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RATIONAL HOMOTOPY PERTURBATION METHOD FOR SOLVING LINEAR QUADRATIC PROBLEMS AND COMPARISON WITH HOMOTOPY PERTURBATION METHOD

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Abstract. *This article proposes a new method to solve the linear optimal control problem, namely the rational homotopy perturbation method (RHPM). RHPM is the promotion of homotopy perturbation method (HPM). First of all, we introduce the RHPM. Then by solving the Riccati equation using RHPM and HPM, we find that the RHPM has the advantages of less iterations, fast convergence and high precision compared with the HPM. Finally we introduce the Pontryagin maximum principle and the basic concept of the linear quadratic optimal control problem, and illustrate the specific process of RHPM solving the linear quadratic optimal control problem by enumerating a second-order control system example. In the process of using RHPM, the difficulty is to find the parameters by nonlinear fitting. This requires us to find the numerical solution of the differential equation firstly, and then use the numerical solution for fitting to find the parameters.*

Keywords: Rational homotopy perturbation method, Homotopy perturbation method, Pontryagin minimum principle, Linear quadratic, Optimal control problems

1. **Introduction.** In the past several decades, the optimal control has played an important role in modern control theory, not only in all areas of physics, but also in ecconmy, aerospace, chemical engineering [1], robotic [2], etc. However, we know that in many cases, analytical solutions to differential equations evolved from optimal control problems are difficult to find. The method of solving differential equation and optimal control problems in current research still has limitations. Nowadays many experts and scholars are based on Pontryagin minimum principle [3] and Hamilton-Jacobi-Bellman equation [4], and put forward some related calculation methods to solve the optimal control problem, such as Adomian decomposition method (ADM [5]), modal series method (MSM [6]), homotopy analysis method (HAM [7,31]), variational iteration method (VIM [8,32]), and homotopy perturbation method (HPM [9]). The HPM is an effective tool for solving various linear and nonlinear problems.

The perturbation method is the most widely used method for solving approximate solutions [18-20]. Many nonlinear problems in science and astronomy have been successfully solved. However, in recent years, the emergence of so-called singular perturbation problems caused by many physical problems, experts and scholars has generated new interest in the equation solving process involving small parameters. Like the HPM [10-17,21-24], it is not based on small parameters, but on artificial parameters. This has greatly improved the original perturbation method. In 2013, Sargolzaei et al. proposed using the homotopy

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perturbation method to solve the optimal control problem of linear time-delay systems. Mainly use the principle of maximum value, the necessary optimal conditions and the homotopy perturbation method to transform the problem into a linear time-invariant two-point boundary value problem [7]. Of course, the HPM still has disadvantages such as slower solution speed, more iterations and larger errors when faced with more types of nonlinear equations with more complex nature [25,26]. To this end, many experts and scholars have revised the HPM, and then proposed the RHPM. RHPM [27-30] is a method based on HPM. It uses the power series quotient to improve the HPM. When solving nonlinear problems, the computational efficiency is also greatly improved, and the accuracy of the solution is also improved. Especially in the optimal control linear quadratic problem solving [33,34], the RHPM is a very effective method. In 2019, Sun proposed the rational homotopy perturbation method used to solve the optimal control problem of linear systems in her master's thesis [40].

In this paper, we mainly apply RHPM to solving the optimal control problem of linear systems. The difference from Sun Yubo is that when solving the optimal control problem, the numerical solution of the differential equation is used instead of the exact analytical solution when fitting the parameters. First, we use RHPM and traditional HPM to solve the Riccati equation and compare them to find that RHPM is closer to the exact solution. Then we use RHPM to solve a second-order control system problem. During the application of RHPM, we solve differential equations and obtain numerical solutions using MATLAB. Next use numerical scatter points to fit and find parameters by using MAPLE and then obtain the final solution of the linear optimal control system. The main framework of this article is as follows. Section 2 mainly introduces the basic concepts of the rational homotopy perturbation method. Section 3 is the proof of convergence of RHPM. Section 4 uses HPM and RHPM respectively to solve a Riccati equation, comparing the two results and exact solution. Section 5 mainly introduces the Pontryagin minimum principle and the basic concept of the linear quadratic optimal control problem, and an example of optimal control of a second-order linear system is given to illustrate the application of RHPM in linear quadratic optimal control. Finally, Section 6 concludes the paper.

2. Rational Homotopy Perturbation Method.

To illustrate the basic ideas of this method, we first consider the following nonlinear differential equations:

$$A(u) - f(r) = 0, \quad u(0) = u_0, \quad r \in \Omega \tag{1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{2}$$

where A is an ordinary differential operator, u_0 is an initial approximation of Equation (1), B is a boundary operator, and $f(r)$ is a known analytic function. Γ is the boundary of Ω, where Ω is the domain. Operator A can be divided into two parts, L and N, where L is linear and N is nonlinear, so (1) can be written in the following form:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega \tag{3}$$

Through homotopy perturbation techniques, we construct a homotopy $v(r, p)$: $\Omega \times [0, 1] \to R$, which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega \tag{4}$$

where p is an embedded parameter and u_0 is the initial approximate solution of (1). Obviously from (4) we can get

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0 \tag{5}$$

When p continuously changes from 0 to 1, $v(r,p)$ changes from the initial approximate solution $u_0(r)$ to $u(r)$. In topology, this change is called deformation, and $L(v) \to L(u_0)$, $A(v) \to f(r)$ is called a homotopy function.

In the HPM, we assume that the solution of (4) is about the power series form of p:

$$V = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + \cdots \tag{6}$$

Setting $p = 1$, we obtain the approximate solution of (1).

$$v(t) = \lim_{p\to 1} V = v_0 + v_1 + v_2 + \cdots$$

However, in the RHPM, we assume that the solution of Equation (4) is in the form of two power series quotients of p, namely:

$$U = \frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i} = \frac{v_0 + pv_1 + p^2 v_2 + \cdots}{w_0 + pw_1 + p^2 w_2 + \cdots} \tag{7}$$

where v_i is an unknown but determinable function, and w_i is a known function for the argument.

In general, we take $w_0 = 1$, $w_i = \alpha_i t^i$.

When $p \to 1$, the approximate solution of (7) is

$$u(t) = \lim_{p\to 1} U = \frac{v_0 + v_1 + v_2 + v_3 + \cdots}{w_0 + w_1 + w_2 + w_3 + \cdots} \tag{8}$$

If both $\lim_{p\to 1} \sum_{i=0}^{\infty} v_i$ and $\lim_{p\to 1} \sum_{i=0}^{\infty} w_i$ limits exist, and $\sum_{i=0}^{\infty} w_i \neq 0$, then the limit of (8) exists.

3. **Proof of Convergence of Rational Homotopy Perturbation Method.** To analyze the convergence of the rational homotopy perturbation method (RHPM), we write (4) in the following form:

$$L(v) = L(u_0) + p[f(r) - N(v) - L(u_0)] = 0 \tag{9}$$

Apply the inverse operator L^{-1} to both sides of (9) and we can get

$$v = u_0 + p\left[L^{-1} f(r) - L^{-1} N(v) - u_0\right] \tag{10}$$

If

$$v = \frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i} \tag{11}$$

substituting (11) into the right side of (10) has the following form:

$$v = u_0 + p\left\{L^{-1} f(r) - \left(L^{-1} N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i}\right] - u_0\right\} \tag{12}$$

When the limit $p \to 1$, the exact solution of (1) can be obtained from (12), namely:

$$\begin{aligned} u &= \lim_{p\to 1}\left(u_0 + p\left\{L^{-1} f(r) - \left(L^{-1} N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i}\right] - u_0\right\}\right) \\ &= L^{-1} f(r) - \left[\sum_{i=0}^{\infty} \left(L^{-1} N\right)\left(\frac{v_i}{\beta}\right)\right], \quad \beta = \sum_{i=0}^{\infty} w_i \end{aligned} \tag{13}$$

We use the Banach theorem to study the convergence of RHPM, which solves the fixed point problem of the nonlinear operator N in (1).

Theorem 3.1. (Convergence condition) [35]. *X and Y are Banach spaces, and N: $X \to Y$ is a contracted nonlinear map, then*

$$\forall w, w^* \in X, \quad ||N(w) - N(w^*)|| \le \gamma ||w - w^*||; \quad 0 < \gamma < 1 \tag{14}$$

According to Banach's fixed point theorem, N has a unique fixed point u, and $N(u) = u$, assuming that the sequence generated by RHPM can be expressed as:

$$W_n = N(W_{n-1}), \quad W_{n-1} = \sum_{i=0}^{n-1} \left(\frac{v_i}{\beta}\right), \quad n = 1, 2, 3, \dots \tag{15}$$

Under the conditions of $W_0 = \left(\frac{v_0}{\beta}\right) \in B_r(u)$ and $B_r(u) = \{w^* \in X | ||w^* - u|| < r\}$, there are

(i) $W_n \in B_r(u)$.
(ii) $\lim_{n\to\infty} W_n = u$ (u is the RHPM analytical approximate solution of (1)).

Prove by inductive method
(i) When $n = 1$

$$||W_1 - u|| = ||N(W_0) - N(u)|| \le \gamma ||w_0 - u|| \tag{16}$$

If $||W_n - u|| \le \gamma^{n-1} ||w_0 - u||$ is an induction hypothesis, then

$$||W_n - u|| = ||N(W_{n-1}) - N(u)|| \le \gamma ||W_{n-1} - u|| \le \gamma^n ||w_0 - u|| \tag{17}$$

Using (i), we can get

$$||W_n - u|| \le \gamma^n ||w_0 - u|| \le \gamma^n r < r \Rightarrow W_n \in B_r(u) \tag{18}$$

(ii) Because $||W_n - u|| \le \gamma^n ||w_0 - u||$ and $\lim_{n\to\infty} \gamma^n = 0$, $\lim_{n\to\infty} ||W_n - u|| = 0$,

$$\lim_{n\to\infty} W_n = u \tag{19}$$

Because W_n is convergent, when $n \to \infty$, $W_n = u$, u is convergent. Therefore, the solution obtained by using RHPM to solve (1) has convergence.

4. **Comparison of RHPM and HPM.** Consider the Riccati equation

$$\begin{cases} y'(t) - y^2(t) + 1 = 0 \\ y(0) = 0 \end{cases} \tag{20}$$

Find the approximate solution of (20).
The exact solution of this differential equation can be obtained as:

$$y(t) = -\tanh(t) \tag{21}$$

4.1. **Implementation of RHPM.** We use the rational homotopy perturbation method to find the approximate solution of (20).

Let $L(v) = \dot{v} + v + 1$, construct a homotopy map:

$$H(v, p) = (1 - p)(\dot{v} + v + 1) + p\left(\dot{v} - v^2 + 1\right) = \dot{v} + v + 1 + p\left(-v^2 - v\right) = 0 \tag{22}$$

If we use $[M, N]$ order approximation in the RHPM method, we can suppose that the solution for system (22) has the following form:

$$V_{[M,N]} = \frac{\sum_{i=0}^{M} p^i v_i}{1 + \sum_{i=1}^{N} \alpha_i t^i p^i} \tag{23}$$

Substitute (23) into Equation (22) and compare the same power coefficients of p:

$$p^0 : \begin{cases} v_0' + v_0 + 1 = 0 \\ v_0(0) = 0 \end{cases}$$

$$p^1 : \begin{cases} v_1' + v_0'\alpha_1 t - v_0\alpha_1 + v_0\alpha_1 t + v_1 + 2\alpha_1 t - v_0^2 - v_0 = 0 \\ v_1(0) = 0 \end{cases}$$

$$p^2 : \begin{cases} v_0'\alpha_2 t^2 + v_1'\alpha_1 t + v_2' - 2v_0\alpha_2 t - v_1\alpha_1 + v_0\alpha_2 t^2 + v_1\alpha_1 t + v_2 + \alpha_1^2 t^2 + 2\alpha_2 t^2 \\ - 2v_0 v_1 - v_0\alpha_1 t - v_1 = 0 \\ v_2(0) = 0 \end{cases}$$

$$\vdots$$

Solving the above formula, we obtain

$$v_0 = e^{-t} - 1$$
$$v_1 = -\alpha_1 t + e^{-t}\alpha_1 t - e^{-2t} - e^{-t}t + e^{-t}$$
$$v_2 = e^{-t}\left(e^{-2t} - e^{-t}(t(\alpha_1 - 2) + 1) + t(\alpha_1 - 1) + t^2\left(\alpha_2 - \alpha_1 + \frac{1}{2}\right) - \alpha_2 t^2 e^t\right)$$

Substituting these results into (23), we obtain

$$\begin{aligned} v_{[2,1]} &= \lim_{p\to 1} V_{[2,1]} = \lim_{p\to 1} \frac{1}{1+\alpha_1 t}\left(v_0 + pv_1 + p^2 v_2\right) \\ &= \frac{1}{1+\alpha_1 t}(v_0 + v_1 + v_2) \\ &= \frac{1}{1+\alpha_1 t}\Big[2e^{-t} + e^{-3t} - e^{-2t}\alpha_1 t + 2e^{-2t}t - 2e^{-2t} + 2e^{-t}\alpha_1 t - 2e^{-t}t \\ &\quad - e^{-t}t^2\alpha_1 + e^{-t}\alpha_2 t^2 + \frac{1}{2}e^{-t}t^2 - \alpha_2 t^2 - \alpha_1 t - 1\Big] \end{aligned} \tag{24}$$

For the exact solution function $y(t) = -\tanh(t)$, the image is centrally symmetric, we only need to study the positive semi-axis of the x-axis, and the function image basically tends to be horizontal after $t = 5$, and little research value. Therefore, we can select the interval $t \in [0, 5]$. Then compare the errors of various methods and exact solutions. In order to make RHPM achieve higher accuracy, we use the NonlinearFit command in MAPLE to perform nonlinear fitting. Select the initial value of t as 0, the step size is 0.1 (the smaller the step size, the more precise), and the boundary value is 5. The specific solution process is as follows:

1) We use MATLAB to find the numerical solution of (20), and get a set of scattered points;

2) Then import these scattered points into the MAPLE, and use the NonlinearFit command to fit;

3) If the error is too large, reduce the step size and repeat the first two steps.

Finally, we can find the adjustment parameter as

$$\alpha_1 = 0.1419, \quad \alpha_2 = -0.0009 \tag{25}$$

The approximate solution of Equation (20) can be obtained by substituting (25) into (24).

$$v_{[2,1]} = \frac{2e^{-t} + e^{-3t} + 1.858e^{-2t}t - 1.716e^{-t}t + 0.359e^{-t}t^2 - 0.1419t - 2e^{-2t} - 0.009t^2 - 1}{0.1419t + 1}$$

4.2. **Implementation of HPM.** In addition, we use the homotopy perturbation method to solve. Let $L(v) = \dot{v} + v + 1$, and construct a homotopy map:

$$\begin{aligned} H(v,p) &= (1-p)(\dot{v}+v+1) + p\left(\dot{v} - v^2 + 1\right) \\ &= \dot{v} + v + 1 + p\left(-v^2 - v\right) = 0 \end{aligned} \tag{26}$$

Suppose

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{27}$$

Substitute (27) into (26) to compare the same power coefficients of p:

$$p^0 : \begin{cases} v_0' + v_0 + 1 = 0 \\ v_0(0) = 0 \end{cases}$$

$$p^1 : \begin{cases} v_1' + v_1 - v_0^2 - v_0 = 0 \\ v_1(0) = 0 \end{cases}$$

$$p^2 : \begin{cases} v_2' + v_2 - 2v_0 v_1 - v_1 = 0 \\ v_2(0) = 0 \end{cases}$$

$$\vdots$$

Solving the above formula can obtain

$$\begin{aligned} v_0 &= e^{-t} - 1 \\ v_1 &= e^{-t} - e^{-t}\left(t + e^{-t}\right) \\ v_2 &= e^{-t}\left[e^{-2t} - t + e^{-t}(2t-1) + \frac{1}{2}t^2\right] \end{aligned}$$

Then its second-order approximate solution is

$$\begin{aligned} u &= \lim_{p\to 1} v = \lim_{p\to 1}\left(v_0 + pv_1 + p^2 v_2\right) \\ &= v_0 + v_1 + v_2 \\ &= 2e^{-t} + e^{-3t} - 2te^{-t} - 2e^{-2t} + 2te^{-2t} + \frac{1}{2}t^2 e^{-t} - 1 \end{aligned} \tag{28}$$

4.3. **Comparison and discussion.** The difference between the RHPM and the HPM is that RHPM leads into adjustment parameters, which can be obtained by fitting using the numerical solution of the differential equation. It is precisely because of the introduction of these parameters that the accuracy of RHPM is improved. The reason for using numerical solutions is that many differential equations cannot be solved analytically. In this section, we have selected a typical Riccati equation with analytical solutions as an example for comparison with the analytical approximate solutions obtained by HPM and RHPM.

It can be seen from Figure 1 and Table 1 that the rational homotopy perturbation method is obviously closer to the exact solution than the homotopy perturbation method, and the rational homotopy perturbation method has smaller error in solving the nonlinear differential equation. The number of iterations is smaller and the solution is more accurate.

5. **Application of Rational Homotopy Perturbation Method in Linear Quadratic Problems.** The linear quadratic optimal control is of importance in modern control theory. It is the most active subject in control theory. In this section, we will use RHPM to solve the linear quadratic optimal control.

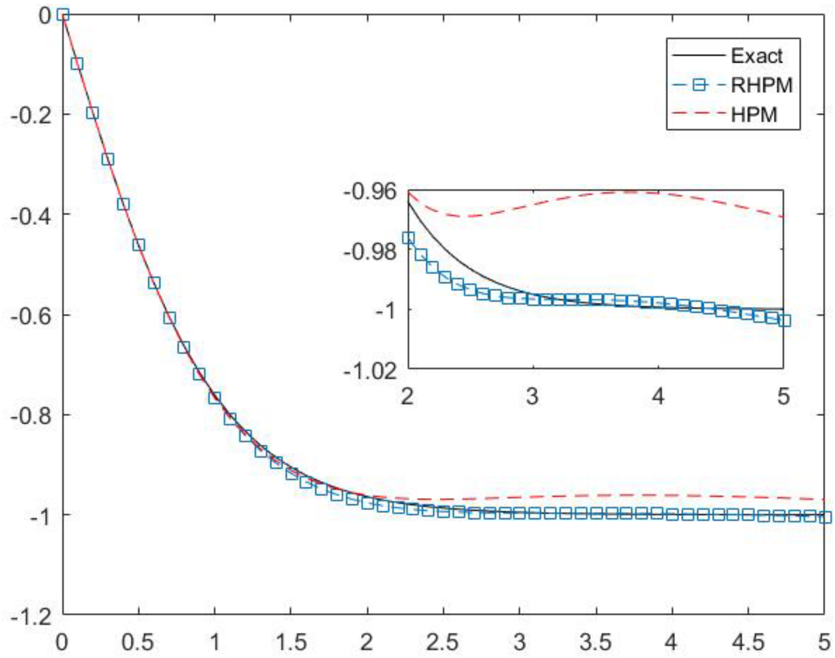

FIGURE 1. Exact solution and approximate solution curve of Equation (20)

TABLE 1. Exact and approximate solutions of Equation (20)

t	Exact	RHPM	HPM
0	0	0	0
0.5	-0.4621	-0.4622	-0.4624
1.0	-0.7616	-0.7670	-0.7663
1.5	-0.9051	-0.9173	-0.9112
2.0	-0.9640	-0.9761	-0.9609
2.5	-0.9866	-0.9935	-0.9690
3.0	-0.9951	-0.9966	-0.9651
3.5	-0.9982	-0.9968	-0.9614
4.0	-0.9993	-0.9979	-0.9613
4.5	-0.9998	-1.0003	-0.9644
5.0	-0.9999	-1.0037	-0.9693

5.1. **Pontryagin maximum principle.** The equation of state of the system is

$$\dot{X} = f(X, U, t), \quad X(t) \in R^n \tag{29}$$

Initial conditions

$$X(t_0) = X_0 \tag{30}$$

Control vector $U(t) \in R^m$, and subject to the following constraints:

$$U \in \Omega \tag{31}$$

Terminal constraint

$$G[X(t_f), t_f] = 0 \tag{32}$$

Indicator function

$$J = \phi[X(t_f), t_f] + \int_{t_0}^{t_f} F(X, U, t)dt \tag{33}$$

Hamiltonian function

$$H(X,U,\lambda,t) = F[X,U,t] + \lambda^T f(X,U,t) \tag{34}$$

To find the optimal control $U^*(t)$, let J take the minimum value. The necessary conditions for J to take the minimum value are $X(t)$, $U(t)$, $\lambda(t)$ and t_f satisfy the following set of equations

1° Regular equation

$$\dot{\lambda} = -\frac{\partial H}{\partial X} \text{ (Coordinated equation)} \tag{35}$$

$$\dot{X} = \frac{\partial H}{\partial \lambda} \text{ (Equation of state)} \tag{36}$$

2° Boundary conditions

$$X(t_0) = X_0, \quad G[X(t_f), t_f] = 0 \tag{37}$$

3° Cross-sectional condition

$$\lambda(t_f) = \frac{\partial \phi}{\partial X(t_f)} + \frac{\partial G^\intercal}{\partial X(t_f)}\nu \tag{38}$$

4° Optimal terminal time condition

$$H(t_f) = -\frac{\partial \phi}{\partial t_f} - \frac{\partial G^\intercal}{\partial t_f}\nu \tag{39}$$

5° On the optimal trajectory $X^*(t)$ and the optimal control $U^*(t)$, the Hamilton function takes the minimum value

$$\min H(X^*, \lambda^*, U, t) = H(X^*, \lambda^*, U^*, t) \tag{40}$$

5.2. **A reference to the linear quadratic problem.** In general, the linear quadratic problem can be expressed as follows: Let the equation of the linear time-varying system be

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + B(t)U(t) \\ Y(t) &= C(t)X(t) \end{aligned} \tag{41}$$

where $X(t)$ is an n-dimensional state vector, $U(t)$ is an m-dimensional control vector, and $Y(t)$ is an l-dimensional output vector; let $U(t)$ be unconstrained.

Let the error vector $e(t)$ be

$$e(t) = Z(t) - Y(t) \tag{42}$$

where $Z(t)$ is a 1-dimensional ideal output vector. Find the optimal control to minimize the following performance indicators

$$J(u) = \frac{1}{2}e^\intercal(t_f)Pe(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [e^\intercal(t)Q(t)e(t) + U^\intercal(t)R(t)U(t)]dt \tag{43}$$

where t_0 is the initial moment, t_f is the end moment, P is an $l \times l$ symmetric semi-positive constant matrix, $Q(t)$ is an $l \times l$ symmetric semi-positive array, and $R(t)$ is an $m \times m$ symmetric positive fixed matrix. Generally, P, $Q(t)$, and $R(t)$ are taken as diagonal arrays.

Here are a few special cases:

1) Regulator problem. At this time, $C(t) = I$ (unit matrix), ideal output $Z(t) = 0$, then $Y(t) = X(t) = -e(t)$. At this time, the problem comes down to using little control. The amount keeps $X(t)$ near zero. It is therefore called a state regulator problem.

2) Servo problems. At this time, $Z(t) \neq 0$, $e(t) = Z(t) - Y(t)$, then $Y(t)$ is followed by $Z(t)$ with a small amount of control, hence the name is track issue.

This article focuses on the state regulator problem, which is known as the following constant system:

$$\begin{aligned} &\dot{X}(t) = AX(t) + BU(t), \quad X(t_0) = X_0, \quad t \in [t_0, t_f] \\ &J = \frac{1}{2}X^{\intercal}(t_f)PX(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(X^{\intercal}QX + U^{\intercal}RU)dt \end{aligned} \tag{44}$$

Find the optimal control $U(t)$ to minimize J, X is n-dimensional, and U is m-dimension. If t is continuous for $\forall t \in [t_0, t_f]$, then the control function $U(t)$ is controllable. When t_f is a finite value, it is a finite continuous state regulator problem; when $t_f \to \infty$, it is a steady state regulator problem. Suppose the control is bounded, U is a closed subset on R^+. By taking the extreme values of the performance index, we can get the optimal control input $u(t)$, where P is a symmetric semi-positive definite matrix and Q and R are constant symmetric positive definite matrices. Or you can change the requirement for Q to Q symmetric semi-positive.

According to the system (44) we consider the following Hamiltonian function:

$$H = \frac{1}{2}(X^{\intercal}QX + U^{\intercal}RU) + \lambda^{\intercal}(AX + BU) \tag{45}$$

where λ is a covariate variable. When $U(t)$ is unconstrained, according to the Pontryagin maximum principle, the optimal control rate can be obtained by the following equation.

$$\frac{\partial H}{\partial U} = RU + B^{\intercal}\lambda = 0 \tag{46}$$

Optimal control rate is

$$U^*(t) = -R^{-1}B^{\intercal}\lambda(t) \tag{47}$$

According to the cross-sectional conditions

$$\lambda(t_f) = \frac{\partial \phi}{\partial X(t_f)} = \frac{\partial}{\partial X(t_f)}\left[\frac{1}{2}X^{\intercal}(t_f)PX(t_f)\right] = PX(t_f) \tag{48}$$

It is found that the coordination state $\lambda(t)$ and the state $X(t)$ are linear in the terminal time t_f, which inspires us to assume

$$\lambda(t) = K(t)X(t) \tag{49}$$

Substituting it into (35) can be obtained

$$U(t) = -R^{-1}B^{\intercal}K(t)X(t) = -G(t)X(t) \tag{50}$$

where $G(t) = R^{-1}B^{\intercal}K(t)$ is called the optimal feedback gain matrix, and $K(t)$ is the $n \times n$ dimensional positive definite symmetric matrix.

Next, our goal is to find $K(t)$ and then find the optimal control. According to the maximum principle, we can find the regular equation

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial X} = -QX(t) - A^{\intercal}\lambda(t) = -QX(t) - A^{\intercal}K(t)X(t) \tag{51}$$

$$\dot{X}(t) = \frac{\partial H}{\partial \lambda} = AX(t) + BU(t) = AX(t) - BR^{-1}B^{\intercal}K(t)X(t) \tag{52}$$

Derive both sides of $\lambda(t) = K(t)X(t)$ and substitute (51):

$$\dot{\lambda}(t) = \dot{K}(t)X(t) + K(t)\dot{X}(t) = -QX(t) - A^{\intercal}K(t)X(t) \tag{53}$$

Substitute (52) into (53)

$$\left[\dot{K}(t) + K(t)A - K(t)BR^{-1}B^{\intercal}K(t) + A^{\intercal}K(t) + Q\right]X(t) = 0 \tag{54}$$

The above formula should be true for any $X(t)$, so the term in square brackets should be zero, which leads to

$$\dot{K}(t) = -K(t)A - A^{\intercal}K(t) + K(t)BR^{-1}B^{\intercal}K(t) - Q \tag{55}$$

The above equation is a nonlinear matrix differential equation of $K(t)$, called the Riccati matrix differential equation. In order to find $K(t)$, we need to know its boundary conditions. We compare (48) and (49)

$$K(t_f) = P \tag{56}$$

In [22], $K(t)$ can be composed of the following forms:

$$K(t) = W(t)V^{-1}(t) \tag{57}$$

where

$$\begin{pmatrix} \dot{V}(t) \\ \dot{W}(t) \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} V(t) \\ W(t) \end{pmatrix} \tag{58}$$

which is

$$\begin{aligned} \dot{V}(t) &= AV(t) - BR^{-1}B^{\intercal}W(t) \\ \dot{W}(t) &= -QV(t) - A^{\intercal}W(t) \end{aligned} \tag{59}$$

with conditions $V(t_f) = I$, $W(t_f) = P$.

5.3. **A case study of rational homotopy perturbation method in linear quadratic problems.** Let the system state equation be [34]

$$\begin{aligned} \dot{x_1} &= x_2, \quad x_1(0) = 1 \\ \dot{x_2} &= u, \quad x_2(0) = 0 \end{aligned} \tag{60}$$

Find the optimal control $u(t)$ to minimize the following performance indicators

$$J = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[x_1^2(t) + u^2(t)\right] dt \tag{61}$$

Solution: Compare the equation of state (60) with the performance indicators (61) and (44).

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P = 0, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = 1 \tag{62}$$

According to (59)

$$\begin{aligned} \dot{V}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V(t) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} W(t) \\ \dot{W}(t) &= -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V(t) - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} W(t) \end{aligned} \tag{63}$$

According to the homotopy perturbation method to construct homotopy functions, there are

$$\begin{aligned} \mathrm{H}_1(V, W, p) &= (1-p)\left(\dot{V}(t) - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V(t)\right) \\ &\quad + p\left(\dot{V}(t) - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V(t) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} W(t)\right) = 0 \\ \mathrm{H}_2(V, W, p) &= (1-p)\left(\dot{W}(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} W(t)\right) \\ &\quad + p\left(\dot{W}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} W(t)\right) = 0 \end{aligned} \tag{64}$$

simplified to

$$
\begin{aligned}
&\dot{V}(t)-\begin{pmatrix}0&1\\0&0\end{pmatrix}V(t)+p\begin{pmatrix}0&0\\0&1\end{pmatrix}W(t)=0\\
&\dot{W}(t)+\begin{pmatrix}0&0\\1&0\end{pmatrix}W(t)+p\begin{pmatrix}1&0\\0&0\end{pmatrix}V(t)=0
\end{aligned}
\tag{65}
$$

According to (7), the approximate solution of the $[M,N]$ order in the rational homotopy perturbation method can be expressed as

$$
V_{[M,N]}=\frac{\sum_{i=0}^{M}p^iv_i(t)}{1+\sum_{i=1}^{N}\alpha_it^ip^i},\quad W_{[M,N]}=\frac{\sum_{i=0}^{M}p^iw_i(t)}{1+\sum_{i=1}^{N}\beta_it^ip^i}
\tag{66}
$$

Substituting (66) into (65), comparing the same power coefficients of p, there are

$$
p^0:\begin{cases}
v_0'(t)-\begin{pmatrix}0&1\\0&0\end{pmatrix}v_0(t)=0\\
w_0'(t)+\begin{pmatrix}0&0\\1&0\end{pmatrix}w_0(t)=0
\end{cases}
$$

$$
p^1:\begin{cases}
v_1'(t)-\begin{pmatrix}0&1\\0&0\end{pmatrix}v_1(t)-v_0(t)\alpha_1+\begin{pmatrix}0&0\\0&1\end{pmatrix}w_0(t)=0\\
w_1'(t)+\begin{pmatrix}0&0\\1&0\end{pmatrix}w_1(t)-w_0(t)\beta_1+\begin{pmatrix}1&0\\0&0\end{pmatrix}v_0(t)=0
\end{cases}
$$

$$
p^2:\begin{cases}
v_2'(t)-\begin{pmatrix}0&1\\0&0\end{pmatrix}v_2(t)+\left[v_1'(t)-\begin{pmatrix}0&1\\0&0\end{pmatrix}v_1(t)\right]\alpha_1t-v_1(t)\alpha_1\\
-2v_0(t)\alpha_2t+\begin{pmatrix}0&0\\0&1\end{pmatrix}w_1(t)=0\\
w_2'(t)+\begin{pmatrix}0&0\\1&0\end{pmatrix}w_2(t)+\left[w_1'(t)+\begin{pmatrix}0&0\\1&0\end{pmatrix}w_1(t)\right]\beta_1t-w_1(t)\beta_1\\
-2w_0(t)\beta_2t+\begin{pmatrix}1&0\\0&0\end{pmatrix}v_1(t)+(2\beta_1t-\alpha_1t)\begin{pmatrix}1&t\\0&0\end{pmatrix}=0
\end{cases}
$$

$$\vdots$$

Initial condition $v_0(0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $w_0(0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Combined with the boundary conditions, we can solve

$$
\begin{cases}
v_0(t)=\begin{pmatrix}1&t\\0&1\end{pmatrix}\\
w_0(t)=\begin{pmatrix}0&0\\0&0\end{pmatrix}
\end{cases}
$$

$$\begin{cases} v_1(t) = \begin{pmatrix} \alpha_1 t & \alpha_1 t^2 \\ 0 & \alpha_1 t \end{pmatrix} \\ w_1(t) = \begin{pmatrix} -t & -\frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{6}t^3 \end{pmatrix} \end{cases}$$

$$\begin{cases} v_2(t) = \begin{pmatrix} \alpha_2 t^2 - \frac{1}{24}t^4 & \alpha_2 t^3 - \frac{1}{120}t^5 \\ -\frac{1}{6}t^3 & \alpha_2 t^2 - \frac{1}{24}t^4 \end{pmatrix} \\ w_2(t) = \begin{pmatrix} -\beta_1 t^2 & -\frac{1}{2}\beta_1 t^3 \\ \frac{1}{2}\beta_1 t^3 & \frac{1}{6}\beta_1 t^4 \end{pmatrix} \end{cases}$$

$$\vdots$$

Substitute the above formula into (66) and let p take the limit $p \to 1$

$$\begin{aligned} V_{[2,1]} &= \frac{1}{1+\alpha_1 t} \begin{pmatrix} -\frac{1}{24}t^4 + \alpha_2 t^2 + \alpha_1 t + 1 & -\frac{1}{120}t^5 + \alpha_2 t^3 + \alpha_1 t^2 + t \\ -\frac{1}{6}t^3 & -\frac{1}{24}t^4 + \alpha_2 t^2 + \alpha_1 t + 1 \end{pmatrix} \\ W_{[2,1]} &= \frac{1}{1+\beta_1 t} \begin{pmatrix} -\beta_1 t^2 - t & -\frac{1}{2}\beta_1 t^3 - \frac{1}{2}t^2 \\ \frac{1}{2}\beta_1 t^3 + \frac{1}{2}t^2 & \frac{1}{6}\beta_1 t^4 + \frac{1}{6}t^3 \end{pmatrix} \end{aligned} \tag{67}$$

According to Formula (57), we obtain

$$G^*(t) = R^{-1}B^{\intercal}W(t)V^{-1}(t) = (G_1^*(t), G_2^*(t)) \tag{68}$$

Among

$$G_1^*(t) = \frac{20t^2(\alpha_1 t+1)\left(t^4+72\alpha_2 t^2+72\alpha_1 t+72\right)}{M+240\alpha_2 t^6+240\alpha_1 t^5+2880{\alpha_2}^2 t^4+5760\alpha_2\alpha_1 t^3+2880{\alpha_1}^2 t^2+5760\alpha_2 t^2+5760\alpha_1 t}$$

$$G_2^*(t) = \frac{-8t^3(\alpha_1 t+1)\left(t^4+120\alpha_2 t^2+120\alpha_1 t+120\right)}{M+240\alpha_2 t^6+240\alpha_1 t^5+2880{\alpha_2}^2 t^4+5760\alpha_2\alpha_1 t^3+2880{\alpha_1}^2 t^2+5760\alpha_2 t^2+5760\alpha_1 t}$$

where $M = t^8 + 240t^4 + 2880$.

First use MATLAB to find the numerical solution of $V(t)$ and $W(t)$. Then, use the NonlinearFit command in MAPLE to perform nonlinear fitting, and fit v_{11} and v_{12} in $V_{[1,2]}$ respectively. The initial value of time t is 0, the boundary value is 1.8, and then the adjustment parameter in v_{11} can be determined as

$$\widehat{\alpha}_1 = 0.006178, \quad \widehat{\alpha}_2 = -0.0004946 \tag{69}$$

In order to distinguish, here we use the symbol of '*', and get the adjustment parameters of v_{12} as:

$$\alpha_1^* = 0.00363, \quad \alpha_2^* = -0.000091 \tag{70}$$

Substituting $\alpha_1 = \frac{\alpha_1^*+\widehat{\alpha}_1}{2}$, $\alpha_2 = \frac{\alpha_2^*+\widehat{\alpha}_2}{2}$ into $G_1^*(t)$, $G_2^*(t)$, get the analytical approximate solution.

Through (65) and boundary conditions we can find the exact solution as:

$$G_1(t) = -\frac{e^{\frac{\sqrt{2}}{2}t}\left[4e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)^2 - e^{\frac{3\sqrt{2}}{2}t} - 2e^{\frac{\sqrt{2}}{2}t} - e^{-\frac{\sqrt{2}}{2}t}\right]}{4e^{\sqrt{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)^2 + 2e^{\sqrt{2}t} + e^{2\sqrt{2}t} + 1} \tag{71}$$

$$G_2(t) = \frac{\sqrt{2}e^{\frac{\sqrt{2}}{2}t}\left[4e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)\sin\left(\frac{\sqrt{2}}{2}t\right) - e^{\frac{3\sqrt{2}}{2}t} + e^{-\frac{\sqrt{2}}{2}t}\right]}{4e^{\sqrt{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)^2 + 2e^{\sqrt{2}t} + e^{2\sqrt{2}t} + 1} \tag{72}$$

It can be seen from Figure 2 and Table 2 that the RHPM has a small error from the exact solution when solving the linear quadratic problem. To minimize J, use the formula

$$\begin{aligned} u(t) &= -Gx(t) = \left[-\frac{1}{2}t^2, -\frac{1}{6}t^3\right] \\ J &= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[x_1^2(t) + u^2(t)\right]dt = 1.774712332 \times 10^{10} + \frac{1}{1280}\pi^5 + \frac{1}{64512}\pi^7 \end{aligned} \tag{73}$$

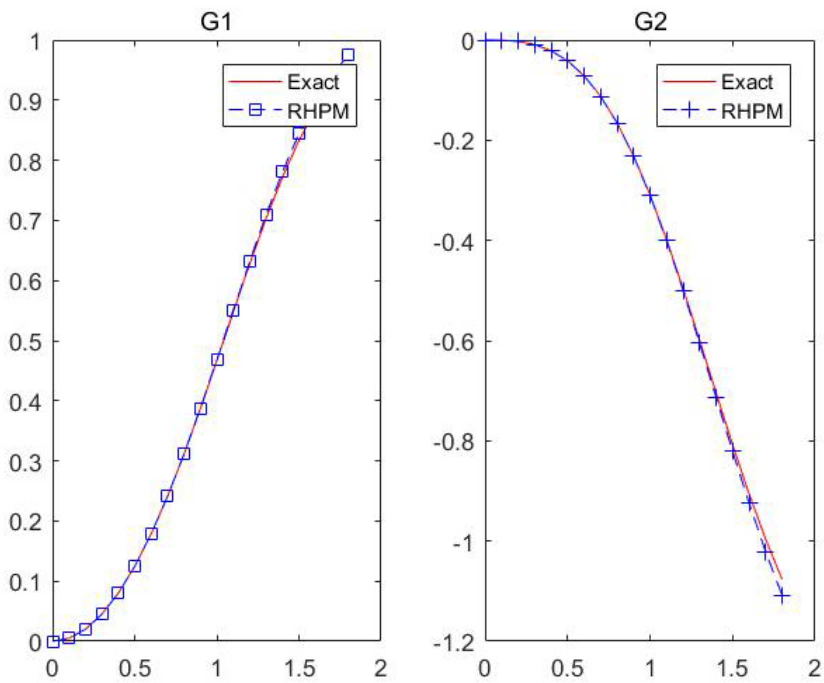

FIGURE 2. Comparison of exact solutions of G1, G2 and RHPM solutions

In the traditional solving of the quadratic optimal control problem of the second-order linear system, the Riccati differential equation is generally used directly to solve the problem. Riccati differential equations have a shaping theory and calculations are more convenient. However, it is difficult to obtain analytical expressions using this method, and the numerical solution can only be calculated using computer programs. Therefore, we use RHPM to solve the above linear quadratic optimal control problem [37]. Through comparison, it is found that RHPM perfectly solves the problem that the Riccati equation [38] cannot be solved in an analytical expression, and the approximate solution obtained by RHPM is quite close to the exact solution. RHPM is a particularly effective method in solving linear optimal control problems.

TABLE 2. Comparison of exact solutions of G1, G2 and RHPM solutions

	G1		G2	
t	RHPM	Exact [34]	RHPM	Exact [34]
0	0	0	0	0
0.2	0.020	0.020	-0.003	-0.003
0.4	0.080	0.080	-0.021	-0.021
0.6	0.178	0.178	-0.071	-0.071
0.8	0.311	0.311	-0.166	-0.165
1.0	0.468	0.467	-0.310	-0.309
1.2	0.631	0.628	-0.499	-0.496
1.4	0.780	0.772	-0.731	-0.704
1.6	0.896	0.884	-0.924	-0.906
1.8	0.973	0.955	-1.109	-1.076

6. **Conclusions.** In this paper, the concept of RHPM and the proof of convergence are introduced in detail. This method is used to solve the approximate solution of nonlinear differential equations. It can be found by comparison with homotopy perturbation method, the rational homotopy perturbation method is much better than the homotopy perturbation method, and its solution speed is fast, the number of iterations is small, and the error is small. Of course, through this paper, it can be found that it is very convenient to solve the linear quadratic problem by using the RHPM.

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REFERENCES

[1] W. L. Garrard and J. M. Jordan, Design of nonlinear automatic flight control systems, *Automatica*, vol.13, no.5, pp.497-505, 1977.

[2] S. Wei, M. Zefran and R. A. DeCarlo, Optimal control of robotic systems with logical constraints: Application to UAV path planning, *IEEE International Conference on Robotics and Automation*, Pasadena, CA, pp.176-181, 2008.

[3] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze et al., *The Mathematical Theory of Optimal Processes*, Interscience, 1962.

[4] R. E. Bellman, *Dynamic Programming*, Dover Publications, Inc., 2003.

[5] A. Fakharian, M. T. H. Beheshti and A. Davari, Solving the Hamilton-Jacobi-Bellman equation using Adomian decomposition method, *International Journal of Computer Mathematics*, vol.87, no.12, pp.2769-2785, 2010.

[6] A. Jajarmi, N. Pariz, S. Effati et al., Solving infinite horizon nonlinear optimal control problems using an extended modal series method, *Frontiers of Information Technology Electronic Engineering*, vol.12, no.8, pp.667-677, 2011.

[7] A. Sargolzaei, K. Kang, S. Noei et al., Assessment of He's homotopy perturbation method for optimal control of linear time-delay systems, *Applied Mathematical Sciences*, vol.7, no.5, pp.349-361, 2013.

[8] S. A. Yousefi, M. Dehghan and A. Lotfi, Finding the optimal control of linear systems via He's variational iteration method, *International Journal of Computer Mathematics*, vol.87, no.4, pp.1042-1050, 2010.

[9] S. Effati and H. Saberi, Solving a class of linear and non-linear optimal control problems by homotopy perturbation method, *IMA Journal of Mathematical Control and Information*, vol.28, no.4, pp.539-553, 2011.

[10] J. Biazar and H. Aminikhah, Study of convergence of homotopy perturbation method for systems of partial differential equations, *Computers Mathematics with Applications*, vol.58, no.11, pp.2221-2230, 2009.

[11] J. Biazar and H. Ghazvini, Convergence of the homotopy perturbation method for partial differential equations, *Nonlinear Analysis: Real World Applications*, vol.10, no.5, pp.2633-2640, 2009.

[12] A. Barari, M. Omidvar, A. Ghotbi and R. Ganji, Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations, *Acta Applicandae Mathematicae*, vol.104, no.2, pp.161-171, 2008.

[13] H. Vazquez-Leal, A. Marin-Hernandez et al., Exploring collision-free path planning by using homotopy continuation methods, *Applied Mathematics and Computation*, vol.219, no.14, pp.7514-7532, 2013.

[14] J. He, Homotopy perturbation method for solving boundary value problems, *Physics Letters A*, vol.350, no.1, pp.87-88, 2006.

[15] J. He, Homotopy perturbation method: A new nonlinear analytical technique, *Applied Mathematics and Computation*, vol.135, no.1, pp.73-79, 2003.

[16] J. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos, Solitons & Fractals*, vol.26, no.3, pp.695-700, 2005.

[17] J. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *International Journal of Non-Linear Mechanics*, vol.35, no.1, pp.37-43, 2000.

[18] Y. Khan, H. Vazquez-Leal and Q. Wu, An efficient iterated method for mathematical biology model, *Neural Computing and Applications*, vol.23, no.3, pp.677-682, 2013.

[19] Y. Khan, Q. Wu, N. Faraz, A. Yildirim and M. Madani, A new fractional analytical approach via a modified Riemann-Liouville derivative, *Applied Mathematics Letters*, vol.25, no.10, pp.1340-1346, 2012.

[20] Y. Khan, Q. Wu, N. Faraz and A. Yildirim, The effects of variable viscosity and thermal conductivity on a thin film flow over a shrinking/stretching sheet, *Computers Mathematics with Applications*, vol.61, no.11, pp.3391-3399, 2011.

[21] M. Madani, M. Fathizadeh, Y. Khan and A. Yildirim, On the coupling of the homotopy perturbation method and Laplace transformation, *Mathematical and Computer Modelling*, vol.53, no.9, pp.1937-1945, 2011.

[22] H. Aminikhah, The combined Laplace transform and new homotopy perturbation methods for stiff systems of ODEs, *Applied Mathematical Modelling*, vol.36, no.8, pp.3638-3644, 2012.

[23] F. I. Compean, D. Olvera, F. J. Campa, L. L. De Lacalle, A. Elias-Zuniga and C. A. Rodriguez, Characterization and stability analysis of a multivariable milling tool by the enhanced multistage homotopy perturbation method, *International Journal of Machine Tools and Manufacture*, vol.57, pp.27-33, 2012.

[24] J. Biazar and B. Ghanbari, The homotopy perturbation method for solving neutral functional-differential equations with proportional delays, *Journal of King Saud University-Science*, vol.24, no.1, pp.33-37, 2012.

[25] J. Biazar and M. Eslami, A new homotopy perturbation method for solving systems of partial differential equations, *Computers Mathematics with Applications*, vol.62, no.1, pp.225-234, 2011.

[26] J. Biazar, M. Eslami and H. Ghazvini, Homotopy perturbation method for systems of partial differential equations, *International Journal of Nonlinear Sciences and Numerical Simulation*, vol.8, no.3, pp.413-418, 2007.

[27] J. Biazar, M. Asadi and F. Salehi, Rational homotopy perturbation method for solving stiff systems of ordinary differential equations, *Applied Mathematical Modelling*, vol.39, no.3, pp.1291-1299, 2015.

[28] H. Vazquez-Leal, A. Sarmiento-Reyes, Y. Khan et al., Rational biparameter homotopy perturbation method and Laplace-Padé coupled version, *Journal of Applied Mathematics*, 2012.

[29] A. Belndez, A. Hernandez, T. Belndez et al., Application of He's homotopy perturbation method to the Duffing-harmonic oscillator, *International Journal of Nonlinear Sciences and Numerical Simulation*, vol.8, no.1, pp.79-88, 2007.

[30] J. He, Homotopy perturbation method for bifurcation of nonlinear problems, *International Journal of Nonlinear Sciences and Numerical Simulation*, vol.6, no.2, pp.207-208, 2005.

[31] M. S. Zahedi and H. S. Nik, On homotopy analysis method applied to linear optimal control problems, *Applied Mathematical Modelling*, vol.37, no.23, pp.9617-9629, 2013.

[32] A. Ghorbani and J. Saberi-Nadjafi, An effective modification of He's variational iteration method, *Nonlinear Analysis: Real World Applications*, vol.10, no.5, pp.2828-2833, 2009.

[33] X. Zhong, H. He, H. Zhang et al., Optimal control for unknown discrete-time nonlinear Markov jump systems using adaptive dynamic programming, *IEEE Transactions on Neural Networks Learning Systems*, vol.25, no.12, pp.2141-2155, 2014.
[34] H. Zhang and Q. Wang, *Optimal Control Theory and Application*, Higher Education Press, 2006.
[35] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Istit. Mat. Univ. Trieste*, vol.6, pp.17-26, 2004.
[36] W. Jia, X. He and L. Guom, The optimal homotopy analysis method for solving linear optimal control problems, *Applied Mathematical Modelling*, vol.45, no.5, pp.865-880, 2017.
[37] Y. Chen and J. Xu, The delayed doubly stochastic linear quadratic optimal control problem, *Mathematical Problems in Engineering*, vol.20, pp.1-10, 2020.
[38] A. Tanikawa, Analysis of the Riccati equation of the optimal filter with disturbance decoupling property for linear stochastic systems, *International Journal of Innovative Computing, Information Control*, vol.15, no.3, pp.811-824, 2019.
[39] M. Najafi and H. Basirzadeh, Optimal control homotopy perturbation method for cancer model, *Journal of Biomathematics*, vol.12, no.3, pp.79-90, 2019.
[40] Y. Sun, *Research on Solving Problem of Nonlinear Equation and Optimal Control Based on RHPM*, Master Thesis, 2019.