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EXISTENCE, UNIQUENESS AND BOUNDEDNESS OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. *It is well known that the study of many processes of the natural sciences can be reduced to solve integro-differential equations with variable boundaries. Recently, studies on certain problems of the environment, such as the corona virus, the emergence of new diseases, and diseases associated with mutations of viruses, have become relevant. A solution to such problems is associated with finding solutions of integro-differential equations. For the last few years, researchers have been paying attention to the newly discovered fractional operators involving nonsingular kernels. The Caputo fractional derivative is the one of these operators which has captured the interest of scientists the most because of the many interesting results reported when this derivative is used in modelling some real-world phenomena. However, the theory of these operators is still to be addressed. In this paper, we establish some new conditions for the existence and uniqueness of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with nonlocal conditions. The desired results are proved by using theory of fractional calculus aid of fixed point theorems due to Banach and Krasnoselskii in Banach spaces.*

Keywords: Volterra-Fredholm integro-differential equation, Caputo fractional derivative, Fixed point method

1. **Introduction.** Fractional calculus is predominately description of the fractional order of integral and derivative operator. It has a lengthy history in mathematics as much as old as differential calculus [1, 2]. Several researchers described that the fractional integral and derivative are suitable for modelling to define the memory and hereditary properties of different substances or system and other real world problem. Many types of fractional operators and definition are obtained. This fact enables the researches to pick up the most convenient fractional derivative for the sake of achieving better results in modeling the real world problem under consideration. Integro-differential equations with nonlocal conditions have attracted the attention of many researchers in the last decades as seen in [1, 2, 3, 4, 5, 6], because of their applications in numerous fields of science, engineering, physics, economy and so on. In the last years, with the development of theorems of fractional integro-differential equations, many authors investigated the existence of solutions of abstract fractional integro-differential equations with nonlocal conditions by using semigroups theorems, solution operator theorems and the relation between solution operators and semigroups constructing by probability density functions as well as fixed point techniques [7, 8, 9, 10, 11, 12, 13].

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Recently, Baleanu et al. [14], by using fixed-point methods, studied the existence and uniqueness of a solution for the nonlinear fractional boundary value problem given by

$$\begin{aligned}&{}^cD^\nu x(t)=f(t,x(t)),\ t\in J=[0,T],\ 0<\nu<1,\\&x(0)=x(T),\ x(0)=\beta_1 x(\eta),\ x(T)=\beta_2 x(\eta),\ 0<\eta<T,\ 0<\beta_1<\beta_2<1.\end{aligned}$$

Devi and Sreedhar [15] used the monotone iterative technique to the Caputo fractional integro-differential equation of the type

$$\begin{aligned}&{}^cD^\nu x(t)=f(t,x(t),I^\nu x(t)),\ t\in J=[0,T],\ 0<\nu<1,\\&x(0)=x_0.\end{aligned}$$

Wang and Zhou [16] studied the Ulam stability and data dependence for a Caputo fractional differential given by

$$\begin{aligned}&{}^cD^\nu x(t)=f(t,x(t)),\ t\in J=[a,+\infty),\ 0<\nu<1,\\&x(a)=\xi.\end{aligned}$$

Dong et al. [17] established the existence and uniqueness of solutions via Banach and Schaude fixed point techniques for the problem given by

$$\begin{aligned}&{}^cD^\nu_{0+}x(t)=f(t,x(t))+\int_0^t G(t,s,x(s))ds,\ t\in J=[0,T],\ 0<\nu\le 1,\\&x(0)=\xi.\end{aligned}$$

Benchohra and Bouriahi [18] investigated existence and stability of solutions for a class of boundary value problem for implicit Caputo fractional differential equations of the type

$$\begin{aligned}&{}^cD^\nu x(t)=f(t,x(t),{}^cD^\nu x(t)),\ t\in J:=[0,T],\ T>0\ ,\ 0<\nu\le 1,\\&x(0)+g(x)=x_0.\end{aligned}$$

In this paper, we extend the results in [16, 17] by proving the existence and uniqueness of solutions for the following nonlinear Caputo fractional Volterra-Fredholm integro-differential equations

$${}^cD^\nu x(t)=f\left(t,x(t),\int_0^t k(t,s)x(s)ds,\int_0^T h(t,s)x(s)ds\right),\ t\in J:=[0,T], \tag{1}$$

$$x(0)+g(x)=x_0, \tag{2}$$

where ${}^cD^\nu$ is the Caputo fractional derivative of order ν, $0<\nu\le 1$, $f:J\times X\times X\times X\longrightarrow X$ is a continuous function, $k,h:J\times J\longrightarrow X$ is a continuous function, and $k_T=\sup\{|k(t,s)|:0\le s\le t\le T\}$, $h_T=\sup\{|h(t,s)|:0\le s\le t\le T\}$, $g(x):C(J,X)\longrightarrow X$, and $x_0\in X$. To prove the existence and uniqueness of solutions, we transform (1) into an equivalent integral equation and then use the Krasnoselskii and Banach fixed point theorems.

The paper is organized as follows. Section 2 presents, as preliminaries, the definition of the fractional derivative, the fractional integral of Riemann-Liouville with respect to another function, and some important results, given as theorems, as well as the spaces in which such operators and theorems are defined. In Section 3, we use the fixed point theorems due to Banach and Krasnoselskii to prove the existence and uniqueness results for the problem (1)-(2). In the special case, when $k=h=g=0$ in the problem (1)-(2) then the results of [16] appear as a special case of our results and when $h=g=0$ in the problem (1)-(2) then the results of [17] appear as a special case of our results. Then, the results presented in this paper extend the main results in [16, 17]. The application of our main results is established in Section 4. Concluding remarks close the paper in Section 5.

2. **Preliminaries.** The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involve the Riemann-Liouville fractional derivative and Caputo derivative [9, 10, 13].

Definition 2.1. [13] *(Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $u \in C([0,T])$ is defined as*

$$J^{\nu}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds,$$

where Γ denotes the Gamma function.

Definition 2.2. [9] *(Caputo fractional derivative). The fractional derivative of $u(t)$ in the Caputo sense is defined by*

$${}^{c}D^{\nu}u(t) = J^{m-\nu}D^{m}u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\nu)}\displaystyle\int_0^t (t-s)^{m-\nu-1}\frac{\partial^m u(s)}{\partial s^m}ds, & m-1<\nu<m, \\ \dfrac{\partial^m u(t)}{\partial t^m}, & \nu = m,\ m \in N, \end{cases}$$

where the parameter ν is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive ν will be considered.

Hence, we have the following properties:

1) $J^{\nu}J^{v}u = J^{\nu+v}u,\ \nu, v > 0.$

2) $J^{\nu}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}u^{\beta+\nu}.$

3) $D^{\nu}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)}u^{\beta-\nu},\ \nu > 0,\ \beta > -1.$

4) $J^{\nu}D^{\nu}u(t) = u(t) - u(a),\ 0 < \nu < 1.$

5) $J^{\nu}D^{\nu}u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^{+})\frac{(t-a)^k}{k!},\ t > 0.$

Definition 2.3. [10] *The Riemann Liouville fractional derivative of order $\nu > 0$ is normally defined as*

$${}^{L}D^{\nu}u(t) = \frac{1}{\Gamma(n-\nu)}\frac{d^m}{dt^m}\int_0^t \frac{u(s)}{(t-s)^{\nu+1-m}}ds,\ m-1<\nu\le m,\ m\in\mathbb{N}.$$

Definition 2.4. [10] *The Caputo derivative of order ν for a function $u : [0,\infty) \longrightarrow \mathbb{R}$ can be written as*

$${}^{c}D^{\nu}u(t) = {}^{L}D^{\nu}\left[u(t) - \sum_{k=0}^{m-1}\frac{t^k}{k}u^{(k)}(0)\right].$$

Theorem 2.1. [19] *Suppose $\nu > 0$, $\tilde{a}(t)$ is a nonnegative function locally integrable on J and $\tilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\tilde{g}(t) \le M$, $t \in J$, and suppose $u(t)$ is nonnegative and locally integrable on J with*

$$u(t) \le \tilde{a}(t) + \tilde{g}(t)\int_0^t (t-s)^{\nu-1}u(s)ds,\ t\in J.$$

Then

$$u(t) \le \tilde{a}(t) + \int_0^t \left[\sum_{m=1}^{\infty}\frac{(\tilde{g}(t)\Gamma(\nu))^m}{\Gamma(m\nu)}(t-s)^{m\nu-1}\tilde{a}(s)\right]ds,\ t\in J.$$

Remark 2.1. *Under the hypothesis of Theorem 2.1, let $\tilde{a}(t)$ be a nondecreasing function on J. Then we have*

$$u(t) \leq \tilde{a}(t)E_\alpha(\tilde{g}(t)\Gamma(\alpha)t^\alpha), \tag{3}$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(t) = \sum_{m=0}^{\infty} \frac{t^\alpha}{\Gamma(m\alpha + 1)}. \tag{4}$$

Lemma 2.1. [13] *(Bochner theorem). A measurable function $f : J \longrightarrow X$ is a Bochner integral if $\|f\|$ is lebegue integrable.*

Lemma 2.2. [13] *(Mazur lemma). If A is a compact subset of X, then its convex closure $co\bar{n}vA$ is compact.*

Lemma 2.3. [10] *(Ascoli-Arzela theorem). Let $S = \{s(t)\}$ be a function family of continuous mappings $s : J \longrightarrow X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in J$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1, 2, \ldots, t \in J)$ in S.*

Theorem 2.2. [13] *(Banach). Let (X, d) be a nonempty complete metric space with $T : X \longrightarrow X$ as a contraction mapping. Then map T has a fixed point $x^* \in X$ such that $Tx^* = x^*$.*

Theorem 2.3. [10] *(Krasnoselskii). Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*
1) $Ax + By \in M$ whenever $x, y \in M$.
2) A is compact and continuous.
3) B is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

3. **Existence and Uniqueness of Solutions.** In this section, we shall give existence and uniqueness results of Equation (1), with the conditions (2). Before starting and proving the main results, we introduce the following hypotheses.
(A1) For each $x, y \in X$, $f(t, x, y, z)$ is strongly measurable w.r.t. t on J.
(A2) For each $t \in J$, $f(t, x, y, z)$ is continuous w.r.t. x and y on X.
(A3) There exist constants $a_f > 0$, $a_g \in (0, 1)$ for arbitrary $u, v, y \in X$ such that

$$\|f(t, u, v, y)\| \leq a_f[1 + \|u\| + \|v\| + \|y\|].$$
$$\|g(u)\| \leq a_g[1 + \|u\|_C].$$

(A4) There exist constants $L_f(\rho) > 0$, $L_g \in (0, 1)$ for arbitrary $u_i, v_i, y_i \in X$, $i = 1, 2$, satisfying $\|u_1\|, \|v_1\|, \|y_1\|, \|u_2\|, \|v_2\|, \|y_2\| \leq \rho$ such that

$$\|f(t, u_1, v_1, y_1) - f(t, u_2, v_2, y_2)\| \leq L_f(\rho)[\|u_1 - u_2\| + \|v_1 - v_2\| + \|y_1 - y_2\|],$$
$$\|g(u) - g(v)\| \leq L_g\|u - v\|_C, \ u, v \in X.$$

(A5) For any $t \in J$, the set

$$K = \left\{(t - s)^{\nu - 1} f\left(s, x(s), \int_0^s k(s, \tau)x(\tau)d\tau, \int_0^T h(s, \tau)x(\tau)d\tau\right) : x \in C(J, X), s \in [0, t]\right\}$$

is relatively compact.

Lemma 3.1. *Let $0 < \nu \leq 1$. Assume that $f \in C[J, X]$. If $x \in C[J, X]$ then x satisfies the problem* (1)-(2) *if and only if x satisfies the mixed type integral equation*

$$\begin{aligned} x(t) &= x_0 - g(x) \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right) ds. \quad (5) \end{aligned}$$

Proof: Define $B_r = \{x \in C(J, X) : \|x\|_C \leq r\}$, for any $r > 0$. Making use of hypotheses (A1)-(A2), we have f is measurable function on J. Now for $x \in B_r$ and $t \in J$, we obtain

$$\begin{aligned} &\int_0^t (t-s)^{\nu-1} \left\| f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)\right\| ds \\ &\leq a_f(1 + r + rTk_T + rTh_T) \left[\frac{T^\nu}{\nu}\right]. \end{aligned}$$

Thus

$$\left\|(t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)\right\|$$

is Lebegue integrable with respect to $s \in [0, t]$, for all $t \in J$ and $x \in B_r$. Then from Bochner's theorem it follows that

$$(t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)$$

is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$. Let $G(\tau, s) = (t-\tau)^{-\nu}|\tau - s|^{\nu-1}$. Since $G(\tau, s)$ is a nonnegative measurable function on $D = [0, t] \times [0, t]$ for $t \in J$, we have

$$\begin{aligned} \int_D G(\tau, s)d\tau &= \int_0^t (t-\tau)^{-\nu} \left(\int_0^\tau (\tau - s)^{\nu-1} ds\right) d\tau + \int_0^t (t-\tau)^{-\nu} \left(\int_\tau^t (s-\tau)^{\nu-1} ds\right) d\tau \\ &\leq \frac{2T}{\nu(1-\nu)}, \end{aligned}$$

and using hypothesis (A3), we obtain

$$(t-\tau)^{-\nu}(\tau - s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)$$

is a Lebesgue integrable function and hence we get

$$\begin{aligned} &{}^L D^\nu \left[I^\nu f\left(t, x(t), \int_0^t k(t,s)x(s)ds, \int_0^T h(t,s)x(s)ds\right)\right] \\ &= f\left(t, x(t), \int_0^t k(t,s)x(s)ds, \int_0^T h(t,s)x(s)ds\right). \end{aligned}$$

We claim that $x(t)$ is absolutely continuous on J. For any disjoint family of open intervals $\{(a_i, b_i)\}_{i=1}^n$ on J with $\sum_{i=1}^n (b_i - a_i) \longrightarrow 0$, we have

$$\begin{aligned} \sum_{i=1}^n \|x(b_i) - x(a_i)\| &\leq \frac{a_f(1 + r + rTk_T + rTh_T)}{\Gamma(\nu)} \sum_{i=1}^n \int_{a_i}^{b_i} (b_i - s)^{\nu-1} ds \\ &\quad + \frac{a_f(1 + r + rTk_T + rTh_T)}{\Gamma(\nu)} \sum_{i=1}^n \int_0^{a_i} \left((a_i - s)^{\nu-1} - (b_i - s)^{\nu-1}\right) ds \\ &\leq \frac{2a_f(1 + r + rTk_T + rTh_T)}{\Gamma(\nu+1)} \sum_{i=1}^n (b_i - a_i)^\nu \end{aligned}$$

$$\longrightarrow 0.$$

Thus $x(t)$ is differential for almost all $t \in J$. According to Definition 2.2, we have

$$\begin{aligned}
{}^cD^\nu x(t) &= {}^cD^\nu \left[I^\nu f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)\right] \\
&= {}^LD^\nu \left[I^\nu f\left(t, x(t), \int_0^t k(t,\tau)x(\tau)d\tau, \int_0^T h(t,\tau)x(\tau)d\tau\right)\right] \\
&\quad - \frac{t^{-\nu}}{\Gamma(1-\nu)}\left[I^\nu f\left(t, x(t), \int_0^t k(t,\tau)x(\tau)d\tau, \int_0^T h(t,\tau)x(\tau)d\tau\right)\right]_{t=0}.
\end{aligned}$$

Since $(t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right)$ is Lebesgue integrable w.r.t. $s \in [0,t]$, for all $t \in J$, we know that $I^\nu f\left(t, x(t), \int_0^t k(t,\tau)x(\tau)d\tau, \int_0^T h(t,\tau)x(\tau)d\tau\right)_{t=0} = 0$ which implies that ${}^cD^\nu x(t) = f\left(t, x(t), \int_0^t k(t,s)x(s)ds, \int_0^T h(t,s)x(s)ds\right)$, a.e. for $t \in J$. Moreover, $x(0) + g(x) = x_0$. Thus, $x \in C(J,X)$ is a solution of the problem (1)-(2). On the other hand, if $x \in C(J,X)$ is a solution of the problem (1)-(2), then x satisfies Equation (5). □

Theorem 3.1. *Suppose the problem (1)-(2) has a solution x on J. If hypothesis (A3) holds, then there exists a constant $\rho > 0$ such that $\|x(t)\| \le \rho$, $\forall t \in J$.*

Proof: By Lemma 3.1, the solution of the problem (1)-(2) is equivalent to the solution of integral Equation (5). Using hypothesis (A3), we have

$$\begin{aligned}
\|x(t)\| &\le \|x_0\| + a_g + a_g\|x\|_C + \frac{a_f T^\nu}{\Gamma(\nu+1)} + \frac{a_f}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\|x(s)\|ds \\
&\quad + \frac{a_f k_T}{\Gamma(\nu)}\int_0^t \frac{(t-\tau)^\nu}{\nu}\|x(\tau)\|d\tau + \frac{a_f h_T}{\Gamma(\nu)}\int_0^T \frac{(t-\tau)^\nu}{\nu}\|x(\tau)\|d\tau \\
&\le \|x_0\| + a_g + a_g\|x\|_C + \frac{a_f T^\nu}{\Gamma(\nu+1)} \\
&\quad + \frac{a_f}{\Gamma(\nu)}\left[1 + \frac{k_T T}{\Gamma(\nu)} + \frac{h_T T}{\Gamma(\nu)}\right]\int_0^T (t-\tau)^{\nu-1}\|x(\tau)\|d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x(t)\|_C \le\ & \frac{\Gamma(\nu+1)(\|x_0\| + a_g) + a_f T^\nu}{(1-a_g)\Gamma(\nu+1)} \\
& + \frac{a_f}{(1-a_g)\Gamma(\nu)}\left[1 + \frac{k_T T}{\Gamma(\nu)} + \frac{h_T T}{\Gamma(\nu)}\right]\int_0^T (t-\tau)^{\nu-1}\|x\|_C d\tau.
\end{aligned}$$

Applying the singular Gronwall inequality stated in Theorem 2.1, we obtain

$$\|x(t)\|_C \le \frac{\Gamma(\nu+1)(\|x_0\| + a_g) + a_f T^\nu}{(1-a_g)\Gamma(\nu+1)}\left[\sum_{n=0}^\infty \frac{\left(a_f T^\nu\left[1 + \frac{k_T T}{\Gamma(\nu)} + \frac{h_T T}{\Gamma(\nu)}\right]\right)^n}{(1-a_g)^n\Gamma(n\nu+1)}\right],$$

where $\sum_{n=0}^\infty \frac{\left(a_f T^\nu\left[1 + \frac{k_T T}{\Gamma(\nu)} + \frac{h_T T}{\Gamma(\nu)}\right]\right)^n}{(1-a_g)^n\Gamma(n\nu+1)}$ is the well known Mittag-Leffler function. Thus there exists a constant $\rho > 0$ such that $\|x(t)\| \le \rho$, for $t \in J$. □

Theorem 3.2. *Assume that hypotheses (A1)-(A4) are fulfilled. If*

$$a_g + \frac{a_f T^{\nu}(1 + k_T T + h_T T)}{\Gamma(\nu + 1)} < 1, \tag{6}$$

$$\gamma_{T,\nu,\rho} = L_g + \frac{T^{\nu} L_f(\rho)(1 + k_T T + h_T T)}{\Gamma(\nu + 1)} < 1, \tag{7}$$

then there exists a unique solution for the problem (1)-(2).

Proof: Consider the operator $\Upsilon : C_\rho \longrightarrow C_\rho$ defined by

$$\begin{aligned}(\Upsilon x)(t) &= x_0 - g(x)\\ &+ \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right) ds,\end{aligned} \tag{8}$$

where $C_\rho := \{x \in C(J, X) : \|x(t)\| \le \rho, t \in J\}$, and

$$\left[\frac{\|x_0\| + a_g + \left(\frac{a_f T^{\nu}[1+k_T T + h_T T]}{\Gamma(\nu+1)}\right)}{1 - \left(a_g + \frac{a_f T^{\nu}[1+k_T T+h_T T]}{\Gamma(\nu+1)}\right)}\right] \le \rho.$$

By Theorem 3.1, it is obvious that Υ is well defined on C_ρ in the sense of Bochner integrable. First we prove that $\Upsilon x \in C_\rho$, for $x \in C_\rho$. For every $x \in C_\rho$, we have

$$\begin{aligned}&\|(\Upsilon x)(t+\delta) - (\Upsilon x)(t)\|\\ \le\ & \frac{a_f(1 + \|x(s)\| + \rho k_T T + \rho h_T T)}{\Gamma(\nu)} \int_0^t \left[(t-s)^{\nu-1} - (t+\delta-s)^{\nu-1}\right] ds\\ &+ \frac{a_f(1 + \|x(s)\| + \rho k_T T + \rho h_T T)}{\Gamma(\nu)} \int_0^{t+\delta} (t+\delta-s)^{\nu-1} ds\\ \le\ & \frac{a_f(1+\rho+\rho k_T T+\rho h_T T)}{\Gamma(\nu)}\left[\frac{t^\nu}{\nu} - \frac{(t+\delta)^\nu}{\nu} + \frac{\delta^\nu}{\nu}\right] + \frac{a_f(1+\rho+\rho k_T T+\rho h_T T)}{\Gamma(\nu)}\left[\frac{\delta^\nu}{\nu}\right]\\ \le\ & \frac{2a_f(1+\rho+\rho k_T T+\rho h_T T)}{\Gamma(\nu+1)}\delta^\nu\\ &\longrightarrow 0 \textbf{ as } \delta \longrightarrow 0.\end{aligned}$$

This shows that $\Upsilon x \in C(J, X)$.

Now, for all $t \in J$ and $x \in C_\rho$, we have

$$\begin{aligned}\|(\Upsilon x)(t)\| &\le \|x_0\| + a_g(1+\rho) + \frac{a_f(1+\rho+\rho k_T T+\rho h_T T)}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds\\ &\le \|x_0\| + (1+\rho)\left[a_g + \frac{a_f(1+k_T T+h_T T)T^\nu}{\Gamma(\nu+1)}\right].\end{aligned} \tag{9}$$

Making use of condition (6) in Equation (9), we obtain $\|(\Upsilon x)(t)\| \le \rho$, which implies that $\Upsilon x \in C_\rho$. Making hypothesis (A4) for any $x, y \in C_\rho$, we have

$$\begin{aligned}\|(\Upsilon x)(t) - (\Upsilon y)(t)\| &\le L_g\|x-y\|_C + \frac{L_f(\rho)}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1}\|x(s)-y(s)\| ds\\ &\quad + \frac{L_f(\rho)T(k_T+h_T)}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1}\|x(s)-y(s)\| ds\\ &\le \left[L_g + \frac{T^\nu L_f(\rho)[1+T(k_T+h_T)]}{\Gamma(\nu+1)}\right]\|x-y\|_C\\ &\le \gamma_{T,\nu,\rho}\|x-y\|_C.\end{aligned}$$

Since $\gamma_{T,\nu,\rho} < 1$, Υ is a contraction map on C_ρ and by applying Banach's contraction mapping principle the operator Υ has a unique fixed point on C_ρ. Hence the problem (1)-(2) has a unique solution, and the proof is completed. □

Theorem 3.3. *Assume that (A1), (A3), (A4) and (A5) hold, and let the condition* (6) *hold, then the problem* (1)-(2) *has at least one solution.*

Proof: Consider the operator Υ defined by Theorem 3.2. We assume that the operator $\Upsilon = P + Q$ on C_ρ

$$\begin{aligned}(Px)(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f\left(s, x(s), \int_0^s k(s,\tau)x(\tau)d\tau, \int_0^T h(s,\tau)x(\tau)d\tau\right) ds,\\ (Qx)(t) &= x_0 - g(x),\ t \in J,\end{aligned}$$

where C_ρ is given in Theorem 3.2. Therefore, to prove the existence of a solution of the problem (1)-(2) is equivalent to proving that the operator $P + Q$ has a fixed point on C_ρ. The proof is divided into several steps.

Step 1. $Px + Qy \in C_\rho$. For every pair $x, y \in C_\rho$, we have

$$\begin{aligned}\|(Px)(t) + (Qy)(t)\| &\leq \|x_0\| + a_g(1+\rho) + \frac{a_f(1+\rho)(1+k_T T + h_T T)T^\nu}{\Gamma(\nu+1)}\\ &\leq \|x_0\| + (1+\rho)\left[a_g + \frac{a_f(1+k_T T + h_T T)T^\nu}{\Gamma(\nu+1)}\right].\end{aligned}$$

Making use of condition (6) in above equation, we obtain $\|(Px)(t) + (Qy)(t)\| \leq \rho$, which implies that $Px + Qy \in C_\rho$.

Step 2. Q is a contraction mapping on C_ρ. For every $y_1, y_2 \in C_\rho$,

$$\|Qy_1 - Qy_2\| = \|g(y_1) - g(y_2)\| \leq L_g\|y_1 - y_2\|_C.$$

From hypothesis (A4), $L_g \in (0,1)$ and hence Q is a contraction mapping.

Step 3. P is a continuous operator. Let $\{x_n\}$ be a sequence of C_ρ such that $x_n \longrightarrow x$ in C_ρ. Then by hypotheses (A2) and (A3), for all $t \in J$, we have

$$\begin{aligned}&\|(Px_n)(t) - (Px)(t)\|\\ &\leq \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} L_f(\rho)\left[\|x_n(s) - x(s)\| + k_T\int_0^s \|x_n(\tau) - x(\tau)\|d\tau\right.\\ &\quad\left. + h_T\int_0^T \|x_n(\tau) - x(\tau)\|d\tau\right] ds\\ &\longrightarrow 0 \textbf{ as } n \longrightarrow \infty.\end{aligned}$$

Thus, $Px_n \longrightarrow Px$ as $n \longrightarrow \infty$ which implies that P is continuous.

Step 4. P is a compact operator. Let $\{x_n\}$ be a sequence of C_ρ.

$$\begin{aligned}\|(Px_n)(t)\| &\leq \frac{a_f(1+\rho+\rho k_T T + \rho h_T T)}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} ds\\ &\leq \frac{a_f(1+\rho+\rho k_T T + \rho h_T T)T^\nu}{\Gamma(\nu+1)}.\end{aligned}$$

Thus $\{x_n\}$ is uniformly bounded.

Now we prove that $\{Px_n\}$ is equicontinuous. For $0 \leq t_1 < t_2 \leq T$, we get

$$\|(Px_n)(t_1) - (Px_n)(t_2)\| \leq \frac{a_f}{\Gamma(\nu)}\int_0^{t_1}\left[(t_1-s)^{\nu-1} - (t_2-s)^{\nu-1}\right](1+\rho+\rho k_T T + \rho h_T T) ds$$

$$
\begin{aligned}
&+\frac{a_f}{\Gamma(\nu)}\int_{t_1}^{t_2}(t_2-s)^{\nu-1}(1+\rho+\rho k_T T+\rho h_T T)ds\\
&\leq \frac{2a_f(1+\rho+\rho k_T T+\rho h_T T)}{\Gamma(\nu+1)}(t_2-t_1)^{\nu}\\
&\longrightarrow 0 \textbf{ as } t_2\longrightarrow t_1.
\end{aligned}
$$

Therefore, $\{Px_n\}$ is equicontinuous. In view of the condition (A5) and Lemma 2.2, we know that $c\bar{o}nv$ is compact. For any $t^*\in J$, we have

$$
\begin{aligned}
(Px_n)(t^*) &= \frac{1}{\Gamma(\nu)}\int_0^{t^*}(t^*-s)^{\nu-1}f\left(s,x_n(s),\int_0^s k(s,\tau)x_n(\tau)d\tau,\int_0^T h(s,\tau)x_n(\tau)d\tau\right)ds\\
&= \frac{1}{\Gamma(\nu)}\lim_{k\longrightarrow\infty}\sum_{i=0}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\nu-1}\\
&\quad\times f\left(\frac{it^*}{k},x_n\left(\frac{it^*}{k}\right),\int_0^{\frac{it^*}{k}}k\left(\frac{it^*}{k},\tau\right)x_n(\tau)d\tau,\int_0^T h\left(\frac{it^*}{k},\tau\right)x_n(\tau)d\tau\right)\\
&= \frac{t^*}{\Gamma(\nu)}\zeta_n,
\end{aligned}
$$

where

$\zeta_n=\lim_{k\longrightarrow\infty}\sum_{i=0}^{k}\frac{1}{k}\left(t^*-\frac{it^*}{k}\right)^{\nu-1}f\left(\frac{it^*}{k},x_n\left(\frac{it^*}{k}\right),\int_0^{\frac{it^*}{k}}k\left(\frac{it^*}{k},\tau\right)x_n(\tau)d\tau,\int_0^T h\left(\frac{it^*}{k},\tau\right)x_n(\tau)d\tau\right).$

Since $c\bar{o}nv$ is convex and compact, we know that $\zeta_n\in c\bar{o}nv$. Hence, for any $t^*\in J$, the set $\{Px_n\}$ $(n=1,2,\ldots)$ is relatively compact. From Ascoli-Arzela theorem every $\{Px_n(t)\}$ contains a uniformly convergent subsequence $\{Px_{nk}(t)\}$ $(k=1,2,3,\ldots)$ on J. Thus, the set $\{Px : x\in C_\rho\}$ is relatively compact. Therefore, the continuity of P and relatively compactness of the set $\{Px : x\in C_\rho\}$ imply that P is a completely continuous operator. By Krasnoselskii's fixed point theorem, we get that $P+Q$ has a fixed point on C_ρ. Hence the problem (1)-(2) has at least one solution. This completes the proof. ☐

4. **Application.** The solution of integro-differential equations has a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, it finally gives an integral equation or an integro-differential equation [20, 21, 22, 23]. In this section, we give the application of our main results established in previous section. We consider the nonlinear Caputo fractional Volterra-Fredholm integro-differential equations (1)-(2), with

$$
\begin{aligned}
f(t,x(t),Kx(t),Hx(t)) &= \frac{e^{-t}|x(t)|}{(5+e^t)(1+|x(t)|)}+\frac{1}{9}\int_0^t\frac{1}{(2+t)^2}x(s)ds\\
&\quad+\frac{1}{9}\int_0^1\frac{1}{(2+t)}x(s)ds.
\end{aligned}
$$

$$
g(x)=\sum_{j=0}^{m}\lambda_j x(t_j),\ x_0=0,\ \nu=\frac{1}{2},\ \lambda_j>0,\ 0<t_1<t_2<\cdots<t_m<1.
$$

For $x,y\in X$ and $t\in J=[0,1]$,

$$
\begin{aligned}
&\|f(t,x(t),Kx(t),Hx(t))-f(t,y(t),Ky(t),Hy(t))\|\\
&\leq \frac{e^{-t}}{(5+e^t)}\|x-y\|+\frac{1}{9}\|Kx-Ky\|+\frac{1}{9}\|Hx-Hy\|
\end{aligned}
$$

$$\leq \frac{1}{6}[\|x-y\| + \|Kx-Ky\| + \|Hx-Hy\|]. \tag{10}$$

Similarly, for all $x \in X$ and each $t \in J$,

$$\begin{aligned}\|f(t,x(t),Kx(t),Hx(t))\| &\leq \frac{1}{6}\left\|\frac{|x(t)|}{(1+|x(t)|)}\right\| + \frac{1}{9}\|Kx\| + \frac{1}{9}\|Hx\| \\ &\leq \frac{1}{6}[\|x\| + \|Kx\| + \|Hx\|]. \end{aligned} \tag{11}$$

Also

$$\|g(x)-g(y)\| \leq \sum_{j=0}^{m}\lambda_j\|x(t_j)-y(t_j)\| \leq \sum_{j=0}^{m}\lambda_j \max_{t_j\in J}\|x(t_j)-y(t_j)\| \tag{12}$$

and

$$\|g(x)\| \leq \left\|\sum_{j=0}^{m}\lambda_j x(t_j)\right\| \leq \|x(t_j)-y(t_j)\| \leq \sum_{j=0}^{m}\lambda_j\|x(t_j)\| \leq \sum_{j=0}^{m}\lambda_j \max_{t_j\in J}\|x(t_j)\|. \tag{13}$$

From (10)-(13), we observe that the assumptions of Theorem 3.2 and Theorem 3.3 can be satisfied by choosing a sufficiently small value of λ_j, and hence the given problem has a solution and this solution is unique.

5. **Conclusions.** In this paper, we establish some new conditions for the existence, uniqueness and boundedness of solutions to the nonlocal Caputo fractional Volterra-Fredholm integro-differential equations in Banach spaces. The desired results are proved by using Ascoli-Arzela theorem, aid of fixed point theorems due to Banach and Krasnoselskii in Banach space. Our results extend and unify many existing results in the literature. This paper contributes to the growth of the fractional calculus, especially in the case fractional differential equations involving a general formulation of Caputo fractional derivative with respect to another function.

The problem considered in this paper can be generalized to a higher dimension involving a general formulation of Hilfer fractional derivative with respect to another function. Also, to study nonlinear fractional systems of Volterra-Fredholm integro-differential equations with nonlocal conditions is a direction which we are working on.

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