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THE BALANCING BIPOLAR CHOQUET INTEGRALS

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Abstract. *Mesiarova-Zemankova et al. have proposed a new type of discrete extension of the Choquet integral that merges positive and negative inputs together, called the balancing Choquet integral for capacities. In this paper, we first introduce an alternative definition of the balancing Choquet integral based on binary-element sets. Then, we propose the balancing bipolar Choquet integral for bi-capacities, which is an extension of balancing Choquet integral. In the second half of the paper, we present a form of the balancing Choquet integral on fuzzy sets, and we extend our model to balancing bipolar Choquet integral on fuzzy sets. Finally, we give basic properties of balancing bipolar Choquet integral on fuzzy sets.*
Keywords: Capacities, Bi-capacities, Balancing Choquet integral, Bipolar balancing Choquet integral, Balancing Choquet integral on fuzzy sets

1. **Introduction.** Choquet integrals defined with respect to capacities [2] (fuzzy measures [3, 4] or non-additive measures [5]) are powerful tools in multiple criteria decision making problems and there are a growing number of publications on the topic (see e.g., [6, 7, 8]). The Choquet integral has been generalized by Wu and Huang [9] on fuzzy sets. Mesiarova-Zemankova et al. [1] have introduced the balancing Choquet integral in order to ensure compensation by the extended discrete Choquet integral. The concept of bi-capacity has been proposed by Grabisch and Labreuche [10] as a generalization of capacity. The bipolar Choquet integral has been proposed in [11] as an extension of the Choquet integral for cases in which the underlying scale is bipolar. In recent literature, bipolarity and its possible applications have been discussed by various researchers [12, 13, 14].

In [15, 16], a new approach for studying the bipolar Choquet integral has been proposed through introducing a concept of ternary-element sets. In this study, we develop this approach to the aggregation on bipolar scales for studying balancing bipolar Choquet integral. This approach is fully different from those methods in previous studies ([1, 9, 11]), and it allows a simple way to introduce new results on bi-capacity and balancing Choquet integral. The proposed results in this paper are as follows.

- The first result introduces a definition of the balancing Choquet integral based on binary-element sets, which is an alternative definition from that defined by Mesiarova-Zemankova et al. [1].
- The second result proposes the balancing bipolar Choquet integral based on ternary-element sets, which is an extension of balancing Choquet integral with respect to bi-capacity.

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- The third result presents a form of balancing Choquet integral on fuzzy sets, which is an alternative approach from that defined by Wu and Huang [9].
- The fourth result proposes a model to the aggregation on bipolar scales for studying the balancing bipolar Choquet integral on fuzzy sets and discusses the basic properties of this integral.

The paper is organized as follows. The next section recalls the basic definitions that are needed in this paper. Section 3 presents the balancing bipolar Choquet integral. In Section 4, we propose the balancing Choquet integral on fuzzy sets. In Section 5, we propose balancing bipolar Choquet integral on fuzzy sets. Finally, the conclusions are made in Section 6. Throughout the paper, we will consider $[0,1]$ to be prototypical unipolar scale, while $[-1,1]$ with 0 as neutral level will be considered as prototypical bipolar scale.

2. **Capacities and Bi-capacities.** In this section, we begin by recalling basic concepts of the equivalent definitions of capacities based on binary-element sets and bi-capacities based on ternary-element sets (for more details, see [15, 16, 17]).

The binary-element set (or simply *bi-element set*) S is a set of the form $S := \{\tau_1, \ldots, \tau_i, \ldots, \tau_n\}$ where $\tau_i = i^+$ or i^-, for all i, $i = 1, \ldots, n$. Hence, we can represent the set of all possible combinations of bi-element sets by

$$Ḅ := \left\{\{\tau_1, \ldots, \tau_i, \ldots, \tau_n\} \,\middle|\, \tau_i \in \left\{i^+, i^-\right\}, \forall i = 1, \ldots, n\right\}.$$

The inclusion relation $\subseteq$ on Ḅ is defined as follows. For $S, T \in$ Ḅ, then, $S \subseteq T$ iff $\forall i = 1, \ldots, n$,

$$\text{if } i^+ \in S \Longrightarrow i^+ \in T. \tag{1}$$

Based on the notion of bi-element sets, the following definition gives an equivalent definition of capacity.

Definition 2.1. *Let Ḅ be the set of all bi-element sets. A set function μ : Ḅ $\to [0,1]$ is called capacity if it satisfies the following requirements:*

1) $\mu(X^+) = \mu(\{1^+, \ldots, n^+\}) = 1$ and $\mu(X^-) = \mu(\{1^-, \ldots, n^-\}) = 0$.

2) $\forall S, T \in$ Ḅ, $S \subseteq T \Longrightarrow \mu(S) \leq \mu(T)$.

The ternary-element set (or simply *ter-element set*) S is a set of the form $S := \{\tau_1, \ldots, \tau_i, \ldots, \tau_n\}$ where $\tau_i = i^+, i^-$, or $i^\emptyset$, $\forall i = 1, \ldots, n$. Hence, we present the set of all possible combinations of ter-elements by

$$Ţ := \left\{\{\tau_1, \ldots, \tau_i, \ldots, \tau_n\} \,\middle|\, \tau_i \in \left\{i^+, i^-, i^\emptyset\right\}, \forall i = 1, \ldots, n\right\}.$$

The order $\sqsubseteq$ on structure of Ţ is given by the following definition.

Definition 2.2. *Suppose S and T are ter-element sets of Ţ. Then, $S \sqsubseteq T$ iff $\forall i = 1, \ldots, n$,*

$$\text{“if } i^+ \in S \text{ implies } i^+ \in T\text{”, and “if } i^\emptyset \in S \text{ implies } i^+ \text{ or } i^\emptyset \in T\text{”}. \tag{2}$$

The following definition is equivalent definition of bi-capacities based on notion of ter-element sets.

Definition 2.3. *Let Ţ be the set of all ter-element sets. A set function ν : Ţ $\to [-1,1]$ is called bi-capacity if it satisfies the following requirements:*

1) $\nu(X^+) = \nu(\{1^+, \ldots, n^+\}) = 1$, $\nu\left(X^\emptyset\right) = \nu\left(\left\{1^\emptyset, \ldots, n^\emptyset\right\}\right) = 0$, and $\nu(X^-) = \nu(\{1^-, \ldots, n^-\}) = -1$,

2) $\forall S, T \in$ Ţ, $S \sqsubseteq T \Longrightarrow \nu(S) \leq \nu(T)$.

In the following definition, and we give an another order relation on structure Ţ, and we will denote it by $\subseteq$, which is an equivalent relation to Bilbao order [18].

Definition 2.4. *Suppose S and T are ter-element sets of Ţ. Then, $S \subseteq T$ iff $\forall i = 1, \dots, n$*

$$\text{“if } i^+ \in S \text{ implies } i^+ \in T\text{” and “if } i^- \in S \text{ implies } i^- \in T\text{”.} \tag{3}$$

In this order, for all S, $T \in$ Ţ, the union, $S \cup T$ of S and T is given by

$$S \cup T = \{\tau_j \vee \tau_k : \tau_j \in S, \tau_k \in T\}, \tag{4}$$

for all $j = 1, \dots, n$, $k = 1, \dots, n$ with $i^+ \vee i^- = i^+$, $i^+ \vee i^\emptyset = i^+$, $i^- \vee i^\emptyset = i^-$, $i = 1, \dots, n$.

3. **The Balancing Bipolar Choquet Integral.** In [1], Mesiarova-Zemankova et al. have proposed the balancing Choquet integral for capacities. This integral is a discrete extension of Choquet integral that merges positive and negative inputs together, thus allowing a compensation effect. In this section, we first define an equivalent expression of “balancing Choquet integral” for capacities μ. Then, we propose a balancing bipolar Choquet integral with respect to bi-capacities ν.

We can describe the balancing Choquet integral for capacity μ based on bi-element sets of real input $\mathbf{a}$ by the following definition.

Definition 3.1. *(i) For a real input $\mathbf{a} = (a_1, \dots, a_i, \dots, a_n)$ we consider a bi-element sets $C_1, \dots, C_p$ with $C_k \subseteq \{1^+, \dots, i^+, \dots, n^+\}$ and $\cup_{k=1}^p C_k = \{1^+, \dots, i^+, \dots, n^+\}$ such that*

1) $|a_i| = |a_j|$ for all $\tau_i, \tau_j \in C_k$,

2) $|a_i| < |a_j|$ for all $\tau_i \in C_k$, $\tau_j \in C_r$ with $k < r$.

Denote $|C_k|_{\mathbf{a}} = |a_i|$ for $\tau_i \in C_k$ and $D_k = \cup_{j=k}^p C_j$. The sets C_k, $k = 1, \dots, p$ are value classes of $\mathbf{a}$.

(ii) Let μ : B̧ $\to [0, 1]$ be a capacity. “The balancing Choquet integral” of input $\mathbf{a}$ for the capacity μ is defined by

$$BCh_\mu(\mathbf{a}) = \sum_{k=1}^{p} |C_k|_{\mathbf{a}} \left[\mu\left(C_k^+ \cup D_k\right) - \mu\left(C_k^- \cup D_{k+1}\right)\right], \tag{5}$$

where $C_k^+, C_k^- \subseteq C_k$ such that $\forall i^+ \in C_k$, $i^+ \in C_k^+$ if $a_i > 0$ and $i^+ \in C_k^-$ if $a_i \leq 0$, with the convention that $D_{p+1} = X^-$.

In expression of Choquet integral, the permutation used in the formula of Choquet integral (see e.g., [2, 5, 19]) need not be unique, but it has no influence on the resulting output. While in the formula of bipolar Choquet integral for bi-capacity ([11]) again permutation need not be unique and then it has influence on the resulting output. For this reason, we propose the balancing bipolar Choquet integral for bi-capacity based on ter-element set, where the permutation has no influence on the resulting output.

For an input vector $\mathbf{a} = (a_1, \dots, a_i, \dots, a_n)$, $a_i \in$ Ŗ, we assume a ter-element set $X^* := \{\tau_1, \dots, \tau_n\}$ with $\tau_i = i^+$ if $a_i > 0$, $\tau_i = i^-$ if $a_i < 0$, and $\tau_i = i^\emptyset$ if $a_i = 0$; for all $i = 1, \dots, n$. Hence, we describe “the balancing bipolar Choquet integral” of bi-capacity ν of real input $\mathbf{a}$ by the following definition.

Definition 3.2. *(i) For an input vector $\mathbf{a} = (a_1, \dots, a_i, \dots, a_n)$ consider sets $C_1, \dots, C_p$ with $C_k \subseteq X^*$ and $\cup_{k=1}^p C_k = X^*$ such that*

1) $|a_i| = |a_j|$ for all $\tau_i, \tau_j \in C_k$,

2) $|a_i| < |a_j|$ for all $\tau_i \in C_k$, $\tau_j \in C_r$ with $k < r$.

Denote $|C_k|_{\mathbf{a}} = |a_i|$ for $\tau_i \in C_k$ and $D_k = \cup_{j=k}^p C_j$.

(ii) Let ν : Ţ $\to [-1, 1]$ be a bi-capacity. “The balancing bipolar Choquet integral” of input $\mathbf{a}$ is given by

$$BCh_\nu(\mathbf{a}) = \sum_{k=1}^{p} |C_k|_{\mathbf{a}}[\nu(D_k) - \nu(D_{k+1})], \tag{6}$$

where $D_{p+1} = X^{\emptyset}$.

For the sake of clarity, let us give the following numerical example.

Example 3.1. *For* $n = 6$, *and suppose* $\mathbf{a} = (3, -8, -3, 5, -8, 3)$. *Then* $X^* = \{1^+, 2^-, 3^-, 4^+, 5^-, 6^+\}$, *and the set of all value classes are* $C_1 = \{1^+, 2^{\emptyset}, 3^-, 4^{\emptyset}, 5^{\emptyset}, 6^+\}$, $C_2 = \{1^{\emptyset}, 2^{\emptyset}, 3^{\emptyset}, 4^+, 5^{\emptyset}, 6^{\emptyset}\}$, $C_3 = \{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^{\emptyset}, 5^-, 6^{\emptyset}\}$.

Using Equation (6) we obtain $BCh_\nu(\mathbf{a}) = 3\left[\nu(\{X^*\}) - \nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^+, 5^-, 6^{\emptyset}\}\right)\right] + 5\left[\nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^+, 5^-, 6^{\emptyset}\}\right) - \nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^{\emptyset}, 5^-, 6^{\emptyset}\}\right)\right] + 8\left[\nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^{\emptyset}, 5^-, 6^{\emptyset}\}\right) - \nu\left(X^{\emptyset}\right)\right] = 3\nu(\{X^*\}) + 2\nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^+, 5^-, 6^{\emptyset}\}\right) + 3\nu\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}, 4^{\emptyset}, 5^-, 6^{\emptyset}\}\right)$.

4. **The Balancing Choquet Integral on Fuzzy Sets.** In this section, we propose a new approach for studying the balancing Choquet integral on fuzzy sets through introducing a notion of fuzzy binary-element sets, which is an alternative approach from that defined by Wu and Huang [9].

For any $S \in \text{B̧}$, we can define the characteristic function by $\chi_S(i) = 1$ if and only if $i^+ \in S$, and $\chi_S(i) = 0$ if and only if $i^- \in S$. The fuzzy bi-element set is a bi-element set with a degree of membership in $[0, 1]$, as shown in the following definition.

Definition 4.1. *The fuzzy bi-element set* $\tilde{S}$ *is the set* $\tilde{S} = \{(\tau_i, \chi_{\tilde{S}}(i)) \,|\tau_i \in \{i^+, i^-\}, i = 1, \ldots, n\}$, *where* $\chi_{\tilde{S}}(i)$ *is called degree of membership of* i *in* $\tilde{S}$ *with* $\chi_{\tilde{S}}(i) \in (0, 1]$ *if and only if* $\tau_i = i^+$, *and* $\chi_{\tilde{S}}(i) = 0$ *if and only if* $\tau_i = i^-$.

For $X := \{1, \ldots, n\}$, let us denote the set of all fuzzy bi-element sets by $\tilde{\text{B̧}}(X)$. The operations on fuzzy bi-element sets of $\tilde{\text{B̧}}(X)$ are introduced in [15]. We will denote by $\tilde{\text{B̧}}$ for a fuzzy algebra of fuzzy bi-element sets, and define the fuzzy algebra as follows.

Definition 4.2. *A nonempty subclass of* $\tilde{\text{B̧}}(X)$ *is called fuzzy algebra* $\tilde{\text{B̧}}$ *if it satisfies the following requirements:*

1) $\{(1^-, 0), \ldots, (n^-, 0)\}$ *and* $= \{(1^+, 1), \ldots, (n^+, 1)\} \in \tilde{\text{B̧}}$,
2) If $\tilde{S}, \tilde{T} \in \tilde{\text{B̧}}$, *then* $\tilde{S} \cup \tilde{T} \in \tilde{\text{B̧}}$,
3) If $\tilde{S} \in \tilde{\text{B̧}}$ *then* $\tilde{S}^c \in \tilde{\text{B̧}}$.

The definition of the fuzzy algebra $\tilde{\text{B̧}}$ permits us to introduce *capacity on fuzzy sets.*

Definition 4.3. *A capacity on fuzzy sets is a fuzzy set function* $\tilde{\mu} : \tilde{\text{B̧}} \to [0, \infty]$ *satisfies the following conditions:*

1) $\tilde{\mu}(\{(1^-, 0), \ldots, (n^-, 0)\}) = 0$,
2) $\tilde{\mu}\left(\tilde{S}\right) \leq \tilde{\mu}\left(\tilde{T}\right)$ *whenever* $\tilde{S}, \tilde{T} \in \tilde{\text{B̧}}$ *with* $\tilde{S} \subseteq \tilde{T}$.

Now, we can extend the balancing Choquet integral to capacity of fuzzy sets $\tilde{\mu}$. The balancing Choquet integral of fuzzy sets is an alternative integral from that introduced by Wu and Huang [9].

Definition 4.4. *(i) For an input vector* $\mathbf{a} = (a_1, \ldots, a_i, \ldots, a_n)$ *we assume a fuzzy bi-element set* $\tilde{S}^+ = \{(i^+, \chi_{\tilde{S}^+}(i)) \,|a_i \in \text{Ŗ}, i \in \{1, \ldots, n\}\}$, *and fuzzy bi-element sets* $\tilde{C}_1, \ldots, \tilde{C}_p$ *with* $\tilde{C}_k \subseteq \tilde{S}^+$ *and* $\cup_{k=1}^{p} \tilde{C}_k = \tilde{S}^+$ *such that* $\chi_{\tilde{C}_k}(i) \leq \chi_{\tilde{C}_r}(j)$ *for all* $i^+ \in \tilde{C}_k$, $j^+ \in \tilde{C}_r$ *with* $k < r$.

Denote $\left[\tilde{C}_k\right]_{\mathbf{a}} = \chi_{\tilde{C}_k}(i)$ *for* $i^+ \in \tilde{C}_k$ *and* $\tilde{D}_k = \cup_{j=k}^{p} \tilde{C}_j$. *The sets* $\tilde{C}_k$, $k = 1, \ldots, p$ *are value classes of* $\mathbf{a}$.

(ii) The balancing Choquet integral of $\mathbf{a}$ *for capacity on fuzzy sets* $\tilde{\mu}$ *is given by*

$$BC_{\tilde{\mu}}(\mathbf{a}) = \sum_{k=1}^{p} \left[\tilde{C}_k\right]_{\mathbf{a}} \left[\tilde{\mu}\left(\tilde{D}_k\right) - \tilde{\mu}\left(\tilde{D}_{k+1}\right)\right], \tag{7}$$

where $\tilde{D}_{p+1} = \tilde{X}^-$.

The special case of Definition 4.4 is the balancing Choquet integral of capacity on classical crisp set whenever $\tilde{\mu}$ is a capacity on classical crisp set.

Example 4.1. *For* $n = 3$, *let us consider fuzzy set* $\tilde{S}$ *with membership function*

$$\chi_{\tilde{S}}(i) = \begin{cases} 0.5 & \text{iff } \tau_i = 1^+, \\ 0.8 & \text{iff } \tau_i = 2^+, \\ 0.2 & \text{iff } \tau_i = 3^+. \end{cases}$$

Hence, $C_1 = \{(1^-, 0), (2^-, 0), (3^+, 0.2)\}$, $C_2 = \{(1^+, 0.5), (2^-, 0), (3^-, 0)\}$, $C_3 = \{(1^-, 0), (2^+, 0.8), (3^-, 0)\}$. *Applying Formula (7) we obtain* $BC_{\tilde{\mu}}(\mathbf{a}) = 0.2[\mu((1^+, 0.5), (2^+, 0.8), (3^+, 0.2)) - \mu((1^+, 0.5), (2^+, 0.8), (3^-, 0))] + 0.5[\mu((1^+, 0.5), (2^+, 0.8), (3^-, 0)) - \mu((1^-, 0), (2^+, 0.8), (3^-, 0))] + 0.8[\mu((1^-, 0), (2^+, 0.8), (3^-, 0)) - \mu(X^-)] = 0.2\mu((1^+, 0.5), (2^+, 0.8), (3^+, 0.2)) + 0.3\mu((1^+, 0.5), (2^+, 0.8), (3^-, 0)) + 0.3\mu((1^-, 0), (2^+, 0.8), (3^-, 0))$.

5. **The Balancing Bipolar Choquet Integral on Fuzzy Sets.** There are several possible symmetric extensions of the Choquet integral and the actual choice of the relevant one depends on the modeled situation and the constraints of decision maker. In this section, we can generalize the definition of the balancing Choquet integral to bi-capacities on fuzzy sets, which is a new approach to the aggregation on bipolar scales for studying the balancing bipolar Choquet integral on fuzzy sets.

In the same way that we presented the balancing Choquet integral on fuzzy sets (Section 4), we can extend the scope of balancing Choquet integral to bi-capacity on fuzzy sets as follows.

Fot $S \in \c{T}$, we can define the characteristic function by $\chi_S(i) = 1$ if and only if $i^+ \in S$, $\chi_S(i) = -1$ if and only if $i^- \in S$, and $\chi_S(i) = 0$ if and only if $i^\emptyset \in S$. The fuzzy ter-element set is a ter-element set with different degree of membership in $[-1, 1]$, as shown in the following definition.

Definition 5.1. *The fuzzy ter-element set* $\tilde{S}$ *is the set* $\tilde{S} = \left\{ (\tau_i, \chi_{\tilde{S}}(i)) \,\middle|\, \tau_i \in \{i^+, i^-, i^\emptyset\},\ i = 1, \ldots, n \right\}$, *where* $\chi_{\tilde{S}}(i)$ *is called degree of membership* i *in* $\tilde{S}$ *with* $\chi_{\tilde{S}}(i) \in [-1, 0)$ *if and only if* $\tau_i = i^-$, $\chi_{\tilde{S}}(i) \in (0, 1]$ *if and only if* $\tau_i = i^+$, *and* $\chi_{\tilde{S}}(i) = 0$ *if and only if* $\tau_i = i^\emptyset$.

For $X := \{1, \ldots, n\}$, let us denote the set of all fuzzy ter-element sets by $\tilde{\c{T}}(X)$, and we define the operations on the sets of $\tilde{\c{T}}(X)$ as follows.

- For $\tilde{S} \in \tilde{\c{T}}(X)$, the complement $\tilde{S}^c$ of $\tilde{S}$ is defined by the following membership function

$$\chi_{\tilde{S}^c}(i) = \begin{cases} 1 - \chi_{\tilde{S}}(i) & \text{if } \tau_i = i^+ \text{ and } \chi_{\tilde{S}}(i) \in (0, 1), \\ 0 & \text{if } \tau_i = i^\emptyset, \\ -1 - \chi_{\tilde{S}}(i) & \text{if } \tau_i = i^- \text{ and } \chi_{\tilde{S}}(i) \in (-1, 0). \end{cases}$$

- For $\tilde{S}, \tilde{T} \in \tilde{\c{T}}(X)$, $\tilde{S} \subseteq \tilde{T}$ holds "if $i^+ \in S$ implies $i^+ \in T$", "if $i^- \in S$ implies $i^- \in T$", and "if $i^\emptyset \in S$ implies i^+, or i^-, or $i^\emptyset \in T$".

- For $\tilde{S}, \tilde{T} \in \tilde{Ţ}(X)$, the union $\tilde{S} \cup \tilde{T}$ of $\tilde{S}$ and $\tilde{T}$ is defined by $\tilde{S} \cup \tilde{T} = \Big\{ \big(\tau_j \vee \tau_k, \chi_{\tilde{S}\cup\tilde{T}}(i)\big) : \tau_j \in \tilde{S}, \tau_k \in \tilde{T} \Big\}$, for all $j = 1, \dots, n$, $k = 1, \dots, n$ with $i^+ \vee i^- = i^+$, $i^+ \vee i^\emptyset = i^+$, $i^- \vee i^\emptyset = i^-$, $i = 1, \dots, n$, and
$$\chi_{\tilde{S}\cup\tilde{T}}(i) = \begin{cases} \chi_{\tilde{S}}(i) & \text{if } \tau_j \vee \tau_k = \tau_j, \\ \chi_{\tilde{T}}(i) & \text{if } \tau_j \vee \tau_k = \tau_k. \end{cases}$$
- For $\tilde{S}, \tilde{T} \in \tilde{Ţ}(X)$, the intersection $\tilde{S} \cap \tilde{T}$ of $\tilde{S}$ and $\tilde{T}$ is defined by $\tilde{S} \cap \tilde{T} = \Big\{ \big(\tau_j \wedge \tau_k, \chi_{\tilde{S}\cap\tilde{T}}(i)\big) : \tau_j \in \tilde{S}, \tau_k \in \tilde{T} \Big\}$, for all $j = 1, \dots, n$, $k = 1, \dots, n$ with $i^+ \wedge i^- = i^-$, $i^+ \wedge i^\emptyset = i^\emptyset$, $i^- \wedge i^\emptyset = i^\emptyset$, $i = 1, \dots, n$, and
$$\chi_{\tilde{S}\cap\tilde{T}}(i) = \begin{cases} \chi_{\tilde{S}}(i) & \text{if } \tau_j \wedge \tau_k = \tau_j, \\ \chi_{\tilde{T}}(i) & \text{if } \tau_j \wedge \tau_k = \tau_k. \end{cases}$$

Let us denote the fuzzy algebra of "fuzzy ter-element sets" by $\tilde{Ţ}$, and we define the fuzzy algebra as follows.

Definition 5.2. *A nonempty subclass of $\tilde{Ţ}(X)$ is fuzzy algebra $\tilde{Ţ}$ if it satisfies the following requirements:*
1) $\{1^-, \dots, n^-\}$, $\{1^\emptyset, \dots, n^\emptyset\}$, and $\{1^+, \dots, n^+\} \in \tilde{Ţ}$,
2) If $\tilde{S}, \tilde{T} \in \tilde{Ţ}$, then $\tilde{S} \cup \tilde{T} \in \tilde{Ţ}$,
3) If $\tilde{S} \in \tilde{Ţ}$ then $\tilde{S}^c \in \tilde{Ţ}$.

The definition of the fuzzy algebra $\tilde{Ţ}$ permits us to introduce *"bi-capacity on fuzzy sets"*.

Definition 5.3. *A bi-capacity on fuzzy sets is a fuzzy set function $\tilde{\nu} : \tilde{Ţ} \to Ŗ$ satisfying the following conditions:*
1) $\tilde{\nu}\left(\{1^\emptyset, \dots, n^\emptyset\}\right) = 0$,
2) $\tilde{\nu}\left(\tilde{S}\right) \leq \tilde{\nu}\left(\tilde{T}\right)$ whenever $\tilde{S}, \tilde{T} \in \tilde{Ţ}$ with $\tilde{S} \subseteq \tilde{T}$.

Now, for an input vector $\mathbf{a} = (a_1, \dots, a_i, \dots, a_n)$, $a_i \in Ŗ$ and $i \in \{1, \dots, n\}$, we assume a ter-element set $\tilde{S}^* = \{(\tau_i, \chi_{\tilde{S}^*}(i))\}$ with $\tau_i = i^+$ if $a_i > 0$, $\tau_i = i^-$ if $a_i < 0$, and $\tau_i = i^\emptyset$ if $a_i = 0$; $\forall i = 1, \dots, n$. The following is definition of balancing bipolar Choquet integral of $\mathbf{a}$ for bi-capacity on fuzzy sets $\tilde{\nu}$.

Definition 5.4. *(i) For an input vector $\mathbf{a} = (a_1, \dots, a_i, \dots, a_n)$ we assume fuzzy ter-element sets $\tilde{C}_1, \dots, \tilde{C}_p$ with $\tilde{C}_k \subseteq \tilde{S}^*$ and $\cup_{k=1}^p \tilde{C}_k = \tilde{S}^*$ such that $\left|\chi_{\tilde{C}_k}(i)\right| \leq \left|\chi_{\tilde{C}_r}(j)\right|$ for all $\tau_i \in \tilde{C}_k$, $\tau_j \in \tilde{C}_r$ with $k < r$.*

And $\left|\tilde{C}_k\right|_{\mathbf{a}} = \left|\chi_{\tilde{C}_k}(i)\right|$ for $\tau_i \in \tilde{C}_k$ and $\tilde{D}_k = \cup_{j=k}^p \tilde{C}_j$. The sets $\tilde{C}_k$, $k = 1, \dots, p$ are value classes of $\mathbf{a}$.

(ii) The balancing bipolar Choquet integral of $\mathbf{a}$ for bi-capacity on fuzzy sets $\tilde{\nu}$ is given by
$$BC_{\tilde{\nu}}(\mathbf{a}) = \sum_{k=1}^{p} \left|\tilde{C}_k\right|_{\mathbf{a}} \left[\tilde{\nu}\left(\tilde{D}_k\right) - \tilde{\nu}\left(\tilde{D}_{k+1}\right)\right], \tag{8}$$
where $\tilde{D}_{p+1} = \tilde{X}^\emptyset$.

For the sake of clarity, let us give the following numerical example.

Example 5.1. *For $n = 3$, let us consider fuzzy ter-element set $\tilde{S}$ with membership function*

$$\chi_{\tilde{S}}(i) = \begin{cases} 0.6 & \text{if } \tau_i = 1^+, \\ -0.5 & \text{if } \tau_i = 2^-, \\ 0.8 & \text{if } \tau_i = 3^+. \end{cases}$$

Then, $\tilde{C}_1 = \{(1^\emptyset, 0), (2^-, -0.5), (3^\emptyset, 0)\}$, $\tilde{C}_2 = \{(1^+, 0.6), (2^\emptyset, 0), (3^\emptyset, 0)\}$, $\tilde{C}_3 = \{(1^\emptyset, 0), (2^\emptyset, 0), (3^+, 0.8)\}$. *Using Equation (8) we obtain,* $BC_{\tilde{\nu}}(\mathbf{a}) = 0.5\big[\tilde{\nu}\big((1^+, 0.6), (2^-, -0.5), (3^+, 0.8)\big) - \tilde{\nu}\left((1^+, 0.6), (2^\emptyset, 0), (3^+, 0.8)\right)\big] + 0.6\left[\tilde{\nu}\left((1^+, 0.6), (2^\emptyset, 0), (3^+, 0.8)\right) - \tilde{\nu}\left((1^\emptyset, 0), (2^\emptyset, 0), (3^+, 0.8)\right)\right] + 0.8\left[\tilde{\nu}\left((1^\emptyset, 0), (2^\emptyset, 0), (3^+, 0.8)\right) - \tilde{\nu}\left(X^\emptyset\right)\right] = 0.5\tilde{\nu}\big((1^+, 0.6), (2^-, -0.5), (3^+, 0.8)\big) + 0.1\tilde{\nu}\left((1^+, 0.6), (2^\emptyset, 0), (3^+, 0.8)\right) + 0.2\tilde{\nu}\big((1^\emptyset, 0), (2^\emptyset, 0), (3^+, 0.8)\big)$.

The special case of Definition 5.4 is the balancing bipolar Choquet integral for bi-capacity on crisp set whenever $\tilde{\nu}$ is a bi-capacity on crisp set. The balancing bipolar Choquet integral on fuzzy sets $\tilde{\nu}$ satisfies the following basic properties.

Proposition 5.1. *To any bi-capacity on fuzzy sets ($\tilde{\nu}$) on $\tilde{Ţ}$, the balancing bipolar Choquet integral*

$$C_{\tilde{\nu}}\big(1_{\tilde{S}}, -1_{\tilde{S}}, 0_{\tilde{S}}\big) = \tilde{\nu}(S), \quad \forall S \in Ţ,\ \tilde{S} \in \tilde{Ţ}.$$

Proof: For input $(1_{\tilde{S}}, -1_{\tilde{S}}, 0_{\tilde{S}})$,

$$\left|\chi_{\tilde{C}_k}(i)\right| = 1 \text{ or } \left|\chi_{\tilde{C}_k}(i)\right| = 0, \quad \forall \tau_i \in \tilde{C}_k$$

and $\tilde{\nu}\left(\tilde{D}_k\right) - \tilde{\nu}\left(\tilde{D}_{k+1}\right) = \tilde{\nu}\left(\{(i^+, \chi_S(i^+)), (i^-, \chi_S(i^-)), (i^\emptyset, 0)\}\right) - \tilde{\nu}\left(\tilde{X}^\emptyset\right) = \tilde{\nu}\big(\{(i^+, \chi_S(i^+)), (i^-, \chi_S(i^-)), (i^\emptyset, 0)\}\big) = \tilde{\nu}\left(\{(i^+, 1), (i^-, -1), (i^\emptyset, 0)\}\right) = \tilde{\nu}\left(\tilde{S}\right) = \tilde{\nu}(S)$.

Hence, from the definition of balancing bipolar Choquet integral for bi-capacity on fuzzy sets (Equation (8)), we have

$$BC_{\tilde{\nu}}\big(1_{\tilde{S}}, -1_{\tilde{S}}, 0_{\tilde{S}}\big) = \sum_{k=1}^{p} \left|\tilde{C}_k\right|_{\mathbf{a}} \left[\tilde{\nu}\left(\tilde{D}_k\right) - \tilde{\nu}\left(\tilde{D}_{k+1}\right)\right]$$

Thus,

$$BC_{\tilde{\nu}}\left(1_{\tilde{S}}, -1_{\tilde{S}}, 0_{\tilde{S}}\right) = \tilde{\nu}(S), \quad \forall S \in Ţ,\ \tilde{S} \in \tilde{Ţ}.$$

The following result shows that "the balancing bipolar Choquet integral for bi-capacity on fuzzy sets" satisfies the monotonicity property.

Proposition 5.2. *To any bi-capacity of fuzzy sets $\tilde{\nu}$ on $\tilde{Ţ}$, $\forall \mathbf{a}, \mathbf{a}' \in Ŗ$, if $a_i \le a'_i$, $\forall i \in \{1, \ldots, n\}$, then $BC_{\tilde{\nu}}(\mathbf{a}) \le BC_{\tilde{\nu}}(\mathbf{a}')$.*

Proof: First, we consider that $a_i < a'_i$, and for all $k \in \{1, \ldots, i-1, i+1, \ldots, n\}$, $a_k = a'_k$. And, we assume that $|\chi_{\tilde{C}_k}(1)| \le \cdots \le |\chi_{\tilde{C}_k}(n)|$ and $|\chi_{\tilde{C}'_k}(1)| \le \cdots \le |\chi_{\tilde{C}'_k}(n)|$. For this case, we prove the monotonicity as follows.

By the balancing bipolar Choquet integral for bi-capacity on fuzzy sets (Equation (8)), we have

$$BC_{\tilde{\nu}}(\mathbf{a}) = \sum_{k=1}^{p} \left|\tilde{C}_k\right|_{\mathbf{a}} \left[\tilde{\nu}\left(\tilde{D}_k\right) - \tilde{\nu}\left(\tilde{D}_{k+1}\right)\right], \tag{9}$$

also

$$BC_{\tilde{\nu}}\left(\mathbf{a}'\right) = \sum_{k=1}^{p} \left|\tilde{C}'_k\right|_{\mathbf{a}'} \left[\tilde{\nu}\left(\tilde{D}'_k\right) - \tilde{\nu}\left(\tilde{D}'_{k+1}\right)\right]. \tag{10}$$

Since $\tilde{D}_k$ and $\tilde{D}_{k+1}$ are the fuzzy ter-element sets and $\tilde{D}_{k+1} \subseteq \tilde{D}_k$, $\tilde{\nu}\left(\tilde{D}_k\right) - \tilde{\nu}\left(\tilde{D}_{k+1}\right) \geq 0$. Similraly, $\tilde{\nu}\left(\tilde{D}'_k\right) - \tilde{\nu}\left(\tilde{D}'_{k+1}\right) \geq 0$.

Now, since $a_i \leq a'_i$ implies $\left|\tilde{C}_k\right|_{\mathbf{a}} \leq \left|\tilde{C}'_k\right|_{\mathbf{a}'}$, it is clear that $BC_{\tilde{\nu}}(\mathbf{a}) \leq BC_{\tilde{\nu}}(\mathbf{a}')$. So, if $a_i < a'_i$ then $BC_{\tilde{\nu}}(\mathbf{a}) \leq BC_{\tilde{\nu}}(\mathbf{a}')$ is proved within the range that the order of $\mathbf{a}$ and $\mathbf{a}'$ does not change. Therfore, by iterating the procedures 2 times at element of the change of the order, if $a_i < a'_i$ then $BC_{\tilde{\nu}}(\mathbf{a}) \leq BC_{\tilde{\nu}}(\mathbf{a}')$. Using this procedure for each i, the result can be proved.

6. **Conclusions.** This paper first presented a framework for extending capacity and balancing Choquet integral for definition of "bi-capacities based on ter-element sets" and "the balancing bipolar Choquet integral". According to this framework, we have introduced an expression of the balancing bipolar Choquet integral. Then we have extended "the balancing bipolar Choquet integral" to fuzzy sets, and we have given the basic properties of this integral. The balancing Choquet integral can be applied in any aggregation when the zero is expected to be the center of symmetry of the input axis, and some kind of the mentioned compensation effect, for example, applications of the balancing Choquet integral in sentiment analysis, especially in sentiment classification. Sentiment classification is mostly focused on methods that assign sentiment degrees (where to each feature a sentiment degree from $[-1, 1]$ is assigned) to individual features (for more details, see [1, 20, 21, 22]). Since the balancing bipolar Choquet integrals were introduced in Sections 4 & 5 as a generalization of the balancing Choquet integral, the application of balancing bipolar Choquet integral in the field sentiment analysis (or, in all decision and evaluation problems where the bipolar scales for input values are taken into account) is an open question for future research.

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