partial P_{e1}/\partial r_1=0$, the strength of the composed cylinder is the highest, and the stratified radius r_1 meets the following equation.

$$a_1r_1^4+b_1r_1^2+c_1=0 \tag{13}$$

where

$$a_1=4A_1\sigma_{S2}\left(B_1B_2R^2+B_1C_2R^2+C_1C_2r^2\right)-2(B_1+C_1)C_2^2\sigma_{S1}r^2$$
$$b_1=8A_1B_2C_1\sigma_{S2}R^2r^2-4B_2C_2(B_1+C_1)\sigma_{S1}R^2r^2$$
$$c_1=-4A_1B_2C_1\sigma_{S2}R^4r^2-2B_2^2(B_1+C_1)\sigma_{S1}R^4r^2$$

If the influences of the difference in tension-compression strength and the intermediate principal stress of the material are not considered, $\alpha_1=\alpha_2=1$ and $b=0$, the expression of the stratified radius can be derived as

$$r_1=\sqrt[4]{\frac{\sigma_{S1}}{\sigma_{S2}}}\times\sqrt{Rr} \tag{14}$$

Thus, the unified strength theory is degraded into Tresca yield criterion, which is consistent with the conclusion in [15]; when the inner and outer layers of cylinder are made of the same material, the stratified radius is calculated as

$$r_1=\sqrt{Rr} \tag{15}$$

which is consistent with the conclusion in [10].

3.2. **Calculation of elastic limit and stratified radius when the inner layer of cylinder is made of fragile material.** If the inner layer of cylinder is made of fragile material, the failure condition of the inner layer of cylinder is that the circumferential stress at the inner wall is $\sigma_\theta=0$, obtaining

$$\left(r^2+r_1^2\right)P_{e2}-2r_1^2q=0 \tag{16}$$

Meanwhile the inner wall of the outer layer of cylinder needs to yield, and Formula (10) is met; the compressive stress between the inner and outer layers of cylinder can be calculated by Formula (12), while the internal pressure of the composed cylinder is

$$P_{e2}=\frac{2\left(1+b\right)\left(R^2-r_1^2\right)r_1^2}{\left(B_2R^2+C_2r_1^2\right)\left(r^2+r_1^2\right)}\sigma_{S2} \tag{17}$$

When $\partial P_{e2}/\partial r_1=0$, the strength of the composed thick-wall cylinder structure is the highest, and the stratified radius can be thus derived which meets the following equation:

$$a_2r_1^4+b_2r_1^2+c_2=0 \tag{18}$$

where,

$$a_2 = 4\left(B_2 + C_2\right)R^2 + 4C_2r^2$$
$$b_2 = 8B_2R^2r^2$$
$$c_2 = -4B_2R^4r^2$$

If the influences of the difference in tension-com-pression strength and the intermediate principal stress of the material are not considered, $\alpha_1 = \alpha_2 = 1$ and $b = 0$; the stratified radius can be calculated as

$$r_1 = \sqrt{\sqrt{(R^2 + r^2)}r - r^2} \tag{19}$$

3.3. **Calculation of interference amount of two-layer composed thick-wall cylinder.** Based on the previous analysis, the calculation expressions of the elastic limit internal pressure, the compressive stress between the inner and outer layers of cylinder, and the stratified radius of the composed thick-wall cylinder are derived, and the calculation expressions of the sleeving pressure and interference amount between the inner and outer layers of cylinder can be further derived. The inner radius and outer radius of the two-layer composed thick-wall cylinder are r and R, respectively; the stratified radius is r_1, and the internal working pressure is $P_{e1}(P_{e2})$; the compressive stress between inner and outer layers of cylinder is q. According to the principle of stress superposition

$$q = P_1 + P_{1e} \tag{20}$$

In the above formula, P_1 is the sleeving pressure, and P_{1e} is the compressive stress generated between the inner and outer layers of cylinder under the action of the internal working pressure. Under the action of the internal working pressure P_{e1} (P_{e2}), the inner layer of thick-wall cylinder is subjected to joint actions of the internal working pressure P_{e1} (P_{e2}) and outer surface compressive stress P_{1e}, and the outer layer of cylinder is subjected to the action of P_{1e} at the inner wall. When the deformations of the inner and outer layers of cylinder at the stratified radius are the same, and the following equation is met

$$\begin{aligned}&\frac{1}{E_2}\left[(1+\nu_2)\frac{R^2r_1P_{1e}}{R^2-r_1^2} + (1-\nu_2)\frac{r_1^3P_{1e}}{R^2-r_1^2}\right]\\ &= \frac{1}{E_1}\left[-(1+\nu_1)\frac{r^2r_1\left(P_{1e}-P_{e1}\right)}{r_1^2-r^2} + (1-\nu_1)\frac{r^2r_1P_{e1} - r_1^3P_{1e}}{r_1^2-r^2}\right]\end{aligned} \tag{21}$$

where ν_1, ν_2 are the Poisson's ratios of the inner and outer layers of cylinder materials, which can be derived by Formula (21)

$$P_{1e} = \frac{2E_2\left(R^2-r_1^2\right)r^2r_1P_{e1}}{E_1\left[(1+\nu_2)r_1R^2 + (1-\nu_2)r_1^3\right](r_1^2-r^2) + E_2\left[(1+\nu_1)r_1r^2 + (1-\nu_1)r_1^3\right](R^2-r_1^2)} \tag{22}$$

From Formulae (12) and (22), q and P_{1e} can be calculated, and substituted into Formula (20) to calculate the sleeving pressure P_1.

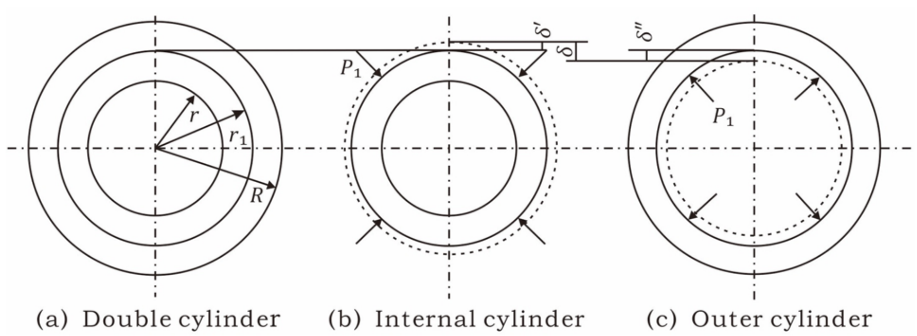


FIGURE 1. Composed thick-wall cylinder

As shown in Figure 1, the outer radius of the inner layer of cylinder is $r_1 + \delta'$, and the inner radius of the outer layer of cylinder is $r_1 - \delta''$; when they are sleeved, the outer surface of the inner cylinder is subjected to the action of the external pressure P_1, and the outer cylinder is subjected to the action of the internal pressure P_1. Therefore, the interference amount is

$$\delta = \frac{P_1 r_1}{E_1}\left(\frac{r_1^2 + r^2}{r_1^2 - r^2} - \nu_1\right) + \frac{P_1 r_1}{E_2}\left(\frac{R^2 + r_1^2}{R^2 - r_1^2} + \nu_2\right) \tag{23}$$

4. **Elastic Limit Analysis and Structural Parameter Calculation of Three-Layer Composed Thick-Wall Cylinder.** Under the action of the elastic limit internal working pressure, if the inner walls of the inner, middle and outer layers of cylinder in the composed thick-wall cylinder fail at the same time, the strength potentials of the three layers of cylinder materials can be brought into full play, that is, the composed thick-wall cylinder with the optimal structure in theory. The stratified radius, elastic limit internal pressure, interference amount, etc., are derived accordingly.

4.1. **Calculation of elastic limit and stratified radius when the inner layer of cylinder is made of ductile material.** If the inner layer of cylinder is made of ductile material, the inner walls of the inner, middle and outer layers of cylinder yield at the same time, obtaining

$$\frac{1 + b - mb\alpha_1}{1 + b}\left[\frac{\left(r_1^2 + r^2\right) P_{e3} - 2r_1^2 q_1}{r_1^2 - r^2}\right] + \frac{mb\alpha_1 + \alpha_1}{1 + b} P_{e3} = \sigma_{S1} \tag{24}$$

$$\frac{1 + b - mb\alpha_2}{1 + b}\left[\frac{\left(r_1^2 + r_2^2\right) q_1 - 2r_2^2 q_2}{r_2^2 - r_1^2}\right] + \frac{mb\alpha_2 + \alpha_2}{1 + b} q_1 = \sigma_{S2} \tag{25}$$

$$\frac{\left(1 + b - mb\alpha_3\right)\left(R^2 + r_2^2\right)}{\left(1 + b\right)\left(R^2 - r_2^2\right)} q_2 + \frac{mb\alpha_3 + \alpha_3}{1 + b} q_2 = \sigma_{S3} \tag{26}$$

In Formulae (24)-(26), P_{e3}, q_1, q_2 are elastic limit internal pressure, compressive stress between inner and middle layers of cylinder, and compressive stress between middle and outer layers of cylinder, respectively. According to Formulae (24)-(26), q_2, q_1, P_{e3} can be derived as

$$q_2 = \frac{\left(1 + b\right)\left(R^2 - r_2^2\right)}{B_3 R^2 + C_3 r_2^2} \sigma_{S3} \tag{27}$$

$$q_1 = \frac{2A_2\left(1 + b\right)\left(R^2 - r_2^2\right) r_2^2}{\left(B_2 r_2^2 + C_2 r_1^2\right)\left(B_3 R^2 + C_3 r_2^2\right)} \sigma_{S3} + \frac{\left(1 + b\right)\left(r_2^2 - r_1^2\right)}{\left(B_2 r_2^2 + C_2 r_1^2\right)} \sigma_{S2} \tag{28}$$

$$P_{e3} = \frac{1 + b}{B_1 r_1^2 + C_1 r^2}\left[\frac{4A_1 A_2\left(R^2 - r_2^2\right) r_1^2 r_2^2 \sigma_{S3}}{\left(B_3 R^2 + C_3 r_2^2\right)\left(B_2 r_2^2 + C_2 r_1^2\right)} + \frac{2A_1\left(r_2^2 - r_1^2\right) r_1^2 \sigma_{S2}}{B_2 r_2^2 + C_2 r_1^2} + \left(r_1^2 - r^2\right)\sigma_{S1}\right] \tag{29}$$

r_1 in the above three formulae is the stratified radius of the inner layer and middle layer of thick-wall cylinder, and r_2 is the stratified radius of the middle layer and outer layer of thick-wall cylinder. According to Formulae (28) and (29), P_{e3}, q_1 are the functions of r_1, r_2, and q_2 is the function of r_2. Assuming that r_1 has been determined, the middle and outer layers of thick-wall cylinder can bear the maximum internal pressure through adjusting r_2, and $\partial q_1/\partial r_2 = 0$; when q_1 is the maximum, r_2 meets the following equation:

$$a_3 r_2^4 + b_3 r_2^2 + c_3 = 0 \tag{30}$$

where,

$$a_3 = 4A_2\sigma_{S3}\left(B_2B_3R^2 + B_2C_3R^2 + C_2C_3r_1^2\right) - 2\left(B_2 + C_2\right)C_3^2\sigma_{S2}r_1^2$$
$$b_3 = 8A_2B_3C_2\sigma_{S3}R^2r_1^2 - 4B_3C_3\left(B_2 + C_2\right)\sigma_{S2}R^2r_1^2$$
$$c_3 = -4A_2B_3C_2\sigma_{S3}R^4r_1^2 - 2B_3^2\left(B_2 + C_2\right)\sigma_{S2}R^4r_1^2$$

when the middle layer and outer layer of materials are the same, $A_2 = A_3$ and $B_2 = B_3$, $C_2 = C_3$, $\sigma_{S2} = \sigma_{S3}$, to derive

$$r_2 = \sqrt{Rr_1} \tag{31}$$

which is consistent with the conclusion in [16]. The above conditions are substituted into Formula (29), and the elastic limit internal pressure of the composed thick-wall cylinder is simplified as

$$P_{e3} = \frac{2A_1\left(1+b\right)\left(2A_2R + B_2R + C_2r_1\right)\left(R - r_1\right)r_1^2}{\left(B_1r_1^2 + C_1r^2\right)\left(B_2R + C_2r_1\right)^2}\sigma_{S2} + \frac{\left(1+b\right)\left(r_1^2 - r^2\right)}{B_1r_1^2 + C_1r_1^2}\sigma_{S1} \tag{32}$$

Thus, the elastic limit P_{e3} of the composed cylinder is the function of the stratified radius r_1, and the calculation expression of r_1 can be derived through $\partial P_{e3}/\partial r_1 = 0$ when P_{e3} is the maximum.

If $\alpha_1 = \alpha_2 = 1$ and $m = 0.5$ (plane strain state), $A_1 = A_2 = 1 + 0.5b$, $B_1 = B_2 = 2 + b = 2A_1$, $C_1 = C_2 = 0$, and the elastic limit of the composed cylinder is simplified as

$$P_{e3} = \frac{2\left(1+b\right)\left(R - r_1\right)}{B_1R}\sigma_{S2} + \frac{\left(1+b\right)\left(r_1^2 - r^2\right)}{B_1r_1^2}\sigma_{S1} \tag{33}$$

The stratified radius r_1 can be derived as

$$r_1 = \sqrt[3]{\frac{\sigma_{S1}}{\sigma_{S2}}Rr^2} \tag{34}$$

which is consistent with the conclusion in [16].

4.2. **Calculation of elastic limit and stratified radius when the inner layer of cylinder is made of fragile material.** When the inner layer of cylinder is made of fragile material, the tensile property is poor, therefore the failure condition of the composed thick-wall cylinder structure is that the circumferential stress of the inner wall is zero, and

$$\frac{\left(r^2 + r_1^2\right)P_{e4} - 2r_1^2q_1}{r_1^2 - r^2} = 0 \tag{35}$$

And the inner walls of the middle and outer layers of cylinder yield at the same time, which can be expressed as Formulae (25) and (26). According to Formulae (25), (26) and (35), the compressive stresses at the inner walls of the outer, middle and outer layers of cylinder are

$$q_2 = \frac{\left(1+b\right)\left(R^2 - r_2^2\right)}{B_3R^2 + C_3r_2^2}\sigma_{S3} \tag{36}$$

$$q_1 = \frac{2A_2\left(1+b\right)\left(R^2 - r_2^2\right)r_2^2}{\left(B_2r_2^2 + C_2r_1^2\right)\left(B_3R^2 + C_3r_2^2\right)}\sigma_{S3} + \frac{\left(1+b\right)\left(r_2^2 - r_1^2\right)}{B_2r_2^2 + C_2r_1^2}\sigma_{S2} \tag{37}$$

$$P_{e4} = \frac{4A_2\left(1+b\right)\left(R^2 - r_2^2\right)r_2^2r_1^2}{\left(B_2r_2^2 + C_2r_1^2\right)\left(B_3R^2 + C_3r_2^2\right)\left(r^2 + r_1^2\right)}\sigma_{S3} + \frac{2\left(1+b\right)\left(r_2^2 - r_1^2\right)r_1^2}{\left(B_2r_2^2 + C_2r_1^2\right)\left(r^2 + r_1^2\right)}\sigma_{S2} \tag{38}$$

The middle and outer layers of thick-wall cylinder yield under the action of q_1; if r_1 is deemed as a constant, q_1 becomes the function of r_2; when q_1 can be derived from $\partial q_1/\partial r_2 = 0$ to get the maximum, r_2 meets the following equation:

$$a_4r_2^4 + b_4r_2^2 + c_4 = 0 \tag{39}$$

where

$$\begin{aligned} a_4 &= 4A_2\sigma_{S3}\left(B_2B_3R^2 + B_2C_3R^2 + C_2C_3r_1^2\right) - 2\left(B_2 + C_2\right)C_3^2 r_1^2\sigma_{S2} \\ b_4 &= 8A_2B_3C_2R^2r_1^2\sigma_{S3} - 4B_3C_3\left(B_2 + C_2\right)R^2r_1^2\sigma_{S2} \\ c_4 &= -4A_2B_3C_2R^4r_1^2\sigma_{S3} - 2B_3^2\left(B_2 + C_2\right)R^4r_1^2\sigma_{S2} \end{aligned}.$$

In general, the elastic limit internal pressure and stratified radius of the three-layer composed thick-wall cylinder are solved according to the following processes: first, taking the value of r_1 from small to large according to the small difference within a certain range; then calculating the corresponding r_2 through Formulae (30) or (39), and calculating the corresponding P_{e3} (P_{e4}) through Formulae (33) and (38); the maximum of P_{e3} (P_{e4}) is the elastic limit internal pressure of the composed thick-wall cylinder, and the corresponding r_1, r_2 are two stratified radii of the composed thick-wall cylinder; finally, q_1, q_2 are calculated according to Formulae (27), (28) or (36), (37).

4.3. **Calculation of interference amount of three-layer composed thick-wall cylinder.** The inner radius and outer radius of three-layer composed thick-wall cylinder are r and R, respectively, and r_1, r_2, q_2, q_1 and P_{e3} (P_{e4}) have been calculated through the previous section. Analysis is performed in the sequence of first sleeving the middle and outer layers of cylinder, and then sleeving the inner layer of cylinder, and the compressive stress q_1 between the inner layer and middle layer of cylinder and the compressive stress q_2 between the middle layer and the outer layer of cylinder can be respectively expressed as

$$q_1 = P_1 + P_{1e} \tag{40}$$

$$q_2 = P_2 + P_{21} + P_{2e} \tag{41}$$

where P_1 is the sleeving compressive stress between the inner layer and middle layer of cylinder, and P_2 is the sleeving compressive stress between the middle layer and outer layer of cylinder; P_{1e} is the compressive stress generated at the inner wall of the middle layer of cylinder under the action of internal working pressure, and P_{2e} is the compressive stress generated at the inner wall of the outer layer of cylinder under the action of internal working pressure; P_{21} is the compressive stress generated at the inner wall of the outer layer of cylinder under the action of P_1. Under the action of P_1, if the displacement of the outer surface of the middle layer of cylinder is equal to that of the inner wall of the outer layer of cylinder, then the following equation is met

$$\begin{aligned} &\frac{1}{E_3}\left[(1+\nu_3)\frac{R^2P_{21}}{R^2 - r_2^2} + (1-\nu_3)\frac{r_2^2P_{21}}{R^2 - r_2^2}\right] \\ &= \frac{1}{E_2}\left[-(1+\nu_2)\frac{r_1^2\left(P_{21} - P_1\right)}{r_2^2 - r_1^2} + (1-\nu_2)\frac{r_1^2P_1 - r_2^2P_{21}}{r_2^2 - r_1^2}\right] \end{aligned} \tag{42}$$

Derive according to Formula (42)

$$P_{21} = \frac{2E_3\left(R^2 - r_2^2\right)r_1^2P_1}{E_2\left[(1+\nu_3)R^2 + (1-\nu_3)r_2^2\right]\left(r_2^2 - r_1^2\right) + E_3\left[(1+\nu_2)r_1^2 + (1-\nu_2)r_2^2\right]\left(R^2 - r_2^2\right)} \tag{43}$$

Under the action of internal working pressure P_e (P_{e3} or P_{e4}), if the displacement of the outer surface of the inner layer of cylinder is equal to that of the inner wall of the middle layer of cylinder, and the displacement of the outer surface of the middle layer of cylinder is equal to that of the inner wall of the outer layer of cylinder, then the following equation set is met

$$\frac{1}{E_2}\left[(1+\nu_2)\frac{r_2^2\left(P_{1e} - P_{2e}\right)}{r_2^2 - r_1^2} + (1-\nu_2)\frac{r_1^2P_{1e} - r_2^2P_{2e}}{r_2^2 - r_1^2}\right]$$

$$= \frac{1}{E_1}\left[(1+\nu_1)\frac{P_e-P_{1e}}{r_1^2-r^2}+(1-\nu_1)\frac{r^2P_e-r_1^2P_{1e}}{r_1^2-r_2^2}\right] \tag{44}$$

$$\frac{1}{E_3}\left[(1+\nu_3)\frac{R^2P_{2e}}{R^2-r_2^2}+(1-\nu_3)\frac{r_2^2P_{2e}}{R^2-r_2^2}\right]$$
$$= \frac{1}{E_2}\left[-(1+\nu_2)\frac{r_1^2\left(P_{2e}-P_{1e}\right)}{r_2^2-r_1^2}+(1-\nu_2)\frac{r_1^2P_{1e}-r_2^2P_{2e}}{r_2^2-r_1^2}\right] \tag{45}$$

Derive according to Formula (45)

$$P_{2e} = \frac{2E_3\left(R^2-r_2^2\right)r_1^2P_{1e}}{E_2\left[(1+\nu_3)R^2+(1-\nu_3)r_2^2\right]\left(r_2^2-r_1^2\right)+E_3\left[(1+\nu_2)r_1^2+(1-\nu_2)r_2^2\right]\left(R^2-r_2^2\right)} \tag{46}$$

Suppose $P_{2e}=nP_{1e}$, and substitute it into Formula (44) and derive

$$P_{1e} = \frac{2E_2\left(r_2^2-r_1^2\right)r^2P_e}{E_2\left[(1+\nu_1)r^2\left(r_2^2-r_1^2\right)+(1-\nu_1)\left(r_1^2r_2^2-r_1^4\right)\right]+E_1\left[(1+\nu_2)r_2^2\left(r_1^2-r^2\right)+(1-\nu_2)\left(r_1^4-r_1^2r^2\right)-2nr_2^2\left(r_1^2-r^2\right)\right]} \tag{47}$$

P_{21}, P_{1e}, P_{2e} can be calculated through Formulae (43), (47), (46).

First, the middle layer of cylinder is sleeved with the outer layer of cylinder, the sum of the outer radius shrinkage of the middle layer of cylinder under the action of external pressure P_2 and the inner radius growth of the outer layer of cylinder under the action of internal pressure is the interference amount between the middle and outer layers of cylinder. According to Lamé Formula, the interference amount can be calculated as follows:

$$\delta_2 = \frac{P_2r_2}{E_2}\left(\frac{r_2^2+r_1^2}{r_2^2-r_1^2}-\nu_2\right)+\frac{P_2r_2}{E_3}\left(\frac{R^2+r_2^2}{R^2-r_2^2}+\nu_3\right) \tag{48}$$

Second, when the inner and middle layers of cylinder are sleeved, the outer radius of the inner layer of cylinder shrinks under the action of the external pressure P_1, and the inner radius of the middle layer of cylinder grows under the actions of the internal pressure P_1 and external pressure P_{21}. As shown in Figure 2, the sum of the outer radius shrinkage of the inner layer of cylinder and the inner radius growth of the middle layer of cylinder is the interference amount between the inner and middle layers of cylinder, and the expression is derived according to Lamé Formula as

$$\delta_1 = \frac{P_1r_1}{E_1}\left(\frac{r^2+r_1^2}{r_1^2-r^2}-\nu_1\right)+\frac{P_1r_1}{E_2}\left(\frac{(1+\nu_2)r_2^2+(1-\nu_2)r_1^2}{r_2^2-r_1^2}\right)-\frac{2r_1r_2^2P_{21}}{E_2\left(r_2^2-r_1^2\right)} \tag{49}$$

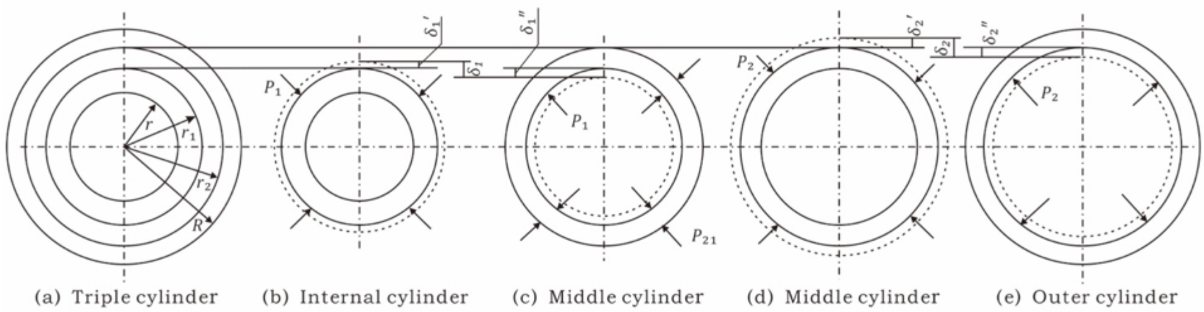


Figure 2. Three-layer composed thick-wall cylinder

5. **Application Example.** There is a high pressure vessel with the inner radius and outer radius of 20 mm and 62.5 mm, which can be simplified as three-layer composed thick-wall cylinder structure: the inner layer is made of fragile material, and $E_1 = 722$ GPa, $\mu_1 = 0.22$, $\sigma_1 = 3000$ MPa; the parameters of middle and outer layers of materials are $E_2 = 197$ GPa, $\mu_2 = 0.3$, $\sigma_2 = 1456$ MPa, $E_3 = 200$ GPa, $\mu_3 = 0.3$, $\sigma_3 = 1226$ MPa, and the elastic limit internal pressure, stratified radius and interference amount of the structure are calculated.

5.1. **Ultimate bearing capacity and structural parameter calculation of composed thick-wall cylinder.** Procedures are prepared as per Section 3 and Section 4 and used for calculating the ultimate bearing capacity, stratified radius, interference amount and other structural parameters of the composed thick-wall cylinder based on unified strength theory. First, irrespective of the difference in tension-compression strength and intermediate principal stress of the material, $\alpha_1 = \alpha_2 = 1$, $b = 0$, and the unified strength theory is degraded into Tresca yield criterion, obtaining the limit internal pressure $P_e = 986.9$ MPa, stratified radius $r_1 = 26.43$ mm, $r_2 = 42.43$ mm and interference amount $\delta_1 = 0.145$ mm, $\delta_2 = 0.126$ mm of the composed thick-wall cylinder. Such results are very close to the optimal solutions obtained in [7] when the outer radius of inner layer of thick-wall cylinder is taken as 25 mm. If the inner radius and outer radius of the inner layer of thick-wall cylinder are predefined as 20 mm, 25 mm, $P_e = 983.63$ MPa, $r_2 = 41.3$ mm, $\delta_1 = 0.116$ mm, $\delta_2 = 0.178$ mm can be calculated according to the method herein, which is almost identical to the result obtained in [7] by Lagrange optimization algorithm.

When only considering the influence of intermediate principal stress of the composed thick-wall cylinder structure, take $\alpha_1 = \alpha_2 = 1$, $\beta = 0.366$, degrade into Mises yield criterion, and then obtain elastic limit internal pressure $P_e = 1135.6$ MPa, stratified radius $r_1 = 26.27$ mm, $r_2 = 42.30$ mm, interference amount $\delta_1 = 0.166$ mm, $\delta_2 = 0.143$ mm. Compared with strength analysis by Tresca yield criterion, the bearing capacity is promoted by 15%. When considering the influences of both difference in tension-compression strength and intermediate principal stress of the material, $\alpha_1 = 0.7$, $\alpha_2 = 0.85$, $b = 0.366$ can be taken, and the bearing capacity of the thick-wall cylinder structure is $P_e = 1247.6$ MPa.

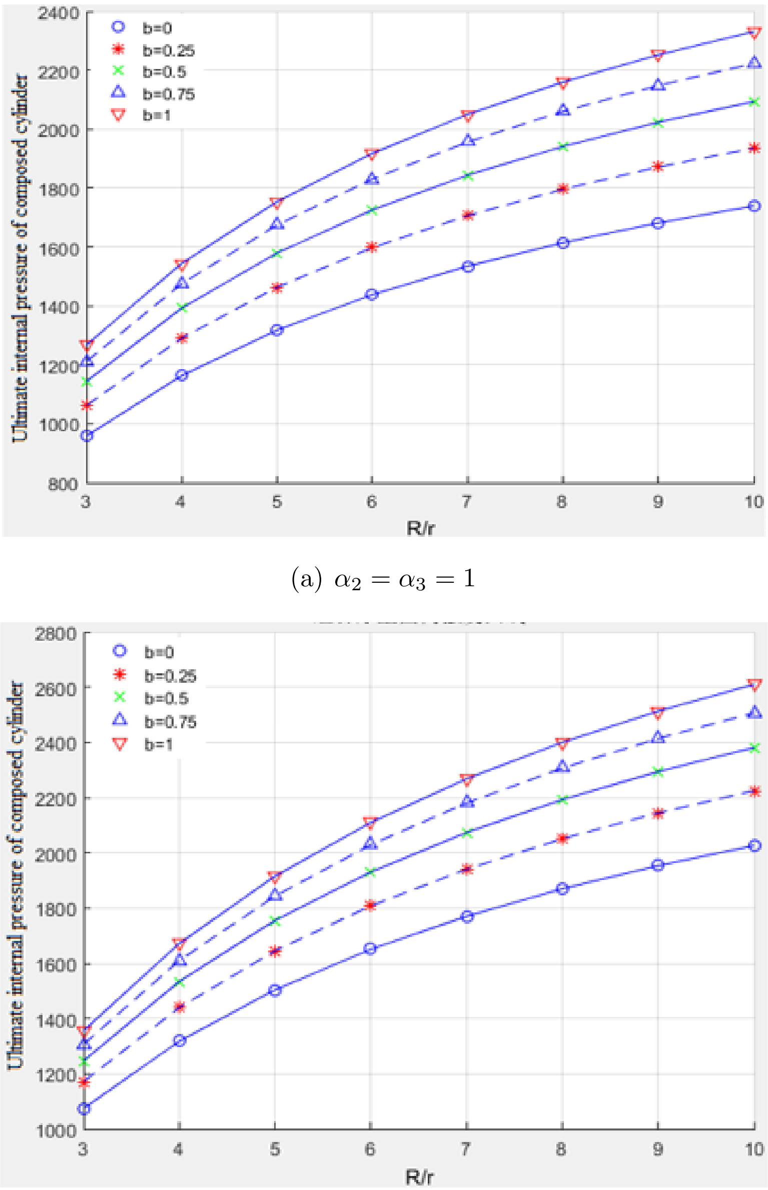
5.2. **Parameter influence analysis.** The elastic limit internal pressure of the composed thick-wall cylinder based on the unified strength theory is influenced by the coefficient of the difference in tension-compression strength α and the intermediate principal stress influence coefficient b of the material; if α and b in Section 5.1 vary, the elastic limit internal pressure of the composed thick-wall cylinder will vary. When the ratio of the inner radius to the outer radius is $R/r = 3 \to 10$, and the intermediate principal stress influence coefficient is $b = 0 \to 1$, whether to consider the difference in tension-compression strength ($\alpha_1 = 0.7$, $\alpha_2 = 0.85$) of the material or not consider the difference in tension-compression strength ($\alpha_2 = \alpha_3 = 1$) of the material, the growth rate of the elastic limit internal pressure of the composed thick-wall cylinder is as shown in Table 1.

According to Figure 3(a), when $\alpha_2 = \alpha_3 = 1$, $R/r = 3$, the elastic limit internal pressure of the composed thick-wall cylinder is increased from 957.6 MPa to 1,266.6 MPa with b gradually increased from 0 to 1, and the growth rate is 32.3%; according to Figure 3(b), when $\alpha_2 = 0.7$, $\alpha_3 = 0.85$, $R/r = 3$, the elastic limit internal pressure of the composed thick-wall cylinder is increased from 1,074.1 MPa to 1,355.4 MPa with b gradually increased from 0 to 1, and the growth rate is 26.2%.

According to Figures 3(a) and 3(b), the elastic limit internal pressure of the composed thick-wall cylinder will increase continuously with the increase of the ratio of inner radius

TABLE 1. Bearing capacity growth rate (percentage) of composed thick-wall cylinder considering the difference in tension-compression strength

R/r	$b=0$	$b=0.25$	$b=0.5$	$b=0.75$	$b=1$
3	12.17	10.38	8.99	7.90	7.01
4	13.29	11.56	10.21	9.14	8.27
5	14.13	12.45	11.13	10.09	9.24
6	14.80	13.15	11.86	10.84	10.00
7	15.34	13.72	12.45	11.44	10.62
8	15.79	14.19	12.94	11.95	11.13
9	16.17	14.59	13.36	12.37	11.56
10	16.49	14.93	13.72	12.74	11.94



(a) $\alpha_2 = \alpha_3 = 1$

(b) $\alpha_2 = 0.7$, $\alpha_3 = 0.85$

FIGURE 3. Influence of parameters (α, b) on ultimate elastic internal pressure of composed thick-walled cylinder

TABLE 2. Variation of bearing capacity of the composed thick-wall cylinder with the ratio of inner radius to outer radius irrespective of the difference in tension-compression strength (percentage)

R/r	$b=0$	$b=0.25$	$b=0.5$	$b=0.75$	$b=1$	Average
$3\to4$	21.47	21.61	21.73	21.82	21.90	21.71
$4\to5$	13.28	13.38	13.45	13.52	13.57	13.44
$5\to6$	9.14	9.218	9.27	9.322	9.36	9.26
$6\to7$	6.73	6.80	6.84	6.88	6.91	6.83
$7\to8$	5.20	5.25	5.29	5.32	5.34	5.28
$8\to9$	4.15	4.20	4.23	4.25	4.28	4.22
$9\to10$	3.41	3.44	3.47	3.49	3.51	3.47

to outer radius. Table 2 shows that, irrespective of the difference in tension-compression strength of the material, the bearing capacity of the composed thick-wall cylinder is averagely increased by 21.71% when the ratio of the inner radius to the outer radius is $R/r = 3 \to 4$, and average increase rate is gradually reduced to 3.47% when the ratio of the inner radius to the outer radius is $R/r = 9 \to 10$; Table 3 shows that, considering the difference in tension-compression strength of the material, the bearing capacity of the composed thick-wall cylinder is averagely increased by 23.05% when the ratio of the inner radius to the outer radius is $R/r = 3 \to 4$, and the average increase rate is gradually reduced to 3.79% when the ratio of the inner radius to the outer radius is $R/r = 9 \to 10$. According to the above analysis, when the ratio of the inner radius to the outer radius is small, increasing the wall thickness of the composed thick-wall cylinder can obviously promote the structural bearing capacity, but the effect will gradually weaken with the increase of the wall thickness, as shown in Tables 2 and 3.

TABLE 3. Variation of bearing capacity of the composed thick-wall cylinder with the ratio of inner radius to outer radius considering the difference in tension-compression strength (percentage)

R/r	$b=0$	$b=0.25$	$b=0.5$	$b=0.75$	$b=1$	Average
$3\to4$	22.68	22.91	23.08	23.22	23.34	23.05
$4\to5$	14.12	14.28	14.41	14.51	14.59	14.38
$5\to6$	9.78	9.90	9.99	10.07	10.13	9.97
$6\to7$	7.24	7.33	7.41	7.46	7.51	7.39
$7\to8$	5.61	5.69	5.75	5.79	5.83	5.73
$8\to9$	4.50	4.56	4.61	4.65	4.68	4.60
$9\to10$	3.70	3.75	3.80	3.83	3.86	3.79

6. **Conclusion.** Based on the unified strength theory, considering the influences of the intermediate principal stress and the difference in tension-compression strength of the material, the unified solutions of the elastic limit internal pressure, stratified radius and interference amount of the two-layer and three-layer composed thick-wall cylinders with the inner layer made of fragile material and ductile material have been derived herein, and the unified solutions are degraded into the specific solutions based on the unified strength theory (such as Tresca and Mises yield criteria) through α and b. For the unified solutions of elastic limit internal pressure of the composed thick-wall cylinder, the influence of the difference in tension-compression strength of the material is considered by parameter α,

the influence of the intermediate principal stress is considered by parameter b, and the influence of structural wall thickness is considered by the ratio of inner radius to outer radius. See the following for details.

1) Considering the difference in tension-compression strength of the material, the bearing capacity growth rate of the composed thick-wall cylinder is related to the intermediate principal stress coefficient b and the ratio R/r of the inner radius to the outer radius, and the value is $7.01\% \rightarrow 16.49\%$.
2) With the intermediate principal stress influence coefficient $b = 0 \rightarrow 1$, when $R/r = 3$, and considering the difference in tension-compression strength of the material, the bearing capacity of the composed thick-wall cylinder is increased by 26.2%; when not considering the difference in tension-compression strength of the material, the bearing capacity is increased by 32.3%, and continues to increase with the increase of R/r.
3) With constant increase of the ratio of inner radius to outer radius, the efficiency that the bearing capacity of the composed thick-wall cylinder is increased with the increase of the overall wall thickness of the structure, is lowered quickly; after the three-layer composed thick-wall cylinder is $R/r = 6$, the efficiency that the strength is improved by increasing the wall thickness, becomes very low.

The analysis shows that, the coefficient difference in tension-compression strength α and the intermediate principal stress influence coefficient b of the material have significant influences on the bearing capacity of the composed thick-wall cylinder, and the strength potential of the material can be brought into full play by using the unified strength theory and considering the influences of the parameters α and b, so as to reduce the size of the composed thick-wall cylinder structure. The plastic limit internal pressure and structural parameters of the composed thick-wall cylinder based on the proposed method will be investigated, and the influence of the coefficient difference in tension-compression strength and intermediate principal stress influence coefficient on the plastic limit of composed thick-wall cylinder will be analyzed in the further research work.

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