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CHARACTERIZATIONS OF SOME REGULARITIES OF ORDERED Γ-SEMIHYPERGROUPS IN TERMS OF INTERVAL-VALUED Q-FUZZY Γ-HYPERIDEALS

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Abstract. *A Q-fuzzy of a nonempty set X is a mapping from $X \times Q$ to the unit closed interval $[0, 1]$. It is well-known that the concept of interval-valued fuzzy sets, introduced by Zadeh, is a generalized concept of fuzzy sets. The concept of interval-valued fuzzy sets and the notion of Q-fuzzy sets are combined so-called interval-valued Q-fuzzy sets. It turns out that the concept of interval-valued Q-fuzzy sets is a generalized notion of Q-fuzzy sets. The purposes of this paper are to introduce interval-valued Q-fuzzy right (resp., left, two-sided) Γ-hyperideals in ordered Γ-semihypergroups and investigate some properties of interval-valued Q-fuzzy right (resp., left, two-sided) Γ-hyperideals. Finally, we characterize some regularities in ordered Γ-semihypergroups by using interval-valued Q-fuzzy right Γ-hyperideals and interval-valued Q-fuzzy left Γ-hyperideals.*
Keywords: Ordered Γ-semihypergroup, Interval-valued Q-fuzzy subset, Interval-valued Q-fuzzy Γ-hyperideal

1. **Introduction.** Hyperstructure theory was introduced in 1934 when Marty defined hypergroups [14]. The theory is widely studied from theoretical point of view and for their applications to many subjects of pure and applied mathematics.

The concept of ordered semihypergroups is one of generalizations of the concept of semihypergroups, first considered by Heidari and Davvaz, in the sense that every semihypergroup can be regarded as an ordered semihypergroup [5]. This concept was studied in different aspects by various researchers. Another generalization of semihypergroups, studied by Kondo and Lekkoksung, is the concept of ordered Γ-semihypergroups [11]. It extended the notion of ordered semihypergroups, and was investigated by many authors [4].

The concept of fuzzy sets was introduced by Zadeh [18]. This notion can be applied in many fields of mathematics, engineering and decision making theory [10, 17]. The notion of fuzzy sets can be extended into several concepts, for example, interval-valued fuzzy sets and intuitionistic fuzzy sets studied by Zadeh and Atanassov, respectively [3, 19]. These extensions are applied to studying algebraic properties of some algebraic systems by various scientists [1, 9]. The notion of Q-fuzzy sets is introduced and used to study BCK/BCI-algebras by Jun [8]. Later on, this notion was applied to investigating properties of other algebraic systems [7, 12, 13, 15].

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In this paper, we combine the notions of interval-valued fuzzy sets and Q-fuzzy sets, so-called interval-valued Q-fuzzy sets. We apply this notion to investigating some properties of ordered Γ-semihypergroups. Indeed, we introduce the concepts interval-valued Q-fuzzy right (resp., left, two-sided) Γ-hyperideals of ordered Γ-semihypergroups. It is known that some particular classes of ordered Γ-semihypergroups can be characterized in terms of fuzzy Γ-hyperideals. We apply more general tools to characterizing these particular classes. In fact, we use the concept of interval-valued Q-fuzzy right and left Γ-hyperideals to characterize regular and intra-regular ordered Γ-semihypergroups.

This paper is organized as follows. The basic information of ordered Γ-semihypergroups is provided. Moreover, we introduced the concept of interval-valued Q-fuzzy sets in Section 2, and applied in Section 3 the concepts of interval-valued Q-fuzzy sets and Γ-hyperideals of ordered Γ-semihypergroups. There were some primary results concerning interval-valued Q-fuzzy left Γ-hyperideals and interval-valued Q-fuzzy right Γ-hyperideals of ordered Γ-semihypergroups are present in this section. In Section 4, we illustrate a characterization of regular and intra-regular ordered Γ-semihypergroups. Finally, Section 5 concludes the paper.

2. Preliminaries.

In this section, we recall some basic terms and definitions of ordered Γ-semihypergroups. For more information about Γ-hyperstructure theory and ordered Γ-semihypergroups, the readers are referred to [6, 11]. Moreover, the concept of interval-valued Q-fuzzy Γ-hyperideals of ordered Γ-semihypergroups is introduced.

Definition 2.1. *A mapping* $\circ\colon S \times S \to \mathcal{P}^*(S)$ *is called a* (binary) hyperoperation *on* S*, where* S *is a nonempty set and* $\mathcal{P}^*(S)$ *denotes the set of all nonempty subsets of* S*.*

A *hypergroupoid* is a structure $\langle S; \circ\rangle$ consisting of a nonempty set S together with a (binary) hyperoperation $\circ$ on S.

Let $\langle S; \circ\rangle$ be a hypergroupoid, A and B be nonempty subsets of S, and $x \in S$. We define

$$A \circ B := \bigcup_{a\in A, b\in B} a \circ b, \quad x \circ A := \{x\} \circ A \quad \text{and} \quad A \circ x := A \circ \{x\}.$$

Definition 2.2. *A hypergroupoid* $\langle S; \circ\rangle$ *satisfying*

$$x \circ (y \circ z) = (x \circ y) \circ z$$

for all $x, y, z \in S$*, is called a* semihypergroup*.*

Definition 2.3 ([2])**.** *Let* S *and* Γ *be two nonempty sets. Then* $\langle S; \Gamma\rangle$ *is called a* Γ-semihypergroup *if every* $\gamma \in \Gamma$ *is a (binary) hyperoperation on* S*, and for every* $\alpha, \beta \in \Gamma$ *and* $x, y, z \in S$ *we have*

$$x\alpha(y\beta z) = (x\alpha y)\beta z.$$

Definition 2.4 ([11])**.** *Let* S *and* Γ *be two nonempty sets and* $\leq$ *be a binary relation on* S*. The structure* $\langle S; \Gamma, \leq\rangle$ *is called an* ordered Γ-semihypergroup *if the following conditions are satisfied:*

1) $\langle S; \Gamma\rangle$ *is a* Γ*-semihypergroup;*

2) $\langle S; \leq\rangle$ *is a partially ordered set;*

3) For every $a, b, c \in S$ *and* $\gamma \in \Gamma$*,* $a \leq b$ *implies* $a\gamma c \leq b\gamma c$ *and* $c\gamma a \leq c\gamma b$*.*

By Definition 2.4, for any $a, b, c, d \in S$ and $\gamma \in \Gamma$, we note that $a\gamma b \leq c\gamma d$ means that for any $x \in a\gamma b$, there is $y \in c\gamma d$ such that $x \leq y$. Moreover, we can see that every ordered semihypergroup is an ordered Γ-semihypergroup in the sense that the set Γ is of cardinality one containing such an associative binary hyperoperation.

From now on, we denote an ordered Γ-semihypergroup $\langle S;\Gamma,\leq\rangle$ by **S**.

Let **S** be an ordered Γ-semihypergroup, A and B be two nonempty subsets of S. Then we define

$$A\Gamma B=\bigcup_{\gamma\in\Gamma}A\gamma B=\bigcup\{a\gamma b\mid a\in A,b\in B\text{ and }\gamma\in\Gamma\}.$$

For $K\subseteq S$, we denote

$$(K]:=\{a\in S\mid a\leq k\text{ for some }k\in K\}.$$

Let **S** be an ordered Γ-semihypergroup. A nonempty subset A of S is said to be a *sub-Γ-semihypergroup* of **S** if $\langle A;\Gamma',\leq'\rangle$ is an ordered Γ-semihypergroup, where $\Gamma':=\{\gamma|_{A\times A}:\gamma\in\Gamma\}$ and $\leq':=\leq\cap(A\times A)$.

Definition 2.5. *Let* **S** *be an ordered* Γ*-semihypergroup. A nonempty subset* A *of* S *is called a* right (resp., left) Γ-hyperideal *of* **S** *if*

1) $A\Gamma S\subseteq A$ *(resp.,* $S\Gamma A\subseteq A$*),*

2) $(A]\subseteq A$*.*

If A is both a right Γ-hyperideal and a left Γ-hyperideal of **S**, then A is called a (*two-sided*) Γ*-hyperideal* of **S**.

By an interval number D on closed unit interval $[0,1]$, we mean an interval $[a^-,a^+]$, where $0\leq a^-\leq a^+\leq 1$. We denote the set of all interval numbers on $[0,1]$ by $D[0,1]$. For $D_1=[a^-,a^+]$, $D_2=[b^-,b^+]\in D[0,1]$, we define

1) $D_1\cap D_2=[\min\{a^-,b^-\},\min\{a^+,b^+\}]$,

2) $D_1\cup D_2=[\max\{a^-,b^-\},\max\{a^+,b^+\}]$,

3) $D_1\leq D_2$ if and only if $a^-\leq b^-$ and $a^+\leq b^+$.

Definition 2.6. *Let* X *and* Q *be given nonempty sets. A* Q-fuzzy *subset of* X *is a function* $A\colon X\times Q\to[0,1]$*. A function* $\widetilde{A}\colon X\times Q\to D[0,1]$ *is called an* interval-valued Q-fuzzy *subset of* X*.*

Let $\widetilde{A}$ be a Q-fuzzy subset of X. Then we can see that $\widetilde{A}$ can be defined as

$$\widetilde{A}(x,q):=\left[A^-(x,q),A^+(x,q)\right]$$

for all $x\in X$ and $q\in Q$, where A^- and A^+ are Q-fuzzy subsets of X.

Let A be a nonempty subset of S. The *characteristic function* $\widetilde{\chi}_A$ of A is an interval-valued Q-fuzzy subset of S defined by

$$\widetilde{\chi}_A(a,q):=\begin{cases}\widetilde{1} & \text{if } a\in A,\\ \widetilde{0} & \text{if } a\notin A\end{cases}$$

for all $a\in S$ and $q\in Q$, where $\widetilde{1}:=[1,1]$ and $\widetilde{0}:=[0,0]$.

For an ordered Γ-semihypergroup **S**, and $a\in S$. We define

$$\mathbf{S}_a:=\{(x,y)\in S\times S:a\leq x\Gamma y\text{ for some }x,y\in S\}.$$

Let $\widetilde{A}$ and $\widetilde{B}$ be two interval-valued Q-fuzzy subsets of an ordered Γ-semihypergroup S. Then, we define the product $\widetilde{A}*\widetilde{B}$ of $\widetilde{A}$ and $\widetilde{B}$ by

$$\left(\widetilde{A}*\widetilde{B}\right)(a,q):=\begin{cases}\bigvee_{(x,y)\in\mathbf{S}_a}\left\{\min\left\{\widetilde{A}(x,q),\widetilde{B}(y,q)\right\}\right\} & \text{if } \mathbf{S}_a\neq\emptyset,\\ \widetilde{0} & \text{otherwise}\end{cases}\qquad(1)$$

for all $a\in S$ and $q\in Q$, where $\widetilde{0}:=[0,0]$.

Let $\left\{\widetilde{A}_i : i \in I\right\}$ be a collection of interval-valued Q-fuzzy sets. We define a new interval-valued Q-fuzzy set $\bigcap_{i\in I} \widetilde{A}_i$ by

$$\bigcap_{i\in I} \widetilde{A}_i(x,q) := \inf_{i\in I} \widetilde{A}_i(x,q) := \left[\inf_{i\in I} A^-(x,q), \inf_{i\in I} A^+(x,q)\right]$$

for all $(x,q) \in X \times Q$.

The concepts of many kinds of Γ-hyperideals play an important role in investigating the structural properties of ordered Γ-semihypergroups. We apply the notion of interval-valued Q-fuzzy sets to the concepts of these Γ-hyperideals in ordered Γ-semihypergroups. In Section 3, we introduce interval-valued Q-fuzzy left and right Γ-hyperideals. It is not difficult to illustrate that every left (resp., right) Γ-hyperideal can be considered as an interval-valued Q-fuzzy left (resp., right) Γ-hyperideals as presented in Lemma 4.3. Moreover, these generalized concepts of left and right Γ-hyperideals are studied by using the operation $*$ defined in (1).

3. Interval-Valued Q-Fuzzy Γ-Hyperideals of Ordered Γ-Semihypergroups.

In this section, we introduce the notion of interval-valued Q-fuzzy right (resp., left, two-side) Γ-hyperideals of ordered Γ-semihypergroups. Some properties and their characterizations are obtained.

Definition 3.1. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. An interval-valued* Q*-fuzzy subset* $\widetilde{A}$ *of* S *is called an* interval-valued Q-fuzzy sub-Γ-semihypergroup *of* $\mathbf{S}$ *if* $\inf_{a\in x\Gamma y}\left\{\widetilde{A}(a,q)\right\} \geq \min\left\{\widetilde{A}(x,q), \widetilde{A}(y,q)\right\}$ *for all* $x,y \in S$ *and* $q \in Q$.

Example 3.1. *Let* $S = \{a,b,c\}$ *and* $Q = \{p,q\}$*. Define the hyperoperation* $\circ$ *on* S *by the following table:*

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$

Define an order on S *as follows:* $\leq\ := \{(a,b),(a,c)\} \cup \{(x,x) : x \in S\}$*. Then,* $\mathbf{S} := \langle S; \Gamma, \leq\rangle$ *is an ordered* Γ*-semihypergroup, where* $\Gamma = \{\circ\}$*. We define an interval-valued* Q*-fuzzy set* $\widetilde{A}$ *of* $\mathbf{S}$ *by*

$$\widetilde{A}(x,p) := \begin{cases} [0.6, 0.8] & \text{if } x = a, \\ [0.6, 0.7] & \text{if } x = b, \\ [0.4, 0.7] & \text{if } x = c, \end{cases}$$

and

$$\widetilde{A}(x,q) := \begin{cases} [0.2, 0.7] & \text{if } x = a, \\ [0.2, 0.5] & \text{if } x = b, \\ [0.4, 0.5] & \text{if } x = c \end{cases}$$

for all $x \in S$*. Then* $\widetilde{A}$ *is an interval-valued* Q*-fuzzy sub-*Γ*-hyperideal of* $\mathbf{S}$.

Definition 3.2. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. An interval-valued* Q*-fuzzy subset* $\widetilde{A}$ *of* S *is called an* interval-valued Q-fuzzy right (resp., left) Γ-hyperideal *of* $\mathbf{S}$ *if for any* $x,y \in S$ *and* $q \in Q$

1) $x \leq y$ *implies* $\widetilde{A}(x,q) \geq \widetilde{A}(y,q)$,

2) $\inf_{a\in x\Gamma y}\left\{\widetilde{A}(a,q)\right\} \geq \widetilde{A}(x,q)$ *(resp.,* $\inf_{a\in x\Gamma y}\left\{\widetilde{A}(a,q)\right\} \geq \widetilde{A}(y,q)$*).*

An interval-valued Q-fuzzy subset of S is called an *interval-valued two-sided* Γ*-hyperideal* of $\mathbf{S}$ if it is both an interval-valued Q-fuzzy left Γ-hyperideal and an interval-valued Q-fuzzy right Γ-hyperideal of $\mathbf{S}$.

Example 3.2. *Let $S = \{a, b, c\}$ and $Q = \{p, q\}$. Define the hyperoperation $\circ$ on S by the following table:*

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a, b\}$	$\{c\}$

Define an order on S as follows: $\leq := \{(a, b)\} \cup \{(x, x) : x \in S\}$. Then, $\mathbf{S} := \langle S; \Gamma, \leq\rangle$ is an ordered Γ-semihypergroup, where $\Gamma = \{\circ\}$. We define an interval-valued Q-fuzzy set $\widetilde{A}$ of $\mathbf{S}$ by

$$\widetilde{A}(x,p) := \begin{cases} [0.3, 0.7] & \text{if } x = a, \\ [0.1, 0.5] & \text{if } x = b, \\ [0.3, 0.7] & \text{if } x = c, \end{cases}$$

and

$$\widetilde{A}(x,q) := \begin{cases} [0.5, 0.8] & \text{if } x = a, \\ [0.2, 0.7] & \text{if } x = b, \\ [0.5, 0.8] & \text{if } x = c \end{cases}$$

for all $x \in S$. Then $\widetilde{A}$ is an interval-valued Q-fuzzy left Γ-hyperideal of $\mathbf{S}$. We note that $\widetilde{A}$ is not an interval-valued Q-fuzzy right Γ-hyperideal of $\mathbf{S}$ since $\inf_{u\in c\circ b}\left\{\widetilde{A}(u,p)\right\} = \widetilde{A}(b,p) < \widetilde{A}(c,p)$.

Example 3.3. *Let $S = \{a, b, c\}$ and $Q = \{p, q\}$. Define the hyperoperation $\circ$ on S by the following table:*

$\circ$	a	b	c
a	$\{b\}$	$\{b\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{b\}$	$\{b, c\}$	$\{c\}$

Define an order on S as follows: $\leq := \{(a, b)\} \cup \{(x, x) : x \in S\}$. Then, $\mathbf{S} := \langle S; \Gamma, \leq\rangle$ is an ordered Γ-semihypergroup, where $\Gamma = \{\circ\}$. We define an interval-valued Q-fuzzy set $\widetilde{A}$ of $\mathbf{S}$ by

$$\widetilde{A}(x,p) := \begin{cases} [0.1, 0.5] & \text{if } x = a, \\ [0.3, 0.7] & \text{if } x = b, \\ [0.1, 0.5] & \text{if } x = c, \end{cases}$$

and

$$\widetilde{A}(x,q) := \begin{cases} [0.2, 0.6] & \text{if } x = a, \\ [0.4, 0.8] & \text{if } x = b, \\ [0.2, 0.6] & \text{if } x = c \end{cases}$$

for all $x \in S$. Then $\widetilde{A}$ is an interval-valued Q-fuzzy right Γ-hyperideal of $\mathbf{S}$.

In ordered Γ-semihypergroups, it is obvious that for any nonempty intersection of Γ-hyperideals is again a Γ-hyperideal. The following result shows a similar behavior of interval-valued Q-fuzzy Γ-hyperideals. In fact, we show that any intersection of interval-valued Q-fuzzy Γ-hyperideals is an interval-valued Q-fuzzy Γ-hyperideals.

Theorem 3.1. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup and* $\left\{\widetilde{A}_i : i \in I\right\}$ *be a family of interval-valued* Q*-fuzzy right (resp., left, two-sided)* Γ*-hyperideals of* $\mathbf{S}$*. If* $\bigcap_{i\in I} \widetilde{A}_i \neq \emptyset$*, then* $\bigcap_{i\in I} \widetilde{A}_i$ *is an interval-valued* Q*-fuzzy right (resp., left, two-sided)* Γ*-hyperideal of* $\mathbf{S}$*.*

Proof: We prove the theorem for interval-valued Q-fuzzy right Γ-hyperideals. In the cases of interval-valued Q-fuzzy left and interval-valued Q-fuzzy two-sided Γ-hyperideals of $\mathbf{S}$ can be done similarly. Let $x, y \in S$ and $q \in Q$. Then, if $x \leq y$, then $\widetilde{A}_i(x,q) \geq \widetilde{A}_i(y,q)$ for all $i \in I$. Thus, $\inf_{i\in I}\left\{\widetilde{A}_i(x,q)\right\} \geq \inf_{i\in I}\left\{\widetilde{A}_i(y,q)\right\}$. That is, $\left(\bigcap_{i\in I} A_i\right)(x,q) \geq \left(\bigcap_{i\in I} A_i\right)(y,q)$. Let $x, y \in S$ and $q \in Q$. Then

$$\inf_{a\in x\Gamma y}\left\{\left(\bigcap_{i\in I}\widetilde{A}_i\right)(a,q)\right\} = \inf_{a\in x\Gamma y}\left\{\inf_{i\in I}\left\{\widetilde{A}_i(a,q)\right\}\right\} \geq \inf_{i\in I}\left\{\widetilde{A}_i(x,q)\right\} = \left(\bigcap_{i\in I}\widetilde{A}_i\right)(x,q).$$

Hence, $\bigcap_{i\in I} \widetilde{A}_i$ is an interval-valued Q-fuzzy right Γ-hyperideal of $\mathbf{S}$. □

We provide characterizations of interval-valued Q-fuzzy sub-Γ-semihypergroups and Q-fuzzy left (resp., right, two-sided) Γ-hyperideals of ordered Γ-semihypergroups in terms of an operation defined in (1) in Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5.

Theorem 3.2. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup, and* $\widetilde{A}$ *be an interval-valued* Q*-fuzzy subset of* S*. Then the following statements are equivalent.*

1) $\widetilde{A}$ *is an interval-valued* Q*-fuzzy sub-*Γ*-semihypergroup of* $\mathbf{S}$*.*

2) $\widetilde{A} * \widetilde{A} \subseteq \widetilde{A}$*.*

Proof: 1) ⇒ 2). Let $x \in S$ and $q \in Q$. If $\mathbf{S}_x = \emptyset$, then $\left(\widetilde{A} * \widetilde{A}\right)(x,q) = \widetilde{0} \leq \widetilde{A}(x,q)$. Suppose that $\mathbf{S}_x \neq \emptyset$. Then

$$\begin{aligned}\left(\widetilde{A} * \widetilde{A}\right)(x,q) &= \bigvee_{(b,c)\in\mathbf{S}_x}\left\{\min\left\{\widetilde{A}(a,q), \widetilde{A}(b,q)\right\}\right\} \leq \bigvee_{(a,b)\in\mathbf{S}_x}\inf_{c\in a\Gamma b}\left\{\widetilde{A}(c,q)\right\} \\ &\leq \bigvee_{(a,b)\in\mathbf{S}_x}\widetilde{A}(x,q) = \widetilde{A}(x,q).\end{aligned}$$

Hence, $\widetilde{A} * \widetilde{A} \subseteq \widetilde{A}$.

2) ⇒ 1). If $\widetilde{A} * \widetilde{A} \subseteq \widetilde{A}$, then for any $x, y \in S$ and $q \in Q$, we have that

$$\begin{aligned}\inf_{a\in x\Gamma y}\left\{\widetilde{A}(a,q)\right\} &\geq \inf_{a\in x\Gamma y}\left\{\left(\widetilde{A} * \widetilde{A}\right)(a,q)\right\} = \inf_{a\in x\Gamma y}\left\{\bigvee_{(b,c)\in\mathbf{S}_a}\left\{\min\left\{\widetilde{A}(b,q), \widetilde{A}(c,q)\right\}\right\}\right\} \\ &\geq \inf_{a\in x\Gamma y}\left\{\min\left\{\widetilde{A}(x,q), \widetilde{A}(y,q)\right\}\right\} = \min\left\{\widetilde{A}(x,q), \widetilde{A}(y,q)\right\}.\end{aligned}$$

Therefore, $\widetilde{A}$ is an interval-valued Q-fuzzy sub-Γ-semihypergroup of $\mathbf{S}$. □

From now on, we simply write $\widetilde{\chi}_S$ by $\widetilde{S}$.

Theorem 3.3. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup and* $\widetilde{A}$ *be an interval-valued* Q*-fuzzy subset of* S*. Then the following conditions are equivalent.*

1) $\widetilde{A}$ *is an interval-valued* Q*-fuzzy left* Γ*-hyperideal of* $\mathbf{S}$*.*

2) $\widetilde{A}$ *satisfies the following conditions.*

(a) $x \leq y$ *implies* $\widetilde{A}(x,q) \geq \widetilde{A}(y,q)$ *for all* $x, y \in S$ *and* $q \in Q$*.*

(b) $\widetilde{S} * \widetilde{A} \subseteq \widetilde{A}$*.*

Proof: 1) ⇒ 2). Let $x \in S$ and $q \in Q$. It is clear that if $\mathbf{S}_x = \emptyset$, then $\left(\widetilde{A} * \widetilde{A}\right)(x,q) = \widetilde{0} \leq \widetilde{A}(x,q)$. Suppose that $\mathbf{S}_x \neq \emptyset$. Then

$$\left(\widetilde{S} * \widetilde{A}\right)(x,q) = \bigvee_{(a,b)\in \mathbf{S}_x} \left\{\min\left\{\widetilde{S}(a,q), \widetilde{A}(b,q)\right\}\right\} = \bigvee_{(a,b)\in \mathbf{S}_x} \left\{\widetilde{A}(b,q)\right\}$$
$$\leq \bigvee_{(a,b)\in \mathbf{S}_x} \left\{\inf_{c\in a\Gamma b} \left\{\widetilde{A}(c,q)\right\}\right\} \leq \bigvee_{(a,b)\in \mathbf{S}_x} \left\{\widetilde{A}(x,q)\right\} = \widetilde{A}(x,q).$$

Hence, $\widetilde{S} * \widetilde{A} \subseteq \widetilde{A}$.

2) ⇒ 1). Let $x, y \in S$ and $q \in Q$. Then,

$$\inf_{a\in x\Gamma y} \left\{\widetilde{A}(a,q)\right\} \geq \inf_{a\in x\Gamma y} \left\{\left(\widetilde{S} * \widetilde{A}\right)(a,q)\right\} = \inf_{a\in x\Gamma y} \left\{\bigvee_{(b,c)\in \mathbf{S}_a} \left\{\min\left\{\widetilde{S}(b,q), \widetilde{A}(c,q)\right\}\right\}\right\}$$
$$\geq \inf_{a\in x\Gamma y} \left\{\min\left\{\widetilde{S}(x,q), \widetilde{A}(y,q)\right\}\right\} = \inf_{a\in x\Gamma y} \left\{\widetilde{A}(y,q)\right\} = \widetilde{A}(y,q).$$

This shows that $\widetilde{A}$ is an interval-valued Q-fuzzy left Γ-hyperideal of $\mathbf{S}$. □

Similar to Theorem 3.3, we obtain the following theorems.

Theorem 3.4. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup and* $\widetilde{A}$ *be an interval-valued* Q*-fuzzy subset of* S*. Then the following conditions are equivalent.*

1) $\widetilde{A}$ *is an interval-valued* Q*-fuzzy right* Γ*-hyperideal of* $\mathbf{S}$.

2) $\widetilde{A}$ *satisfies the following conditions.*

(b) $x \leq y$ *implies* $\widetilde{A}(x,q) \geq \widetilde{A}(y,q)$ *for all* $x, y \in S$ *and* $q \in Q$.

(a) $\widetilde{A} * \widetilde{S} \subseteq \widetilde{A}$.

Theorem 3.5. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup and* $\widetilde{A}$ *be an interval-valued* Q*-fuzzy subset of* S*. Then the following conditions are equivalent.*

1) $\widetilde{A}$ *is an interval-valued* Q*-fuzzy* Γ*-hyperideal of* $\mathbf{S}$.

2) $\widetilde{A}$ *satisfies the following conditions.*

(a) $x \leq y$ *implies* $\widetilde{A}(x,q) \geq \widetilde{A}(y,q)$ *for all* $x, y \in S$ *and* $q \in Q$.

(b) $\widetilde{S} * \widetilde{A} \subseteq \widetilde{A}$ *and* $\widetilde{A} * \widetilde{S} \subseteq \widetilde{A}$.

4. **Some Regularities Characterizations.** In this section we characterize regular and intra-regular ordered Γ-semihypergroups in terms of interval-valued Q-fuzzy right Γ-hyperideals and interval-valued Q-fuzzy left Γ-hyperideals. Firstly, we recall particular classes of ordered Γ-semihypergroups: regular and intra-regular.

An ordered Γ-semihypergroup $\mathbf{S}$ is *regular* if, for each element $a \in S$, there exists an element $x \in S$ such that $a \leq a\Gamma x\Gamma a$ [16]. Equivalent definition:

1) $A \subseteq (A\Gamma S\Gamma A]$, for all $A \subseteq S$.
2) $a \in (a\Gamma S\Gamma a]$ for all $a \in S$.

An ordered Γ-semihypergroup $\mathbf{S}$ is *intra-regular* if, for each element $a \in S$, there exist elements $x, y \in S$ such that $a \leq x\Gamma a\Gamma a\Gamma y$ [11]. Equivalent definition:

1) $A \subseteq (S\Gamma A\Gamma A\Gamma S]$, for all $A \subseteq S$.
2) $a \in (S\Gamma a\Gamma a\Gamma S]$ for all $a \in S$.

We recall a characterization of regular and intra-regular ordered Γ-semihypergroups by right and left Γ-hyperideals as follows.

Lemma 4.1 ([16])**.** *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. Then the following conditions are equivalent.*

1) $\mathbf{S}$ *is regular.*

2) $R \cap L = (R\Gamma L]$ *for every right* Γ*-hyperideal* R *and every left* Γ*-hyperideal* L *of* $\mathbf{S}$*.*

Lemma 4.2 ([16])**.** *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. Then the following conditions are equivalent.*

1) $\mathbf{S}$ *is intra-regular.*

2) $R \cap L \subseteq (L\Gamma R]$ *for every right* Γ*-hyperideal* R *and every left* Γ*-hyperideal* L *of* $\mathbf{S}$*.*

The following lemmas can be proven straightforwardly. The proof is skipped.

Lemma 4.3. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup, and* R *and* L *be a right and a left* Γ*-hyperideal of* $\mathbf{S}$*, respectively. Then* $\widetilde{\chi}_R$ *and* $\widetilde{\chi}_L$ *are an interval-valued* Q*-fuzzy right and an interval-valued* Q*-fuzzy left* Γ*-hyperideal of* $\mathbf{S}$*, respectively.*

Lemma 4.4. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup and* A, B *be nonempty subsets of* S*. Then we have*

1) $\widetilde{\chi}_A \cap \widetilde{\chi}_B = \widetilde{\chi}_{A\cap B}$*;*

2) $\widetilde{\chi}_A * \widetilde{\chi}_B = \widetilde{\chi}_{(A\Gamma B]}$*.*

Lemma 4.5. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. Let* $\widetilde{A}$ *and* $\widetilde{B}$ *be an interval-valued* Q*-fuzzy right and an interval-valued* Q*-fuzzy left* Γ*-hyperideal of* $\mathbf{S}$*, respectively. Then* $\widetilde{A} * \widetilde{B} \subseteq \widetilde{A} \cap \widetilde{B}$*.*

Proof: Let $x \in S$ and $q \in Q$. It is clear that $\left(\widetilde{A} * \widetilde{B}\right)(x,q) = \widetilde{0} \leq \left(\widetilde{A} \cap \widetilde{B}\right)(x,q)$ if $\mathbf{S}_x = \emptyset$. Suppose that $\mathbf{S}_x \neq \emptyset$. Then

$$\begin{aligned}\left(\widetilde{A} * \widetilde{B}\right)(x,q) &= \bigvee_{(u,v)\in\mathbf{S}_x} \left\{\min\left\{\widetilde{A}(u,q), \widetilde{B}(v,q)\right\}\right\} \\ &\leq \bigvee_{(u,v)\in\mathbf{S}_x} \left\{\min\left\{\inf_{a\in u\Gamma v}\left\{\widetilde{A}(a,q)\right\}, \inf_{a\in u\Gamma v}\left\{\widetilde{B}(a,q)\right\}\right\}\right\} \\ &\leq \bigvee_{(u,v)\in\mathbf{S}_x} \left\{\min\left\{\widetilde{A}(x,q), \widetilde{B}(x,q)\right\}\right\} \\ &= \min\left\{\widetilde{A}(x,q), \widetilde{B}(x,q)\right\}.\end{aligned}$$

Hence, we have that $\widetilde{A} * \widetilde{B} \subseteq \widetilde{A} \cap \widetilde{B}$. □

Now, we are ready to characterize regular and intra-regular ordered Γ-semihypergroups. The following theorem provides a characterization of regular ordered Γ-semihypergroups in terms of interval-valued Q-fuzzy right and left Γ-hyperideals.

Theorem 4.1. *Let* $\mathbf{S}$ *be an ordered* Γ*-semihypergroup. Then the following statements are equivalent.*

1) $\mathbf{S}$ *is regular.*

2) $\widetilde{A} * \widetilde{B} = \widetilde{A} \cap \widetilde{B}$ *for any interval-valued* Q*-fuzzy right* Γ*-hyperideal* $\widetilde{A}$ *and interval-valued* Q*-fuzzy left* Γ*-hyperideal* $\widetilde{B}$ *of* $\mathbf{S}$*.*

Proof: 1) ⇒ 2). Let $\widetilde{A}$ and $\widetilde{B}$ be an interval-valued Q-fuzzy right and an interval-valued Q-fuzzy left Γ-hyperideal of $\mathbf{S}$, respectively. Suppose that $a \in S$ and $q \in Q$. Since $\mathbf{S}$ is regular, there exists $x \in S$ such that $a \leq a\Gamma x\Gamma a$. Thus, $\mathbf{S}_a \neq \emptyset$. Then, we have

$$\left(\widetilde{A} * \widetilde{B}\right)(a,q) = \bigvee_{(u,v)\in\mathbf{S}_a} \left\{\min\left\{\widetilde{A}(u,q), \widetilde{B}(v,q)\right\}\right\} \geq \min\left\{\inf_{c\in a\Gamma x}\widetilde{A}(c,q), \widetilde{B}(a,q)\right\}$$

$$\geq \min\left\{\widetilde{A}(a,q), \widetilde{B}(a,q)\right\} = \left(\widetilde{A}\cap\widetilde{B}\right)(a,q).$$

Thus, $\widetilde{A}\cap\widetilde{B} \subseteq \widetilde{A}*\widetilde{B}$. By Lemma 4.5, we obtain that $\widetilde{A}*\widetilde{B} = \widetilde{A}\cap\widetilde{B}$.

2) $\Rightarrow$ 1). Let R and L be a right and a left Γ-hyperideal of **S**, respectively. Then $\widetilde{\chi}_R$ and $\widetilde{\chi}_L$ are an interval-valued Q-fuzzy right and an interval-valued Q-fuzzy left Γ-hyperideal of **S**, respectively. By hypothesis, we obtain $\widetilde{\chi}_R * \widetilde{\chi}_L = \widetilde{\chi}_R \cap \widetilde{\chi}_L$. By Lemma 4.4, we obtain that $\widetilde{\chi}_R * \widetilde{\chi}_L = \widetilde{\chi}_{(R\Gamma L]}$ and $\widetilde{\chi}_R \cap \widetilde{\chi}_L = \widetilde{\chi}_{R\cap L}$. Thus, $\widetilde{\chi}_{(R\Gamma L]} = \widetilde{\chi}_{R\cap L}$. Now, we let $a \in (R\Gamma L]$. Then $\widetilde{\chi}_{(R\Gamma L]}(a) = \widetilde{1}$, which implies that $\widetilde{\chi}_{A\cap B}(a) = \widetilde{1}$. Therefore, we obtain $a \in R\cap L$. Thus, we have $(R\Gamma L] \subseteq R\cap L$. On the other hand, it is not difficult to see that $R\cap L \subseteq (R\Gamma L]$ and, thus, $R\cap L = (R\Gamma L]$. By Lemma 4.1, **S** is regular. □

The concepts of interval-valued Q-fuzzy left and right Γ-hyperideals can also be used to characterize intra-regular ordered Γ-semihypergroups as the following theorem present.

Theorem 4.2. *Let* **S** *be an ordered* Γ*-semihypergroup. Then the following statements are equivalent.*

1) **S** *is intra-regular.*

2) $\widetilde{B}\cap\widetilde{A} \subseteq \widetilde{B}*\widetilde{A}$ *for any interval-valued Q-fuzzy right* Γ*-hyperideal* $\widetilde{A}$ *and interval-valued Q-fuzzy left* Γ*-hyperideal* $\widetilde{B}$ *of* **S**.

Proof: 1) $\Rightarrow$ 2). Let $\widetilde{A}$ and $\widetilde{B}$ be an interval-valued Q-fuzzy right and an interval-valued Q-fuzzy left Γ-hyperideal of **S**, respectively. Suppose that $a \in S$ and $q \in Q$. Since **S** is intra-regular, there exist $x, y \in S$ such that $a \leq x\Gamma a\Gamma a\Gamma y$. Thus, $\mathbf{S}_a \neq \emptyset$. Then

$$\left(\widetilde{B}*\widetilde{A}\right)(a,q) = \bigvee_{(u,v)\in\mathbf{S}_a}\left\{\min\left\{\widetilde{B}(u,q), \widetilde{A}(v,q)\right\}\right\} \geq \min\left\{\inf_{c\in x\Gamma a}\widetilde{B}(c,q), \inf_{d\in a\Gamma y}\widetilde{A}(d,q)\right\}$$

$$\geq \min\left\{\widetilde{B}(a,q), \widetilde{A}(a,q)\right\} = \left(\widetilde{B}\cap\widetilde{A}\right)(a,q).$$

Thus, we have $\widetilde{B}\cap\widetilde{A} \subseteq \widetilde{B}*\widetilde{A}$.

2) $\Rightarrow$ 1). Let R and L be a right and a left Γ-hyperideal of **S**, respectively. Then $\widetilde{\chi}_R$ and $\widetilde{\chi}_L$ are an interval-valued Q-fuzzy right and an interval-valued Q-fuzzy left Γ-hyperideal of **S**, respectively. Then by hypothesis, we obtain $\widetilde{\chi}_R \cap \widetilde{\chi}_L \subseteq \widetilde{\chi}_L * \widetilde{\chi}_R$. By Lemma 4.4, we have $\widetilde{\chi}_L * \widetilde{\chi}_R = \widetilde{\chi}_{(L\Gamma R]}$ and $\widetilde{\chi}_R \cap \widetilde{\chi}_L = \widetilde{\chi}_{R\cap L}$. Thus, $\widetilde{\chi}_{R\cap L} \subseteq \widetilde{\chi}_{(L\Gamma R]}$. Let $a \in R\cap L$. Then $\widetilde{\chi}_{R\cap L}(a) = \widetilde{1}$, which implies that $\widetilde{\chi}_{(L\Gamma R]}(a) = \widetilde{1}$. Therefore, we obtain $a \in (L\Gamma R]$. Thus, we have $R\cap L \subseteq (L\Gamma R]$. By Lemma 4.2, **S** is intra-regular. □

5. **Conclusion.** The concept of interval-valued Q-fuzzy Γ-hyperideals of ordered Γ-semihypergroups is introduced. Their characterizations are provided in terms of the characteristic functions. Particular classes of ordered Γ-semihypergroups: regular and intra-regular, are characterized in terms of interval-valued Q-fuzzy right and interval-valued Q-fuzzy left Γ-hyperideals of an ordered Γ-semihypergroup. We can ask for other characterizations of these particular classes in our future work.

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REFERENCES

[1] S. Abdullah, M. Aslam and K. Hila, Interval valued intuitionistic fuzzy sets in Γ-semihypergroups, *International Journal of Machine Learning and Cybernetics*, vol.7, no.2, pp.217-228, 2016.

[2] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, *Carpathian Journal of Mathematics*, vol.26, no.1, pp.11-23, 2010.

[3] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol.20, no.1, pp.87-96, 1986.

[4] B. Davvaz and S. Omidi, C-Γ-hyperideal theory in ordered Γ-semihypergroups, *Journal of Mathematical and Fundamental Sciences*, vol.49, no.2, pp.181-192, 2017.

[5] D. Heidari and B. Davvaz, On ordered hyperstructures, *University Politehnica of Bucharest Scientific Bulletin – Series A – Applied Mathematics and Physics*, vol.73, no.2, pp.85-96, 2011.

[6] D. Heidari, S. O. Dehkordi and B. Davvaz, Γ-semihypergroups and their properties, *University Politehnica of Bucharest Scientific Bulletin – Series A – Applied Mathematics and Physics*, vol.72, no.1, pp.195-208, 2010.

[7] N. Hidayat, I. Yanti, Z. Fitriah, D. Miftah, A. Anam, S. I. Rahman and F. Hidayat, On cartesian product of intuitionistic Q-fuzzy ideal on ordered semigroup, *Journal of Physics: Conference Series*, vol.1494, 2020.

[8] Y. B. Jun, Q-fuzzy subalgebras of BCK/BCI-algebras, *Scientiae Mathematicae Japonicae*, vol.53, no.2, pp.317-322, 2001.

[9] S. Kar, K. P. Shum and P. Sarkar, Interval-valued prime fuzzy ideals of semigroups, *Lobachevskii Journal of Mathematics*, vol.34, no.1, pp.11-19, 2013.

[10] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, *Information Sciences*, vol.171, nos.1-3, pp.13-28, 2005.

[11] M. Kondo and N. Lekkoksung, On intra-regular ordered Γ-semihypergroups, *International Journal of Mathematical Analysis*, vol.7, no.28, pp.1379-1386, 2013.

[12] S. Lekkoksung, On semiprime and prime Q-fuzzy ideals in ordered semigroups, *International Journal of Mathematical Analysis*, vol.6, no.5, pp.397-404, 2012.

[13] S. Lekkoksung, On Q-fuzzy ideals in ordered semigroups, *International Journal of Pure and Applied Mathematics*, vol.92, no.3, pp.369-379, 2014.

[14] F. Marty, On a generalization of the notion of groups, *Proc. of the 8th Congress des Mathematiciens Scandinaves*, pp.45-49, 1934.

[15] W. Nakkhasen, On Q-fuzzy hyperideals of semihyperrings, *International Journal of Mathematics and Computer Science*, vol.14, no.3, pp.535-546, 2019.

[16] S. Omidi, B. Davvaz and C. Abdioğlu, Some properties of quasi-Γ-hyperideals and hyperfilters in ordered Γ-semihypergroups, *Southeast Asian Bulletin of Mathematics*, vol.42, no.2, pp.223-242, 2018.

[17] Y. Xie, J. Liu and L. Wang, H_∞ filtering design for Takagi-Sugeno fuzzy model with immeasurable premise variables by applying a switching method, *ICIC Express Letters*, vol.14, no.3, pp.257-264, 2020.

[18] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, pp.338-353, 1965.

[19] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, *Information Sciences*, vol.8, no.3, pp.199-249, 1975.