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A COMMON FIXED POINT THEOREM OF FOUR MAPS IN CONE (STRONG) B-METRIC SPACES

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Abstract. *In this thesis, we establish a common fixed point theorem for four self-mappings in cone b-metric space which shows that four functions satisfying the compatibility property have a unique common fixed point under weaker contraction conditions than in [Roshan et al., 2014]. In the same setting, we obtain a similar result in cone strong b-metric space. In order to illustrate the validity and superiority of our result, we construct an example on a normal cone which is used widely in the vector k-norm functions and the matrix Ky Fan k-norm functions.*

Keywords: Common fixed point, Cone b-metric space, Cone strong b-metric space, Cone compatible

1. **Introduction.** The famous Banach Contraction Mapping Principle was proposed in 1922, which gave a sufficient condition for the existence and uniqueness of a self-mapping on metric space. Since some real problems were boiled down to fixed point theory, many fixed point results were investigated. Generally, there are two main aspects in the research of fixed point theory. One is to weaken the self-mapping's properties and the other is to expand the research area to some other spaces such as graph metric, b-metric and cone metric space.

The notion of common fixed point can trace back to [1, 2]. In recent years, a lot of common fixed point results are discovered by scholars. Alqahtani et al. proved some common fixed point theorems for two self-mappings in the context of a complete b-metric space [10], and Roshan et al. introduced four mappings and proved some common fixed point results for four mappings [6]. For more results, one can see [9, 10, 15, 16, 17, 18], etc. Some of them are in b-metric space (e.g., [6, 9, 10]) and some are in cone metric space (e.g., [16, 17, 18]). However, so far, there is no common fixed point result in cone b-metric spaces. The concept of b-metric space was first introduced in [3, 4]. As a quasi-metric space, b-metric space plays a key role in the research of Gromov hyperbolic metric spaces [5]. Many scholars studied the fixed point theory in b-metric space in recent years. Bakhtin [3] and Czerwik [4] introduced the notion of b-metric space. Then more and more scholars get more relevant results (e.g., [6, 7, 8, 9, 10]). The discontinuity of b-metric brings a lot of inconvenience to the research. Then, the strong b-metric was proposed by [11] and some scholars studied the fixed point theory in strong b-metric space (e.g., [12, 13]). Cone metric space was first introduced in [14], which generalized the metric space to partial ordered Banach space. Cone b-metric space was first introduced in [15], which can be seen as a combination of b-metric space and cone metric space. The latest research results show that fixed points can also be used in decision-making [20]. In view

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of the importance of common fixed point theory and cone b-metric space, this paper aims at establishing common fixed point theorems of four self-mappings satisfying generalized contraction principle in cone b-metric space and cone strong b-metric space respectively. In the last part of the thesis, we take a cone which is widely used in optimization problems as an example to show the validity of our results.

2. **Problem Statement and Preliminaries.**

Definition 2.1. [16] *Let $\mathbb{B}$ be a real Banach space and θ be the zero element in $\mathbb{B}$. A subset $\mathcal{C}$ of $\mathbb{B}$ is called a cone if and only if the following conditions are satisfied:*

(i) $\mathcal{C}$ is closed, nonempty and $\mathcal{C} \neq \{\theta\}$;

(ii) $a, b \in \Re$, $a, b \geq 0$, $x, y \in \mathcal{C} \Rightarrow ax + by \in \mathcal{C}$;

(iii) $\mathcal{C} \cap (-\mathcal{C}) = \{\theta\}$.

Moreover, for $x, y \in \mathbb{B}$, we say that $x \preceq y$ if $y - x \in \mathcal{C}$. $\mathcal{C}$ is called a normal cone if there is a real number $L \geq 0$ such that

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq L\|y\|.$$

Here we list some properties about normal cone used in this thesis, which can be obtained directly from Definition 2.1:

(i) $x \in \mathbb{B}$, $\theta \preceq x \Rightarrow x \in \mathcal{C}$;

(ii) $x, y \in \mathbb{B}$, $x \preceq y$, $a \in \mathbb{R}$, $a > 0 \Rightarrow ax \preceq ay$;

(iii) $x, y, z \in \mathbb{B}$, $x \preceq y \Rightarrow x + z \preceq y + z$;

(iv) $a, b \in \Re$, $0 < a \leq b$, $x \in \mathcal{C} \Rightarrow ax \preceq bx$;

(v) $x \prec y$ stands for $y - x$ is a point of the interior of $\mathcal{C}$, that is, $x \prec y$ implies $x \preceq y$ and $x \neq y$.

Definition 2.2. [15] *Let $\mathbb{X}$ be a nonempty set and $\mathbb{B}$ be a real Banach space with cone $\mathcal{C}$. A vector-valued function $D : \mathbb{X} \times \mathbb{X} \to \mathbb{B}$ is said to be a cone b-metric on $\mathbb{X}$ if the following conditions are satisfied: for all $x, y, z \in \mathbb{X}$ and $k \geq 1$,*

(i) $\theta \preceq D(x, y)$, and $D(x, y) = \theta$ if and only if $x = y$;

(ii) $D(x, y) = D(y, x)$;

(iii) $D(x, z) \preceq k[D(x, y) + D(y, z)]$ (The least number k satisfying this equality is called the b-metric coefficient).

In order to remedy the lack of continuity of b-metric, the notion of strong b-metric was proposed by [11]. We here extend it to cone metric space.

Definition 2.3. [15] *Let $\mathbb{X}$ be a nonempty set and $\mathbb{B}$ be a real Banach space with cone $\mathcal{C}$. A vector-valued function $D : \mathbb{X} \times \mathbb{X} \to \mathbb{B}$ is said to be a cone strong b-metric on $\mathbb{X}$ if the following conditions are satisfied: for all $x, y, z \in \mathbb{X}$ and $k \geq 1$,*

(i) $\theta \preceq D(x, y)$, and $D(x, y) = \theta$ if and only if $x = y$;

(ii) $D(x, y) = D(y, x)$;

(iii) $D(x, z) \preceq D(x, y) + kD(y, z)$.

Remark 2.1. *(i) Whether D is a cone b-metric or a cone strong b-metric, we know $D(x, y) \in \mathcal{C}$ for all $x, y \in \mathbb{X}$. That is, the two expressions $D : \mathbb{X} \times \mathbb{X} \to \mathbb{B}$ and $D : \mathbb{X} \times \mathbb{X} \to \mathcal{C}$ are actually the same. (ii) If $k = 1$, both concepts of cone b-metric and cone strong b-metric coincide with the standard cone metric. (iii) If $\mathbb{B} = \mathcal{C} = [0, \infty)$, the cone b-metric coincides with the standard b-metric and the cone strong b-metric coincides with the standard strong b-metric.*

Definition 2.4. *Let $(\mathbb{X}, D)$ be a cone (strong) b-metric space. We say that $\{x_n\}$ is* *(i) a cone Cauchy sequence if for* $\epsilon \in \mathbb{B}$ *with* $\epsilon \succeq \theta$,

$$\exists N > 0,\ \forall n, m > N \Rightarrow D(x_n, x_m) \preceq \epsilon;$$

(ii) a cone convergent sequence if for $\epsilon \in \mathbb{B}$ *with* $\epsilon \succeq \theta$ *and* $x \in \mathbb{X}$,

$$\exists N > 0,\ \forall n > N \Rightarrow D(x_n, x) \preceq \epsilon \ \left(\text{denoted by } \lim_{n\to\infty} x_n = x\right).$$

A cone (strong) b-metric space $\mathbb{X}$ *is said to be complete if every Cauchy sequence in* $\mathbb{X}$ *is convergent.*

Definition 2.5. f *and* g *are self-mappings on a cone (strong) b-metric space* $(\mathbb{X}, D)$. $\{x_n\}$ *is a sequence such that* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ *for some* $t \in \mathbb{X}$. *Then* $\{f, g\}$ *is called a cone compatible pair if* $\lim_{n\to\infty} D(fg(x_n), gf(x_n)) = \theta$.

Noting that strong b-metric is continuous [11], we have cone strong b-metric is also continuous (see Lemma 2.1).

Lemma 2.1. *Let* $(\mathbb{X}, D)$ *be a cone strong b-metric space. Suppose that* $\{x_n\}$ *is cone convergent to* x *and* $\{y_n\}$ *is cone convergent to* y, *it holds that* $\lim_{n\to\infty} D(x_n, y_n) = D(x, y)$.

Noting that b-metric does not need to be continuous, we need the following two results in this thesis about upper and lower limits (see Lemma 2.2 and Lemma 2.3) and we omit the proofs which are similar to [19].

Lemma 2.2. *Let* $(\mathbb{X}, D)$ *be a cone b-metric space. If there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *with* $\lim_{n\to\infty} x_n = t$ *for some* $t \in \mathbb{X}$ *such that* $\lim_{n\to\infty} D(x_n, y_n) = D(x, y)$, *then we have* $\lim_{n\to\infty} y_n = t$.

Lemma 2.3. *Let* $(\mathbb{X}, D)$ *be a cone b-metric space with b-metric coefficient* $k \geq 1$. *Suppose that the sequence* $\{x_n\} \subseteq \mathbb{X}$ *satisfies* $\lim_{n\to\infty} x_n = x$. *Then, for* $z \in \mathbb{X}$, *it holds that*

$$\frac{1}{k} D(x, z) \preceq \liminf_{n\to\infty} D(x_n, z) \preceq \limsup_{n\to\infty} D(x_n, z) \preceq k D(x, z).$$

And, for $\{y_n\}$ *which is cone convergent to* y, *it holds that*

$$\frac{1}{k^2} D(x, y) \preceq \liminf_{n\to\infty} D(x_n, y_n) \preceq \limsup_{n\to\infty} D(x_n, y_n) \preceq k^2 D(x, z).$$

3. **Main Results.**

Theorem 3.1. *Suppose that the nonempty cone b-metric space* $(\mathbb{X}, D)$ *is complete with b-metric coefficient* $k \geq 1$ *where* $D : \mathbb{X} \times \mathbb{X} \to \mathcal{C} \subseteq \mathbb{B}$ *and* $\mathcal{C}$ *is a normal cone of the Banach space* $\mathbb{B}$. *Let* f, g, S *and* T *be the self-mappings on* $(\mathbb{X}, D)$ *with* $f(\mathbb{X}) \subseteq T(\mathbb{X})$ *and* $g(\mathbb{X}) \subseteq S(\mathbb{X})$. *Suppose that* S *and* T *are continuous, and* $\{f, S\}$ *and* $\{g, T\}$ *are cone compatible pairs satisfying*

$$\begin{aligned} D(f(x), g(y)) \preceq\ & \lambda_1 D(S(x), T(y)) + \lambda_2 D(f(x), T(y)) + \lambda_3 D(S(x), g(y)) \\ & + \lambda_4 D(f(x), S(x)) + \lambda_5 D(g(y), T(y)) \end{aligned} \tag{1}$$

for all $x, y \in \mathbb{X}$, *where* $\lambda_i \geq 0, 1 = 1, 2, 3, 4, 5$ *and*

$$k(\lambda_1 + 2k \max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5) < 1,\ k^4(\lambda_1 + \lambda_2 + \lambda_3) < 1,\ k^3(\lambda_2 + \lambda_4) < 1. \tag{2}$$

Then the mappings f, g, S *and* T *have a unique common fixed point.*

Proof: First we see that

$$\lambda_1 D(S(x),T(y)) + \lambda_2 D(f(x),T(y)) + \lambda_3 D(S(x),g(y)) + \lambda_4 D(f(x),S(x)) + \lambda_5 D(g(y),T(y)) \in \mathcal{C}.$$

Then we take four steps to complete the proof.

Step 1: construct a sequence $\{x_n\}$

Taking a point $a_0 \in \mathbb{X}$. Since $f(\mathbb{X}) \subseteq T(\mathbb{X})$, there is a point $a_1 \in \mathbb{X}$ such that $T(a_1) = f(a_0)$. Since $g(\mathbb{X}) \subseteq S(\mathbb{X})$, there is a point $a_2 \in \mathbb{X}$ such that $g(a_1) = S(a_2)$. Repeating this method, we can construct a sequence $\{a_n\}$ as follows. For $a_0 \in \mathbb{X}$ and for $\forall n$:

there exist $a_1 \in \mathbb{X}$ such that $T(a_1) = f(a_0)$ and $a_2 \in \mathbb{X}$ such that $g(a_1) = S(a_2)$;
there exist $a_3 \in \mathbb{X}$ such that $T(a_3) = f(a_2)$ and $a_4 \in \mathbb{X}$ such that $g(a_3) = S(a_4)$;
...
there exist $a_{2n+1} \in \mathbb{X}$ such that $T(a_{2n+1}) = f(a_{2n})$ and $a_{2n+2} \in \mathbb{X}$ such that $g(a_{2n+1}) = S(a_{2n+2})$.

Define a sequence $\{x_n\}$ with

$$x_{2n} = T(a_{2n+1}) = f(a_{2n}),\ x_{2n+1} = g(a_{2n+1}) = S(a_{2n+2}).$$

Step 2: prove $\{x_n\}$ is a cone Cauchy sequence

Step 2.1: prove $D(x_{2n}, x_{2n+1}) \preceq (\lambda_1 + 2k\lambda_3 + \lambda_4 + \lambda_5) D(x_{2n-1}, x_{2n})$

$$\begin{aligned} D(x_{2n}, x_{2n+1}) &= D(f(a_{2n}), g(a_{2n+1})) \\ &\preceq \lambda_1 D(S(a_{2n}), T(a_{2n+1})) + \lambda_2 D(f(a_{2n}), T(a_{2n+1})) + \lambda_3 D(S(a_{2n}), g(a_{2n+1})) \\ &\quad + \lambda_4 D(f(a_{2n}), S(a_{2n})) + \lambda_5 D(g(a_{2n+1}), T(a_{2n+1})) \\ &= \lambda_1 D(x_{2n-1}, x_{2n}) + \lambda_2 D(x_{2n}, x_{2n}) + \lambda_3 D(x_{2n-1}, x_{2n+1}) + \lambda_4 D(x_{2n}, x_{2n-1}) \\ &\quad + \lambda_5 D(x_{2n+1}, x_{2n}). \end{aligned}$$

From Definition 2.2 (iii), we get

$$\begin{aligned} D(x_{2n}, x_{2n+1}) \preceq\ & \lambda_1 D(x_{2n-1}, x_{2n}) + k\lambda_3 D(x_{2n-1}, x_{2n}) + k\lambda_3 D(x_{2n}, x_{2n+1}) \\ & + \lambda_4 D(x_{2n}, x_{2n-1}) + \lambda_5 D(x_{2n+1}, x_{2n}). \end{aligned} \tag{3}$$

Now, we prove that $D(x_{2n}, x_{2n+1}) \preceq D(x_{2n-1}, x_{2n})$, for each $n \in \mathbb{N}$. If $D(x_{2n-1}, x_{2n}) \prec D(x_{2n}, x_{2n+1})$ for some $n \in \mathbb{N}$, from (3) we have

$$\begin{aligned} & D(x_{2n}, x_{2n+1}) \\ \preceq\ & \lambda_1 D(x_{2n+1}, x_{2n}) + 2k\lambda_3 D(x_{2n+1}, x_{2n}) + \lambda_4 D(x_{2n}, x_{2n+1}) + \lambda_5 D(x_{2n+1}, x_{2n}) \\ =\ & (\lambda_1 + 2k\lambda_3 + \lambda_4 + \lambda_5) D(x_{2n}, x_{2n+1}) \\ \preceq\ & (\lambda_1 + 2k\max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5) D(x_{2n}, x_{2n+1}) \\ \preceq\ & D(x_{2n}, x_{2n+1}) \end{aligned}$$

which is a contradiction. So, we have $D(x_{2n}, x_{2n+1}) \preceq D(x_{2n-1}, x_{2n})$ for each $n \in \mathbb{N}$. Moreover, by (3) we have

$$D(x_{2n}, x_{2n+1}) \preceq (\lambda_1 + 2k\lambda_3 + \lambda_4 + \lambda_5) D(x_{2n-1}, x_{2n}). \tag{4}$$

Step 2.2: prove $D(x_{2n}, x_{2n-1}) \preceq (\lambda_1 + 2k\lambda_2 + \lambda_4 + \lambda_5) D(x_{2n-1}, x_{2n-2})$

Using the similar technique with Step 2.1, we have

$$\begin{aligned} & D(x_{2n}, x_{2n-1}) \\ =\ & D(f(a_{2n}), g(a_{2n-1})) \\ \preceq\ & \lambda_1 D(S(a_{2n}), T(a_{2n-1})) + \lambda_2 D(f(a_{2n}), T(a_{2n-1})) + \lambda_3 D(S(a_{2n}), g(a_{2n-1})) \end{aligned}$$

$$
\begin{aligned}
&+ \lambda_4 D(f(a_{2n}), S(a_{2n})) + \lambda_5 D(g(a_{2n-1}), T(a_{2n-1})) \\
&= \lambda_1 D(x_{2n-1}, x_{2n-2}) + \lambda_2 D(x_{2n}, x_{2n-2}) + \lambda_3 D(x_{2n-1}, x_{2n-1}) + \lambda_4 D(x_{2n}, x_{2n-1}) \\
&+ \lambda_5 D(x_{2n-1}, x_{2n-2}) \\
&= \lambda_1 D(x_{2n-1}, x_{2n-2}) + \lambda_2 D(x_{2n}, x_{2n-2}) + \lambda_4 D(x_{2n}, x_{2n-1}) + \lambda_5 D(x_{2n-1}, x_{2n-2}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
D(x_{2n}, x_{2n-1}) \preceq\ & \lambda_1 D(x_{2n-1}, x_{2n-2}) + k\lambda_2 D(x_{2n-1}, x_{2n}) + k\lambda_2 D(x_{2n-1}, x_{2n-2}) \\
& + \lambda_4 D(x_{2n-1}, x_{2n}) + \lambda_5 D(x_{2n-1}, x_{2n-2}).
\end{aligned}
$$

By the similar deduction, we have

$$D(x_{2n}, x_{2n-1}) \preceq D(x_{2n-1}, x_{2n-2}).$$

Moreover, the following inequality holds

$$D(x_{2n}, x_{2n-1}) \preceq (\lambda_1 + 2k\lambda_2 + \lambda_4 + \lambda_5) D(x_{2n-1}, x_{2n-2}). \tag{5}$$

Step 2.3: prove $\{x_n\}$ converges a point x

From (4) and (5) we have

$$D(x_n, x_{2n-1}) \preceq \lambda D(x_{n-1}, x_{n-2}), \quad n \geq 2$$

where $\lambda = \lambda_1 + 2k \max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5$. Then we have $\lambda < 1$ and

$$D(x_n, x_{n-1}) \preceq \cdots \preceq \lambda^{n-1} D(x_1, x_0). \tag{6}$$

For $n > m$, by Definition 2.2 (iii) we have

$$D(x_m, x_n) \preceq kD(x_m, x_{m+1}) + k^2 D(x_{m+1}, y_{m+2}) + \cdots + k^{n-m} D(x_{n-1}, x_n).$$

Hence, from (6) and $k\lambda < 1$, we have

$$
\begin{aligned}
D(x_n, x_m) &\preceq \left(k\lambda^m + k^2\lambda^{m+1} + \cdots + k^{n-m}\lambda^{n-1}\right) D(x_1, x_0) \\
&\preceq k\lambda^m \left[1 + k\lambda + (k\lambda)^2 + \cdots\right] D(x_1, x_0) \\
&= \frac{k\lambda^m}{1 - k\lambda} D(x_1, x_0).
\end{aligned}
$$

By Definition 2.1, we get

$$\|D(x_n, x_m)\| \leq L \frac{k\lambda^m}{1 - k\lambda} \|D(x_1, x_0)\|.$$

Thus, $\{x_n\}$ is a cone Cauchy sequence, denoted by $\lim_{n\to\infty} x_n = x$.

Step 3: prove x is a common fixed point of f, g, S and T

We know

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x. \tag{7}$$

Step 3.1: prove x is a fixed point of S

By (7),

$$\lim_{n\to\infty} T(x_{2n+1}) = \lim_{n\to\infty} f(x_{2n}) = \lim_{n\to\infty} S(x_{2n+2}) = \lim_{n\to\infty} g(x_{2n+1}) = x. \tag{8}$$

Due to the cone compatibility of $\{f, S\}$, we have

$$\lim_{n\to\infty} D(fS(x_{2n}), Sf(x_{2n})) = \theta. \tag{9}$$

Noticing that S is continuous, we have

$$\lim_{n\to\infty} Sf(x_{2n}) = S\left(\lim_{n\to\infty} f(x_{2n})\right) = S(x).$$

From Lemma 2.2, we have $\lim_{n\to\infty} fS(x_{2n}) = S(x)$. By (1), we have

$$\begin{aligned} & D(fS(a_{2n}), g(a_{2n+1})) \\ & \preceq \lambda_1 D\left(S^2(a_{2n}), T(a_{2n+1})\right) + \lambda_2 D(fS(a_{2n}), T(a_{2n+1})) + \lambda_3 D\left(S^2(a_{2n}), g(a_{2n+1})\right) \\ & \quad + \lambda_4 D\left(fS(a_{2n}), S^2(a_{2n})\right) + \lambda_5 D(g(a_{2n+1}), T(a_{2n+1})). \end{aligned} \tag{10}$$

Then,

$$\begin{aligned} & \liminf_{n\to\infty} D(fS(a_{2n}), g(a_{2n+1})) \\ & \preceq \limsup_{n\to\infty} \left[\lambda_1 D\left(S^2(a_{2n}), T(a_{2n+1})\right) + \lambda_2 D(fS(a_{2n}), T(a_{2n+1})) + \lambda_3 D\left(S^2(a_{2n}), g(a_{2n+1})\right) \right. \\ & \quad \left. + \lambda_4 D\left(fS(a_{2n}), S^2(a_{2n})\right) + \lambda_5 D(g(a_{2n+1}), T(a_{2n+1}))\right]. \end{aligned}$$

Using Lemma 2.3 we get

$$\begin{aligned} & \frac{D(S(x), x)}{k^2} \\ & \preceq k^2[\lambda_1 D(S(x), x) + \lambda_2 D(S(x), x) + \lambda_3 D(S(x), x) + \lambda_4 D(S(x), S(x)) + \lambda_5 D(x, x)] \\ & = k^2[\lambda_1 D(S(x), x) + \lambda_2 D(S(x), x) + \lambda_3 D(S(x), x)] \\ & = k^2(\lambda_1 + \lambda_2 + \lambda_3) D(S(x), x) \\ & = \frac{(a_1 + a_2 + a_3)k^4}{k^2} D(S(x), x). \end{aligned}$$

Then, we have

$$D(S(x), x) \preceq k^4(\lambda_1 + \lambda_2 + \lambda_3) D(S(x), x) \preceq D(S(x), x) \Rightarrow D(S(x), x) = \theta$$

which implies $S(x) = x$.

Step 3.2: prove x is a fixed point of T

Since $\{g, T\}$ is cone compatible, by (8) we have

$$\lim_{n\to\infty} D(gT(a_{2n+1}), Tg(a_{2n+1})) = \theta.$$

According to continuity of T, we get

$$\lim_{n\to\infty} T^2(a_{2n+1}) = T(x) = \lim_{n\to\infty} Tg(a_{2n+1}).$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} gT(a_{2n}) = T(x).$$

By (1), we can get

$$\begin{aligned} & D(f(a_{2n}), gT(a_{2n+1})) \\ & \preceq \lambda_1 D\left(S(a_{2n}), T^2(a_{2n+1})\right) + \lambda_2 D\left(f(a_{2n}), T^2(a_{2n+1})\right) + \lambda_3 D(S(a_{2n}), gT(a_{2n+1})) \\ & \quad + \lambda_4 D(f(a_{2n}), S(a_{2n})) + \lambda_5 D\left(gT(a_{2n+1}), T^2(a_{2n+1})\right). \end{aligned}$$

Using Lemma 2.3, by the similar deduction of Step 3.1, we get

$$\begin{aligned} & \frac{D(x, T(x))}{k^2} \\ & \preceq k^2[\lambda_1 D(x, T(x)) + \lambda_2 D(x, T(x)) + \lambda_3 D(x, T(x)) + \lambda_4 D(x, x) + \lambda_5 D(T(x), T(x))] \\ & = k^2[\lambda_1 D(x, T(x)) + \lambda_2 D(x, T(x)) + \lambda_3 D(x, T(x))] \\ & = k^2(\lambda_1 + \lambda_2 + \lambda_3) D(x, T(x)) \\ & = \frac{(\lambda_1 + \lambda_2 + \lambda_3)k^4}{k^2}. \end{aligned}$$

Then, $D(x, T(x)) = \theta$, i.e., $T(x) = x$.

Step 3.3: prove x is a common fixed point of f and g

Again from (1), we have

$$D(f(x), g(a_{2n+1})) \preceq \lambda_1 D(S(x), T(a_{2n+1})) + \lambda_2 D(f(x), T(a_{2n+1})) + \lambda_3 D(S(x), g(a_{2n+1})) + \lambda_4 D(f(x), S(x)) + \lambda_5 D(g(a_{2n+1}), T(a_{2n+1})).$$

By $S(x) = x$ and Lemma 2.3, we have

$$\begin{aligned} &\frac{D(f(x), x)}{k} \\ &\preceq k\lambda_1 D(S(x), x) + k\lambda_2 D(f(x), x) + k\lambda_3 D(S(x), x) + k\lambda_4 D(f(x), S(x)) + k^2\lambda_5 D(x, x) \\ &= k(\lambda_2 + \lambda_4) D(f(x), x) \\ &\preceq \frac{k^3(\lambda_2 + \lambda_4)}{k^2} D(f(x), x) \end{aligned}$$

which implies $f(x) = x$. By (1) and $T(x) = f(x) = S(x) = x$, we have

$$D(x, g(x)) = D(f(x), g(x)) \preceq (\lambda_2 + \lambda_5) D(x, g(x)).$$

Hence $x = g(x)$. Thus, $S(x) = T(x) = g(x) = f(x) = x$.

Step 4: prove x is the unique common fixed point

If there exists another common fixed point y in $\mathbb{X}$ for f, g, S and T, then

$$\begin{aligned} D(x, y) &= D(f(x), g(y)) \\ &\preceq \lambda_1 D(S(x), T(y)) + \lambda_2 D(f(x), T(y)) + \lambda_3 D(S(x), g(y)) \\ &\quad + \lambda_4 D(f(x), S(x)) + \lambda_5 D(g(y), T(y)) \\ &= (\lambda_1 + \lambda_2 + \lambda_3) D(x, y). \end{aligned}$$

As $\lambda_1 + \lambda_2 + \lambda_3 < 1$, then $x = y$. Therefore, x is the unique common fixed point of f, g, S and T.

Remark 3.1. *Theorem 3.1 shows that the four self-mappings have unique common fixed points as long as the coefficients in the contraction condition satisfy the condition of (2). In the process of proof, step 1-step 3 show the existence of common fixed point, step 1-step 2 provide the algorithm of calculating common fixed point, and step 4 proves the uniqueness of common fixed point.*

Corollary 3.1. [6] *Suppose that the nonempty b-metric space $(\mathbb{X}, d)$ is complete with b-metric coefficient $k \geq 1$. Let f, g, S and T be the self-mappings on $(\mathbb{X}, d)$ with $f(\mathbb{X}) \subseteq T(\mathbb{X})$ and $g(\mathbb{X}) \subseteq S(\mathbb{X})$. Suppose that S and T are continuous, and $\{f, S\}$ and $\{g, T\}$ are compatible pairs satisfying*

$$d(f(x), g(y)) \leq \frac{1}{k^4}(a_1 d(S(x), T(y)) + a_2 d(f(x), T(y)) + a_3 d(S(x), g(y)) + a_4 d(f(x), S(x)) + a_5 d(g(y), T(y))) \tag{11}$$

for all $x, y \in \mathbb{X}$, where $a_i \geq 0, 1 = 1, 2, 3, 4, 5$ and

$$(a_1 + \alpha a_2 + \beta a_3 + a_4 + a_5) < 1 \text{ where } \alpha + \beta = 2 \text{ for } \alpha, \beta \in \{0, 1, 2\}. \tag{12}$$

Then the mappings f, g, S and T have a unique common fixed point.

Proof: It is obvious that $\mathfrak{R}$ is a Banach space and $\mathfrak{R}^+$ is a normal cone of $\mathfrak{R}$. Let $\mathcal{C} = \mathfrak{R}^+$, $\mathbb{B} = \mathfrak{R}$. Then the b-metric d can be rewritten as $d : \mathbb{X} \times \mathbb{X} \to \mathcal{C} \subseteq \mathbb{B}$ and $(\mathbb{X}, d)$ is a cone b-metric space.

Examining (1) and (11), we set $\lambda_i = \frac{a_i}{k^4}$, i.e., $a_i = k^4\lambda_i$. By (12), we have

$$k^4(\lambda_1 + \alpha\lambda_2 + \beta\lambda_3 + \lambda_4 + \lambda_5) < 1.$$

Thus, the condition (2) holds, that is,

$$k(\lambda_1 + 2k\max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5) < 1,\ k^4(\lambda_1 + \lambda_2 + \lambda_3) < 1,\ k^3(\lambda_2 + \lambda_4) < 1.$$

By Theorem 3.1, we complete the proof.

Remark 3.2. *Corollary 3.1 shows that Theorem 3.1 generalized the result obtained in* [6].

Theorem 3.2. *Suppose that the nonempty cone strong b-metric space $(\mathbb{X}, D)$ is complete where $D : \mathbb{X} \times \mathbb{X} \to \mathcal{C} \subseteq \mathbb{B}$ and $\mathcal{C}$ is a normal cone of the Banach space $\mathbb{B}$. Let f, g, S and T be the self-mappings on $(\mathbb{X}, D)$ with $f(\mathbb{X}) \subseteq T(\mathbb{X})$ and $g(\mathbb{X}) \subseteq S(\mathbb{X})$. Suppose that S and T are continuous, and $\{f, S\}$ and $\{g, T\}$ are cone compatible pairs satisfying (1) for all $x, y \in \mathbb{X}$, where $\lambda_i \geq 0, 1 = 1, 2, 3, 4, 5$ and*

$$k[\lambda_1 + (k+1)\max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5] < 1. \tag{13}$$

Then the mappings f, g, S and T have a unique common fixed point.

Proof: We construct a sequence $\{x_n\}$ with the same method in Theorem 3.1. We can get

$$D(x_{2n}, x_{2n+1}) \preceq \lambda_1 D(x_{2n-1}, x_{2n}) + \lambda_3 D(x_{2n-1}, x_{2n}) + k\lambda_3 D(x_{2n}, x_{2n+1}) + \lambda_4 D(x_{2n}, x_{2n-1}) + \lambda_5 D(x_{2n+1}, x_{2n})$$

and

$$D(x_{2n}, x_{2n-1}) \preceq \lambda_1 D(x_{2n-1}, x_{2n-2}) + \lambda_2 D(x_{2n-1}, x_{2n}) + k\lambda_2 D(x_{2n-1}, x_{2n-2}) + \lambda_4 D(x_{2n}, x_{2n-1}) + \lambda_5 D(x_{2n-1}, x_{2n-2}).$$

Meanwhile, we get

$$D(x_{2n}, x_{2n+1}) \preceq [\lambda_1 + (k+1)\lambda_3 + \lambda_4 + \lambda_5] D(x_{2n-1}, x_{2n})$$

and

$$D(x_{2n}, x_{2n-1}) \preceq [\lambda_1 + (k+1)\lambda_2 + \lambda_4 + \lambda_5] D(x_{2n-1}, x_{2n-2}).$$

Hence,

$$D(x_n, x_{n-1}) \preceq \lambda D(x_{n-1}, x_{n-2}),\ n \geq 2$$

where $\lambda = \lambda_1 + (k+1)\max\{\lambda_2, \lambda_3\} + \lambda_4 + \lambda_5$, we get $\lambda < 1$ and

$$D(x_n, x_{n-1}) \preceq \cdots \preceq \lambda^{n-1} D(x_1, x_0).$$

For $n > m$, by Definition 2.3 (iii) we have

$$D(x_m, x_n) \preceq D(x_m, x_{m+1}) + kD(x_{m+1}, x_{m+2}) + \cdots + k^{n-m-1} D(x_{n-1}, x_n).$$

Hence, from $k\lambda < 1$, we have

$$D(x_n, x_m) \preceq \left(\lambda^m + k\lambda^{m+1} + \cdots + k^{n-m-1}\lambda^{n-1}\right) D(x_1, x_0) \preceq \frac{\lambda^m}{1 - k\lambda} D(x_1, x_0).$$

By Definition 2.1, we get

$$\|D(x_n, x_m)\| \leq L\frac{\lambda^m}{1 - k\lambda}\|D(x_1, x_0)\|.$$

Thus, $\{x_n\}$ is a cone Cauchy sequence, denoted by $\lim_{n\to\infty} x_n = x$. Notice that (7)-(10) hold. By Lemma 2.1, taking the limit of (10), we get

$$\begin{aligned} &D(S(x), x) \\ &\preceq \lambda_1 D(S(x), x) + \lambda_2 D(S(x), x) + \lambda_3 D(S(x), x) + \lambda_4 D(S(x), S(x)) + \lambda_5 D(x, x) \\ &= (\lambda_1 + \lambda_2 + \lambda_3) D(S(x), x) \end{aligned}$$

which implies $S(x) = x$.

Similarly, we get

$$\begin{aligned}&D(T(x),x)\\ \preceq\ &\lambda_1 D(T(x),x)+\lambda_2 D(T(x),x)+\lambda_3 D(T(x),x)+\lambda_4 D(T(x),T(x))+\lambda_5 D(x,x)\\ =\ &(\lambda_1+\lambda_2+\lambda_3)D(T(x),x)\end{aligned}$$

and

$$D(f(x),x)\preceq(\lambda_2+\lambda_4)D(f(x),x)$$
$$D(g(x),x)\preceq(\lambda_2+\lambda_5)D(g(x),x).$$

Thus, we have $S(x)=T(x)=g(x)=f(x)=x$.

The proof of uniqueness is the same as that of Step 4 in Theorem 3.1.

Remark 3.3. *From the proof of the theorem, we can see that the algorithm of common fixed point is consistent with that of cone b-metric space. However, due to the continuity of strong b-metric, the contraction condition (13) is much weaker than (2).*

4. **Applications.** We research the following set

$$\mathcal{C}=\left\{(x,t)\in\Re^p\times\Re: e^Tx\le lt, \mathbf{0}\le x\le te\right\}$$

where "$(\mathbf{0},0)$" is the zero vector, "e" is the vector of which all elements are 1's and $1\le l\le p$. The set $\mathcal{C}$ appears widely in optimization problems related to the vector k-norm functions and the matrix Ky Fan k-norm functions.

Theorem 4.1. *The set $\mathcal{C}$ is a normal cone of $\Re^p\times\Re$.*

Proof: We first show that $\mathcal{C}$ is a cone. Obviously, $\mathcal{C}$ is closed, nonempty and $\mathcal{C}\neq(\mathbf{0},0)$. For

$$\forall\lambda\ge 0,\ \forall(x,t),(y,\tau)\in\mathcal{C},$$

we obtain $\lambda(x,t)=(\lambda x,\lambda t)\in\mathcal{C}$ and $(x+y,t+\tau)\in\mathcal{C}$. Hence, for $\forall a,b\in\Re$, $a,b\ge 0$, it holds that $a(x,t)+b(y,\tau)\in\mathcal{C}$. If there is a point $(x,t)\in\mathcal{C}\cap(-\mathcal{C})$, we have $\mathbf{0}\le x\le te$ and $\mathbf{0}\le -x\le -te$. Hence, $(x,t)=(\mathbf{0},0)$, i.e.,

$$\mathcal{C}\cap(-\mathcal{C})=\{(\mathbf{0},0)\}.$$

Now we show $\mathcal{C}$ is normal. According to the definition of the normal cone, $(\mathbf{0},0)\preceq(x,t)\preceq(y,\tau)$ implies $(x,t),(y,\tau),(y-x,\tau-t)\in\mathcal{C}$. From the definition of $\mathcal{C}$, we obtain $\mathbf{0}\le y-x\le(\tau-t)e$. Therefore, $0\le t\le\tau$, $0\le x_i\le y_i$ and

$$x_1^2+\cdots+x_p^2+t^2\le y_1^2+\cdots+y_p^2+\tau^2,$$

which implies $\|x\|\le L\|y\|$ with $L=1$.

Example 4.1. *Define $D:[0,1]\times[0,1]\to\mathcal{C}$ as follows: for $x,y\in[0,1]$,*

$$D(x,y)=\begin{pmatrix}\frac{1}{p}|x-y|^2\\ \frac{1}{p}|x-y|^2\\ \cdots\\ \frac{1}{p}|x-y|^2\\ |x-y|^2\end{pmatrix}$$

Now, we will prove D is a cone b-metric on $[0,1]$ with the b-metric coefficient $k=2$. For $x,y\in[0,1]$, we have

$$\begin{cases} 0 \le \frac{1}{p}|x-y|^2 \le |x-y|^2; \\ p\frac{1}{p}|x-y|^2 \le l|x-y|^2 \end{cases}$$

which implies

$$D(x,y)\in\mathcal{C},\ i.e.,\ D(x,y)\succeq(\mathbf{0},0).$$

It is clear that $D(x,y)=D(y,x)$ and $D(x,y)=(\mathbf{0},0)\Leftrightarrow x=y$. For $z\in[0,1]$, we show

$$D(x,y)\preceq 2[D(x,z)+D(y,z)]$$

Noting that $|x-y|^2\le 2\left(|x-z|^2+|y-z|^2\right)$, we obtain

$$2\left(\frac{1}{p}|x-z|^2+\frac{1}{p}|y-z|^2\right)-\frac{1}{p}|x-y|^2=\frac{1}{p}\left(2|x-z|^2+2|y-z|^2-|x-y|^2\right)\ge 0.$$

Meanwhile,

$$\begin{aligned} p\left(\frac{2}{p}|x-z|^2+\frac{2}{p}|y-z|^2-\frac{1}{p}|x-y|^2\right) &= 2|x-z|^2+2|y-z|^2-|x-y|^2 \\ &\le l\left(2|x-z|^2+2|y-z|^2-|x-y|^2\right). \end{aligned}$$

Hence, we have $2(D(x,z)+D(y,z))-D(x,y)\in\mathcal{C}$, i.e., $D(x,y)\preceq 2[D(x,z)+D(y,z)]$.

Define the mappings f, g, S, T, for all $x\in[0,1]$ by $f(x)=\left(\frac{x}{3}\right)^{16}$, $g(x)=\left(\frac{x}{3}\right)^{8}$, $S(x)=\left(\frac{x}{3}\right)^{8}$, $T(x)=\left(\frac{x}{3}\right)^{4}$. It is obvious that $f\subseteq T$ and $g\subseteq S$.

Suppose, $\{x_n\}$ is a sequence in $[0,1]$ satisfying

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} S(x_n)=t,\ \textit{for some}\ t\in[0,1].$$

Then, we have $\lim_{n\to\infty} x_n=0$ and $t=0$. Then

$$\lim_{n\to\infty} D(f(S(x_n)),S(f(x_n)))=\lim_{n\to\infty} D\left(\left[\frac{\left(\frac{x_n}{3}\right)^8}{3}\right]^{16},\left[\frac{\left(\frac{x_n}{3}\right)^{16}}{3}\right]^{8}\right)=(\mathbf{0},0).$$

Thus, $\{f,S\}$ is a cone compatible pair. In the same way, we know $\{g,T\}$ is also a cone compatible pair.

For $x,y\in[0,1]$, we have

$$D(f(x),g(y))=\begin{pmatrix} \frac{1}{p}\left[\left(\frac{x}{3}\right)^8+\left(\frac{y}{3}\right)^4\right]^2\cdot\left[\left(\frac{x}{3}\right)^8-\left(\frac{y}{3}\right)^4\right]^2 \\ \frac{1}{p}\left[\left(\frac{x}{3}\right)^8+\left(\frac{y}{3}\right)^4\right]^2\cdot\left[\left(\frac{x}{3}\right)^8-\left(\frac{y}{3}\right)^4\right]^2 \\ \cdots \\ \frac{1}{p}\left[\left(\frac{x}{3}\right)^8+\left(\frac{y}{3}\right)^4\right]^2\cdot\left[\left(\frac{x}{3}\right)^8-\left(\frac{y}{3}\right)^4\right]^2 \\ \left[\left(\frac{x}{3}\right)^8+\left(\frac{y}{3}\right)^4\right]^2\cdot\left[\left(\frac{x}{3}\right)^8-\left(\frac{y}{3}\right)^4\right]^2 \end{pmatrix}$$

and

$$D(S(x),T(y)) = \begin{pmatrix} \frac{1}{p}\left[\left(\frac{x}{3}\right)^8 - \left(\frac{y}{3}\right)^4\right]^2 \\ \frac{1}{p}\left[\left(\frac{x}{3}\right)^8 - \left(\frac{y}{3}\right)^4\right]^2 \\ \cdots \\ \frac{1}{p}\left[\left(\frac{x}{3}\right)^8 - \left(\frac{y}{3}\right)^4\right]^2 \\ \left[\left(\frac{x}{3}\right)^8 - \left(\frac{y}{3}\right)^4\right]^2 \end{pmatrix}.$$

Taking $\lambda_1 \in \left(\left(\frac{1}{81^2} + \frac{1}{81}\right)^2, \frac{1}{16}\right)$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, *we have*

$$\lambda_1 D(S(x),T(y)) - D(f(x),g(y))$$

$$= \begin{pmatrix} \frac{1}{p}\left[(x/3)^8 - (y/3)^4\right]^2 \cdot \left[\lambda_1 - ((x/3)^8 + (y/3)^4)^2\right] \\ \frac{1}{p}\left[(x/3)^8 - (y/3)^4\right]^2 \cdot \left[\lambda_1 - ((x/3)^8 + (y/3)^4)^2\right] \\ \cdots \\ \frac{1}{p}\left[(x/3)^8 - (y/3)^4\right]^2 \cdot \left[\lambda_1 - ((x/3)^8 + (y/3)^4)^2\right] \\ \frac{1}{p}\left[(x/3)^8 - (y/3)^4\right]^2 \cdot \left[\lambda_1 - ((x/3)^8 + (y/3)^4)^2\right] \end{pmatrix} \in \mathcal{C}$$

which implies $D(f(x),g(y)) \preceq \lambda_1 D(S(x),T(y))$ *and* $2^4\lambda_1 = 16\lambda_1 < 1$. *Thus, the condition of Theorem 3.1 is satisfied. We know that* f, g, S *and* T *have a unique common fixed point* $x = 0$.

Example 4.2. *Let us consider the following example. Let*

$$\mathcal{C} = \{(x,y) \in \mathfrak{R} \times \mathfrak{R} : x \geq 0,\ y \geq 0\}.$$

It is obvious that $\theta = (0,0)^T$ *is the zero element in* $\mathfrak{R} \times \mathfrak{R}$. *For* $\xi = (\xi_1, \xi_2)^T$, $\eta = (\eta_1, \eta_2)^T \in \mathfrak{R} \times \mathfrak{R}$, *if* $\theta \preceq \xi \preceq \eta$, *then* $\xi, \eta, \eta - \xi \in \mathcal{C}$ *and then* $\xi_i \geq 0$, $\eta_i \geq 0$, $\eta_i - \xi_i \geq 0$, $i = 1, 2$. *Thus, we have*

$$\|\xi\| = \sqrt{\xi_1^2 + \xi_2^2} \leq \sqrt{\eta_1^2 + \eta_2^2} \leq \|\eta\|.$$

Thus, $\mathcal{C}$ *is a normal cone of* $\mathfrak{R} \times \mathfrak{R}$.

Define $D : [0,1] \times [0,1] \to \mathcal{C}$ *as follows: for* $x, y \in [0,1]$,

$$D(x,y) = \begin{pmatrix} |x-y| \\ |x-y|^2 \end{pmatrix}$$

By the deduction similar to Example 4.1, we know that D *is a cone strong b-metric on* $[0,1]$ *with the strong b-metric coefficient* $k = 2$.

Define the f, g, S, *and* T *on* $[0,1]$ *by* $f(x) = \left(\frac{x}{2}\right)^8$, $g(x) = \left(\frac{x}{2}\right)^4$, $S(x) = \left(\frac{x}{2}\right)^4$, $T(x) = \left(\frac{x}{2}\right)^2$. *It is obvious that* $f \subseteq T$, $g \subseteq S$, *and* $\{f, S\}$ *and* $\{g, T\}$ *are cone compatible* *pairs. For $x, y \in [0,1]$, we have*

$$D(f(x),g(y)) = \begin{pmatrix} \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right| \cdot \left[\left(\frac{x}{2}\right)^4 + \left(\frac{y}{2}\right)^2\right] \\ \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right|^2 \cdot \left[\left(\frac{x}{2}\right)^4 + \left(\frac{y}{2}\right)^2\right]^2 \end{pmatrix}$$

and

$$D(S(x),T(y)) = \begin{pmatrix} \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right| \\ \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right|^2 \end{pmatrix}.$$

Taking $\lambda_1 \in \left(\frac{1}{2^4} + \frac{1}{2^2}, \frac{1}{2}\right)$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, we have

$$\lambda_1 D(S(x),T(y)) - D(f(x),g(y)) = \begin{pmatrix} \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right| \cdot \left[\lambda_1 - \left(\left(\frac{x}{2}\right)^4 + \left(\frac{y}{2}\right)^2\right)\right] \\ \left|\left(\frac{x}{2}\right)^4 - \left(\frac{y}{2}\right)^2\right|^2 \cdot \left[\lambda_1 - \left(\left(\frac{x}{2}\right)^4 + \left(\frac{y}{2}\right)^2\right)^2\right] \end{pmatrix} \in \mathcal{C}$$

which implies $D(f(x),g(y)) \preceq \lambda_1 D(S(x),T(y))$ and $2\lambda_1 < 1$. Thus, the condition of Theorem 3.2 is satisfied. We know that f, g, S and T have a unique common fixed point $x = 0$.

Remark 4.1. *$D(x,y)$ defined in Example 4.1 is a cone b-metric not a cone strong b-metric. The D defined in Example 4.2 is a strong b-metric. These two examples illustrate the validity of Theorem 3.1 and Theorem 3.2 respectively.*

5. **Conclusions.** This paper establishes a common fixed point theorem for four self-mappings in cone b-metric space under weaker contraction conditions. And, we obtain a similar result in cone strong b-metric space. Two examples are constructed to illustrate the validity of our result. In future research, we will study common fixed point theory in other spaces, such as cone S-metric spaces.

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