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MINIMUM VARIANCE ESTIMATE WITH EQUALITY CONSTRAINTS BASED ON MATRIX-CONVEX-COMBINATION

Yuanlong Yue*, Bo Shao and Xin Zuo

Research Institute of Automation
China University of Petroleum
No. 18, Fuxue Road, Changping District, Beijing 102249, P. R. China
*Corresponding author: yueyuanlong@cup.edu.cn; Shao_Bo_x@163.com; zuox@cup.edu.cn

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Abstract. *The minimum variance state estimation for unconstrained systems is equivalent to the error covariance optimal state estimation, but now the optimal criterion of error covariance in unconstrained systems is still used to solve the state estimation of constrained systems, ignoring the variance of state estimation. This paper introduces the concept of matrix-convex-combination and derives the minimum variance fusion estimation based on optimal matrix-convex-combination. Then it is extended to linear equality constrained dynamic systems, and the minimum variance estimate with equality constraints based on matrix-convex-combination (C-MCCV) is proposed, which will reduce the covariance of constraint state estimation. At the same time, the statistical properties of C-MCCV are compared with that of the error covariance optimal constrained Kalman estimation (C-KP), and it is verified that C-MCCV is equivalent to C-KP when the state estimation converges within the constrained region. In this case, the equality constrained state estimation can be approximately regarded as unconstrained state estimation. The use of this algorithm is demonstrated on a simple vehicle tracking problem.*

Keywords: Equality constraint state estimation, Matrix-convex-combination, Optimal matrix-convex-combination, Error covariance, Covariance

1. **Introduction.** Kalman filter [1] uses the measurement data containing noise to update the state prediction value, and then recursively solves the optimal estimation of the error covariance of the system state, which is widely used in navigation and positioning [2,3], mapping and remote sensing [4], stochastic process [5], multi-sensor fusion measurement [6], and other fields [7]. However, for some systems with known state variables, the traditional Kalman filter does not make full use of the constraint information, which limits the filtering performance. For some systems with equality or inequality constraints, the constraints can be incorporated into the Kalman filter to improve the filtering performance [8,9].

Systems with state constraints are widely used in engineering applications, such as image tracking [10], fault diagnosis [11], vision systems [12], target tracking [13], robotics [14,15], navigation and positioning [16], mobile mapping [17], and other fields. Therefore, it is extremely important to study the state estimation of dynamic system with state constraints, and many scholars have proposed the solution method of constraint state estimation. Alouani and Blair proposed the pseudo measurement method [18] to study the constraint state estimation in consideration of the kinematics constraint information of the target, which improved the tracking performance. Wen and Durrant-Whyte proposed a dimensionality reduction method by reducing the model parameterization and eliminating the relevant state quantities [19]. Hewett et al. proposed a robust null space method for

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state-space estimation with linear equality constraints [20]. Simon and Chia summarized the equality constrained state estimation methods in detail, compared the advantages and disadvantages of various methods, and gave the most widely used projection method [21].

However, the existing literature mainly focuses on incorporating constraint information into state estimation, ignoring the solving method of state estimation. Existing state estimation solution methods for constrained systems follow the idea of optimal error covariance of the unconstrained Kalman estimation, ignoring the variance of the system state under constraints. In this paper, the fusion estimation of optimal matrix convex combination is derived by defining the matrix-convex-combination of vectors and applied to the state estimation solution of the equality constraint system.

Unlike conventional state constraint estimation methods that seek error covariance optimality, the C-MCCV seeks the optimal covariance of constraint estimation, that is, the degree of state dispersion in the constraint region. Various applications of this work are possible. For example, in dynamic system state estimation with soft constraints and navigation trajectory tracking with time-varying constraints, the practical effect of the work will be better. And this work can also be applied in state feedback control systems that limit the change of control signals. Meanwhile, the algorithm proposed in this paper can be extended to the field of fusion estimation. It may be more effective for some fusion estimations which have special requirements for discrete states in the constrained area, such as the fusion estimation of the drop point of artillery shells, and GPS positioning measurement.

The structure of this paper is as follows. In Section 2, we define matrix-convex-combinations of vectors, show that the existing algebraic-convex-combinations of vectors is a special case of matrix-convex-combination, and derive the minimum variance fusion estimation based on optimal matrix-convex-combination from matrix-convex-combination of random variables. Section 3 proposes the minimum variance estimate with equality constraints based on matrix-convex-combination (C-MCCV) for linear equality constrained systems from the perspective of optimal state variance. Section 4 compares the statistical properties of C-MCCV to that of unconstrained Kalman estimation and compares the statistical properties of C-MCCV to that of error covariance optimally constrained Kalman estimation (C-KP). Section 5 presents some simulation results, and Section 6 provides some concluding remarks.

2. **Minimum Variance Fusion Estimation Based on Optimal Matrix-Convex-Combination.**

2.1. **Definition of matrix-convex-combination.**

Assume that a vector $\mathbf{X} \in \mathbb{R}^n$ and m vectors $\mathbf{X}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$ satisfy

$$\mathbf{X} = \lambda_1 \mathbf{X}_1 + \lambda_2 \mathbf{X}_2 + \cdots + \lambda_m \mathbf{X}_m \tag{1}$$

$\mathbf{X}$ is called the convex combination of m vectors $\mathbf{X}_i$ [22], where $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{m} \lambda_i = 1$. In this paper, the convex combination represented by Equation (1) is called the algebraic-convex-combination, and (1) is called the algebraic-convex-combination of vectors.

Definition 2.1. *Assume that a vector $\mathbf{X} \in \mathbb{R}^n$ and m vectors $\mathbf{X}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$ satisfy*

$$\mathbf{X} = \mathbf{A}_1 \mathbf{X}_1 + \mathbf{A}_2 \mathbf{X}_2 + \cdots + \mathbf{A}_m \mathbf{X}_m \tag{2}$$

$\mathbf{X}$ *is called the matrix-convex-combination of m vectors $\mathbf{X}_i$, where the combination coefficient $\mathbf{A}_i$ is positive semidefinite matrix and $\sum_{i=1}^{m} \mathbf{A}_i = \mathbf{I}$, $\mathbf{I}$ is the identity matrix.*

In particular, when $\mathbf{A}_i = k_i\mathbf{I}$, the matrix-convex-combination of vectors is equivalent to the algebraic-convex-combination of vectors, but the matrix-convex-combination is still based on the algebraic-convex-combination, which is more general than the algebraic-convex-combination. The algebraic-convex-combination of m vectors $\mathbf{X}_i$ is located in the region or boundary of the convex set composed of m vectors. However, when the combination coefficient is matrix $\mathbf{A}_i$, $\mathbf{X}$ can be taken outside the convex set region composed of m vectors. Compared with algebraic-convex-combination, the convex set region of matrix-convex-combination is extended from the original bounded region to the full-space convex set, which means the range of seeking the optimal solution is expanded to full space.

2.2. **The optimal matrix-convex-combination of random variables.**

Definition 2.2. *Assume m n-dimensional random variables $\mathbf{X}_i \sim F(\mathbf{X})$, $i = 1, 2, \ldots, m$, where $\mathbf{X}_i \sim F(\mathbf{X})$ is Gaussian distribution and $\mathbf{X}_i$ are independent of each other. An n-dimensional random variable $\mathbf{X}$ can be given by*

$$\mathbf{X} = \mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_2 + \cdots + \mathbf{A}_m\mathbf{X}_m \tag{3}$$

We call $\mathbf{X}$ the matrix-convex-combination of m random variables $\mathbf{X}_i$, where the combination coefficient $\mathbf{A}_i$ is positive semidefinite matrix and $\sum_{i=1}^{m} \mathbf{A}_i = \mathbf{I}$, $\mathbf{I}$ is the identity matrix.

Take two n-dimensional random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ as an example. The matrix-convex-combination is $\mathbf{X} = \mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_2$. From the perspective of probability, the value of $\mathbf{X}$ requires both $\mathbf{X}_1$ and $\mathbf{X}_2$ to be assigned. In this case, $\mathbf{X}$ means that $\mathbf{X}_1$ and $\mathbf{X}_2$ happen simultaneously. Expanding to m n-dimensional random variables, $\mathbf{X}$ in (3) means that these random variables occur simultaneously.

Considering the uncertainty of the coefficient $\mathbf{A}_i$, the matrix-convex-combination of random variables has various forms, and the smaller the variance of $\mathbf{X}$, the more stable the variable is. Therefore, when the variance of $\mathbf{X}$ is the smallest, $\mathbf{X}$ is called the optimal matrix-convex-combination of $\mathbf{X}_i$.

The state variable $\mathbf{X} \in \mathbb{R}^n$ of the system may be represented by different characterization methods at the same time. Although the characterization methods are different, they can all reflect the state variable. Each characterization method has some uncertainty, and the index describing the degree of dispersion of the characterization method is the variance in probability theory. Assuming that multiple characterization methods of the state variable $\mathbf{X}$ exist simultaneously, the optimal variance characterization of $\mathbf{X}$ can be obtained from the optimal matrix-convex-combination of random variables.

2.3. **The minimum variance fusion estimation based on optimal matrix-convex-combination.**

Lemma 2.1. [23]. *Assume that $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$. If $\mathbf{A} > 0$ and $\mathbf{B} \geq 0$, then*

$$\mathbf{A}^{-1} \geq (\mathbf{A} + \mathbf{B})^{-1} \tag{4}$$

Theorem 2.1. *Assuming that the system state variable $\mathbf{X}$ has estimated values $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ at the same time, then the optimal variance estimation $\hat{\mathbf{X}}$ can be given by*

$$\hat{\mathbf{X}} = \mathbf{A}\hat{\mathbf{X}}_1 + (\mathbf{I} - \mathbf{A})\,\hat{\mathbf{X}}_2 \tag{5}$$

$$\left[Cov\left(\hat{\mathbf{X}}\right)\right]^{-1} = \left[Cov\left(\hat{\mathbf{X}}_1\right)\right]^{-1} + \left[Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1} \tag{6}$$

$\hat{\mathbf{X}}$ *is called the minimum variance unbiased fusion estimation based on optimal matrix-convex-combination, where the combination coefficient* $\mathbf{A}$ *and the variance of* $\hat{\mathbf{X}}$ *satisfy*

$$\mathbf{A} = Cov\left(\hat{\mathbf{X}}_2\right)\left[Cov\left(\hat{\mathbf{X}}_1\right) + Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1} \tag{7}$$

$$Cov\left(\hat{\mathbf{X}}\right) < \min\left[Cov\left(\hat{\mathbf{X}}_1\right), Cov\left(\hat{\mathbf{X}}_2\right)\right] \tag{8}$$

Appendix A provides the necessary proof. We can see that $\hat{\mathbf{X}}$ is the minimum variance estimate, and it means that $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ occur simultaneously. Compared with single $\hat{\mathbf{X}}_1$ or $\hat{\mathbf{X}}_2$, $\hat{\mathbf{X}}$ has less variance, and is more reliable, and more accurate. In this case, we can consider that the fusion estimation based on optimal matrix-convex-combination, $\hat{\mathbf{X}}$, is the optimal state estimate of $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$.

3. Minimum Variance Estimate with Equality Constraints Based on Matrix-Convex-Combination.

The most widely used state estimation method for unconstrained systems is Kalman filtering. Kalman filtering follows the error covariance optimal criterion to solve the state estimation. For unconstrained systems, state minimum variance estimation is equivalent to the error covariance optimal estimation. However, for dynamic systems with constrained information, there are two methods of error covariance optimal estimation and variance optimal estimation. The existing literature mainly focuses on incorporating constraint information into state estimation, ignoring the solving method of state estimation. The optimal criterion of error covariance in unconstrained state estimation, not the minimum variance criterion of constraint state estimation, is still used to solve the constrained state estimation.

Based on the fusion estimation of the optimal matrix-convex-combination, a new algorithm, the minimum variance estimate with equality constraints based on matrix-convex-combination, is presented in this section. Unlike the existing state constrained estimation methods, it seeks the optimal covariance of the constraint estimation.

3.1. Unconstrained state estimation.

Consider the linear time-invariant system given by

$$\mathbf{X}_k = \mathbf{A}_k\mathbf{X}_{k-1} + \boldsymbol{w}_k \tag{9}$$

$$\mathbf{Z}_k = \mathbf{H}_k\mathbf{X}_k + \boldsymbol{v}_k \tag{10}$$

where k is the time index, $\mathbf{X}_k$ is the state vector at k, $\mathbf{Z}_k$ is the measurement, $\mathbf{A}_k$ is the state transfer matrix, $\mathbf{H}_k$ is the observation matrix, and $\boldsymbol{w}_k$ is the system process noise, $\boldsymbol{v}_k$ is the measurement noise, both are uncorrelated zero-mean Gaussian white noise, $\mathbf{Q}_k$ and $\mathbf{R}_k$ are the positive semidefinite covariance matrices of the process noise and the measurement noise. We assume that the following conditions are satisfied

$$E(\mathbf{X}_0) = \bar{\mathbf{X}} \tag{11}$$

$$E(\boldsymbol{w}_k) = 0 \tag{12}$$

$$E(\boldsymbol{v}_k) = 0 \tag{13}$$

$$E\left(\boldsymbol{w}_k\boldsymbol{v}_k^{\mathrm{T}}\right) = 0 \tag{14}$$

$$E\left[\left(\mathbf{X}_0 - \bar{\mathbf{X}}\right)\left(\mathbf{X}_0 - \bar{\mathbf{X}}\right)^{\mathrm{T}}\right] = \mathbf{P}_0 \tag{15}$$

$$E\left(\boldsymbol{w}_k\boldsymbol{w}_m^{\mathrm{T}}\right) = \mathbf{Q}_k\delta_{km} \quad \delta_{km} = \begin{cases} 0, & k \neq m \\ 1, & k = m \end{cases} \tag{16}$$

$$E\left(\boldsymbol{v}_k\boldsymbol{v}_m^{\mathrm{T}}\right) = \mathbf{R}_k\delta_{km} \quad \delta_{km} = \begin{cases} 0, & k \neq m \\ 1, & k = m \end{cases} \tag{17}$$

where $\mathbf{X}_0$ is the initial state of the system, $\bar{\mathbf{X}}$ is the expectation of the state vector and δ_{km} is the Kronecker delta function. The unconstrained Kalman estimate is given by

$$\hat{\mathbf{X}}_{k/k-1} = \mathbf{A}_k \hat{\mathbf{X}}_{k-1} \tag{18}$$

$$\mathbf{P}_{k/k-1} = \mathbf{A}_k \mathbf{P}_{k-1} \mathbf{A}_k^{\mathrm{T}} + \mathbf{Q}_k \tag{19}$$

$$\mathbf{K} = \mathbf{P}_{k/k-1} \mathbf{H}_k^{\mathrm{T}} \left[\mathbf{H}_k \mathbf{P}_{k/k-1} \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k \right]^{-1} \tag{20}$$

$$\hat{\mathbf{X}}_{k/k} = \hat{\mathbf{X}}_{k/k-1} + \mathbf{K} \left(\mathbf{Z}_k - \mathbf{H}_k \hat{\mathbf{X}}_{k/k-1} \right) \tag{21}$$

$$\mathbf{P}_k = \left(\mathbf{I} - \mathbf{K}\mathbf{H}_k \right) \mathbf{P}_{k/k-1} \tag{22}$$

where $\hat{\mathbf{X}}_{k/k}$ is the unconstrained Kalman estimate, $\mathbf{K}$ is the Kalman gain, $\hat{\mathbf{X}}_{k/k-1}$ is the state estimate at moment k obtained from moment $k-1$, $\mathbf{P}_{k/k-1}$ is the error variance of $\hat{\mathbf{X}}_{k/k-1}$, and $\mathbf{P}_k$ is the error variance of $\hat{\mathbf{X}}_{k/k}$.

3.2. **Minimum variance estimate of matrix-convex-combination on equality constraints.** Considering the dynamic system of (9), we obtain the additional constraint

$$\mathbf{D}_k \mathbf{X}_k = \boldsymbol{d}_k \tag{23}$$

where $\mathbf{D}_k \in \mathbb{R}^{s \times n}$ is the constraint matrix, $\boldsymbol{d}_k$ is the constraint vector, and s is the number of constraints. We assume that $\mathbf{D}_k$ is the full rank matrix. If $\mathbf{D}_k$ is not a row full rank matrix, it means that there are redundant constraints in the system. We can make $\mathbf{D}_k$ a row full rank matrix by eliminating the linearly related rows in $\mathbf{D}_k$ and the corresponding elements in the constraint vector.

Now consider the nonlinear state constraint

$$g(\mathbf{X}_k) = \boldsymbol{d}_k \tag{24}$$

We can expand the nonlinear state constraints about a constrained state estimate $\tilde{\mathbf{X}}_k$, which means the nonlinear constraint is linearized at $\tilde{\mathbf{X}}_k$. We have

$$g\left(\tilde{\mathbf{X}}_k\right) + g'\left(\tilde{\mathbf{X}}_k\right)\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right) \approx \boldsymbol{d}_k \tag{25}$$

which indicate that

$$g'\left(\tilde{\mathbf{X}}_k\right)\mathbf{X}_k \approx \boldsymbol{d}_k + g'\left(\tilde{\mathbf{X}}_k\right)\tilde{\mathbf{X}}_k - g\left(\tilde{\mathbf{X}}_k\right) \tag{26}$$

So now we have an approximately linear constraint that is of the form of (23) where $\mathbf{D}_k$ is replaced with $g'\left(\tilde{\mathbf{X}}_k\right)$ and $\boldsymbol{d}_k$ is replaced with $\boldsymbol{d}_k + g'\left(\tilde{\mathbf{X}}_k\right)\tilde{\mathbf{X}}_k - g\left(\tilde{\mathbf{X}}_k\right)$. At this time, the state estimation of nonlinear constraints is transformed into linear equation constrained state estimation. Therefore, this paper is based on the linear equality constraint described by Equation (23).

For the equality constraint (23), we can regard it as the perfect measurement of the state vector, and the measurement space is Φ. In this case, assume that there exists another measurement space Φ' that satisfies $\boldsymbol{d}'_k = \mathbf{D}_k \mathbf{X}_k + \boldsymbol{\varepsilon}$. With the continuous improvement of measurement accuracy, the noise $\boldsymbol{\varepsilon}$ gradually disappears, and Φ' approaches Φ. Finally, the two are equivalent for perfect measurement. That is, $\lim_{\Phi' \to \Phi} \boldsymbol{d}'_k = \boldsymbol{d}_k$ and $\lim_{\Phi' \to \Phi} Cov(\boldsymbol{\varepsilon}) = 0$.

For the measurement space Φ', the state vector has two measurement values $\tilde{\mathbf{X}}_1 = \boldsymbol{d}'_k$ and $\tilde{\mathbf{X}}_2 = \mathbf{D}_k \hat{\mathbf{X}}_k$ at the same time, where $\hat{\mathbf{X}}_k$ is the unconstrained Kalman estimate with variance $\hat{\mathbf{V}}_k$. Therefore, the minimum variance fusion estimation based on optimal matrix-convex-combination $\mathbf{X}_k$ satisfies

$$E(\mathbf{X}_0) = \bar{\mathbf{X}} \mathbf{D}_k \mathbf{X}_k = \bar{\mathbf{K}} \boldsymbol{d}'_k + \left(\mathbf{I} - \bar{\mathbf{K}}\right) \mathbf{D}_k \hat{\mathbf{X}}_k \tag{27}$$

$$\bar{\mathbf{K}} = Cov\left(\mathbf{D}_k\hat{\mathbf{X}}_k\right)\left[Cov\left(\mathbf{D}_k\hat{\mathbf{X}}_k\right) + Cov(\boldsymbol{d}'_k)\right]^{-1} \tag{28}$$

From (6) and (8) we can obtain

$$[Cov(\mathbf{D}_k\mathbf{X}_k)]^{-1} = \left[Cov\left(\mathbf{D}_k\hat{\mathbf{X}}_k\right)\right]^{-1} + [Cov(\boldsymbol{d}'_k)]^{-1} \tag{29}$$

$$Cov(\mathbf{D}_k\mathbf{X}_k) < Cov\left(\mathbf{D}_k\hat{\mathbf{X}}_k\right) \tag{30}$$

where $Cov(\boldsymbol{d}'_k) = Cov(\boldsymbol{\varepsilon})$, $Cov\left(\mathbf{D}_k\hat{\mathbf{X}}_k\right) = \mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}$ and $Cov(\mathbf{D}_k\mathbf{X}_k) = \mathbf{D}_k\mathbf{V}_k\mathbf{D}_k^{\mathrm{T}}$, $\mathbf{V}_k$ is the variance of $\mathbf{X}_k$, so (27) and (29) can be written as

$$\begin{aligned}\mathbf{X}_k &= \left(\mathbf{D}_k^{\mathrm{T}}\mathbf{D}_k\right)^{-1}\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{I} - \bar{\mathbf{K}}\right)\mathbf{D}_k\hat{\mathbf{X}}_k + \left(\mathbf{D}_k^{\mathrm{T}}\mathbf{D}_k\right)^{-1}\mathbf{D}_k^{\mathrm{T}}\bar{\mathbf{K}}\boldsymbol{d}'_k \\ &= \hat{\mathbf{X}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left[\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}} + Cov(\boldsymbol{\varepsilon})\right]^{-1}\left(\boldsymbol{d}'_k - \mathbf{D}_k\hat{\mathbf{X}}_k\right)\end{aligned} \tag{31}$$

$$\mathbf{V}_k = \left(\mathbf{D}_k^{\mathrm{T}}\mathbf{D}_k\right)^{-1}\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{I} - \bar{\mathbf{K}}\right)\mathbf{D}_k\hat{\mathbf{V}}_k \tag{32}$$

When Φ' and Φ are identical, i.e., under the equation constraint (23), the minimum variance estimate with equality constraints based on matrix-convex-combination $\tilde{\mathbf{X}}_k = \lim_{\Phi'\to\Phi}\mathbf{X}_k$, that is

$$\begin{aligned}\tilde{\mathbf{X}}_k &= \lim_{\Phi'\to\Phi}\left[\hat{\mathbf{X}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left[\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}} + Cov(\boldsymbol{\varepsilon})\right]^{-1}\left(\boldsymbol{d}'_k - \mathbf{D}_k\hat{\mathbf{X}}_k\right)\right] \\ &= \hat{\mathbf{X}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{d}_k - \mathbf{D}_k\hat{\mathbf{X}}_k\right)\end{aligned} \tag{33}$$

We define the minimum variance estimate with equality constraints based on matrix-convex-combination (C-MCCV) as follows

$$\tilde{\mathbf{X}}_k^{C\text{-}MCCV} = \hat{\mathbf{X}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{d}_k - \mathbf{D}_k\hat{\mathbf{X}}_k\right) \tag{34}$$

and the error covariance optimal constrained Kalman estimation (C-KP) is given as

$$\tilde{\mathbf{X}}_k^{C\text{-}KP} = \hat{\mathbf{X}}_k - \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\left(\mathbf{D}_k\hat{\mathbf{X}}_k - \boldsymbol{d}_k\right) \tag{35}$$

Compared to the above two equations, we can see that $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ replaces $\hat{\mathbf{P}}_k$ in $\tilde{\mathbf{X}}_k^{C\text{-}KP}$ with $\hat{\mathbf{V}}_k$. As far as the projection method is concerned, C-MCCV projects the estimates into the constraint space through the variance-optimal criterion, while C-KP projects through the error covariance-optimal criterion. Both methods can be regarded as special cases of the projection method.

4. **Properties of the Minimum Variance Estimate with Equality Constraints Based on Matrix-Convex-Combination.**

Theorem 4.1. *The variance of* $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ *is less than the covariance of* $\hat{\mathbf{X}}_k$*, and the error variance of* $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ *is greater than the error covariance of* $\hat{\mathbf{X}}_k$*.*

Proof: We define the variance of $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ as $\tilde{\mathbf{V}}_k^{C\text{-}MCCV}$, and from (32) we can obtain

$$\begin{aligned}\tilde{\mathbf{V}}_k^{C\text{-}MCCV} &= \lim_{\Phi'\to\Phi}\mathbf{V}_k = \lim_{\Phi'\to\Phi}\left[\left(\mathbf{D}_k^{\mathrm{T}}\mathbf{D}_k\right)^{-1}\mathbf{D}_k^{\mathrm{T}}(\mathbf{I} - \mathbf{K})\,\mathbf{D}_k\hat{\mathbf{V}}_k\right] \\ &= \hat{\mathbf{V}}_k - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k\end{aligned} \tag{36}$$

We know that $\mathbf{D}_k$ is full rank and $\hat{\mathbf{V}}_k$ is positive semidefinite, so $\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k$ is also positive semidefinite, which gives $\tilde{\mathbf{V}}_k^{C\text{-}MCCV} < \hat{\mathbf{V}}_k$.

We see from (34) that

$$\tilde{\mathbf{X}}_k^{C\text{-}MCCV} = \hat{\mathbf{X}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\left(\boldsymbol{d}_k - \mathbf{D}_k\hat{\mathbf{X}}_k\right) = \hat{\mathbf{X}}_k - \mathbf{F}\hat{\mathbf{X}}_k + \mathbf{N}\boldsymbol{d}_k \tag{37}$$

where $\mathbf{N} = \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}$ and $\mathbf{F} = \mathbf{N}\times\mathbf{D}_k$.

We define the error variance of $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ as $\tilde{\mathbf{P}}_k^{C\text{-}MCCV}$, and we can obtain

$$\begin{aligned}
\tilde{\mathbf{P}}_k^{C\text{-}MCCV} &= E\left[\left(\tilde{\mathbf{X}}_k^{C\text{-}MCCV} - \mathbf{X}_k\right)\left(\tilde{\mathbf{X}}_k^{C\text{-}MCCV} - \mathbf{X}_k\right)^{\mathrm{T}}\right] \\
&= E\left[\left(\hat{\mathbf{X}}_k - \mathbf{F}\hat{\mathbf{X}}_k + \mathbf{N}\boldsymbol{d}_k - \mathbf{X}\right)\left(\hat{\mathbf{X}}_k - \mathbf{F}\hat{\mathbf{X}}_k + \mathbf{N}\boldsymbol{d}_k - \mathbf{X}\right)^{\mathrm{T}}\right] \\
&= E\left[\hat{\mathbf{X}}\hat{\mathbf{X}}^{\mathrm{T}}\right] - E\left[\hat{\mathbf{X}}\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\right] + E\left[\hat{\mathbf{X}}d^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\right] - E\left[\hat{\mathbf{X}}\mathbf{X}^{\mathrm{T}}\right] \\
&\quad - E\left[\mathbf{F}\hat{\mathbf{X}}\hat{\mathbf{X}}^{\mathrm{T}}\right] + E\left[\mathbf{F}\hat{\mathbf{X}}\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\right] - E\left[\mathbf{F}\hat{\mathbf{X}}d^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\right] + E\left[\mathbf{F}\hat{\mathbf{X}}\mathbf{X}^{\mathrm{T}}\right] \\
&\quad + E\left[\mathbf{N}d\hat{\mathbf{X}}^{\mathrm{T}}\right] - E\left[\mathbf{N}d\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\right] + E\left[\mathbf{N}dd^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\right] - E\left[\mathbf{N}d\mathbf{X}^{\mathrm{T}}\right] \\
&\quad - E\left[\mathbf{X}\hat{\mathbf{X}}^{\mathrm{T}}\right] + E\left[\mathbf{X}\hat{\mathbf{X}}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\right] - E\left[\mathbf{X}d^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\right] + E\left[\mathbf{X}\mathbf{X}^{\mathrm{T}}\right] \\
&= \hat{\mathbf{P}}_k + \mathbf{F}\mathbf{P}_k\mathbf{F}^{\mathrm{T}} \\
&= \hat{\mathbf{P}}_k + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k
\end{aligned} \tag{38}$$

where $\hat{\mathbf{P}}_k$ is the error variance of $\hat{\mathbf{X}}_k$.

The rightmost term in the above equation is positive semidefinite since $\hat{\mathbf{V}}_k$ is a covariance matrix and $\mathbf{D}_k$ is full rank. That is $\tilde{\mathbf{P}}_k^{C\text{-}MCCV} > \hat{\mathbf{P}}_k$.

Theorem 4.2. *The variance of $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ is less than or equal to the covariance of $\tilde{\mathbf{X}}_k^{C\text{-}KP}$. And the equation holds if and only if the state estimate converges to the constrained region. However, the error variance of $\tilde{\mathbf{X}}_k^{C\text{-}MCCV}$ is greater than the error covariance of $\tilde{\mathbf{X}}_k^{C\text{-}KP}$.*

Proof: Now we define the error variance of $\tilde{\mathbf{X}}_k^{C\text{-}KP}$ as $\tilde{\mathbf{P}}_k^{C\text{-}KP}$. We know that $\tilde{\mathbf{P}}_k^{C\text{-}KP} < \hat{\mathbf{P}}_k$, we see from Theorem 4.1 that $\tilde{\mathbf{P}}_k^{C\text{-}MCCV} > \hat{\mathbf{P}}_k$ so we can obtain

$$\tilde{\mathbf{P}}_k^{C\text{-}MCCV} > \tilde{\mathbf{P}}_k^{C\text{-}KP} \tag{39}$$

We see from (36) that

$$\tilde{\mathbf{V}}_k^{C\text{-}MCCV} = \hat{\mathbf{V}}_k - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k \tag{40}$$

We see from (35) that

$$\begin{aligned}
\tilde{\mathbf{X}}_k^{C\text{-}KP} &= \hat{\mathbf{X}}_k - \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\left(\mathbf{D}_k\hat{\mathbf{X}}_k - \boldsymbol{d}_k\right) \\
&= \hat{\mathbf{X}}_k - \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{X}}_k + \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\boldsymbol{d}_k \\
&= \hat{\mathbf{X}}_k - \mathbf{F}'\hat{\mathbf{X}}_k + \mathbf{N}'\boldsymbol{d}_k
\end{aligned} \tag{41}$$

where $\mathbf{N}' = \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}$ and $\mathbf{F}' = \mathbf{N}'\times\mathbf{D}_k$.

Define the variance of $\tilde{\mathbf{X}}_k^{C\text{-}KP}$ as $\tilde{\mathbf{V}}_k^{C\text{-}KP}$. We next obtain

$$\begin{aligned}
\tilde{\mathbf{V}}_k^{C\text{-}KP} &= E\left[\left(\tilde{\mathbf{X}}_k^{C\text{-}KP} - \tilde{\bar{\bar{\mathbf{X}}}}_k\right)\left(\tilde{\mathbf{X}}_k^{C\text{-}KP} - \tilde{\bar{\bar{\mathbf{X}}}}_k\right)^{\mathrm{T}}\right] \\
&= E\left[\left(\hat{\mathbf{X}}_k - \mathbf{F}'\hat{\mathbf{X}}_k + \mathbf{N}'\boldsymbol{d}_k - \tilde{\bar{\bar{\mathbf{X}}}}\right)\left(\hat{\mathbf{X}}_k - \mathbf{F}'\hat{\mathbf{X}}_k + \mathbf{N}'\boldsymbol{d}_k - \tilde{\bar{\bar{\mathbf{X}}}}\right)^{\mathrm{T}}\right] \\
&= \hat{\mathbf{V}}_k - \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{P}}_k \\
&\quad + \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{P}}_k
\end{aligned} \tag{42}$$

Subtracting $\tilde{\mathbf{V}}_k^{C\text{-}MCCV}$ from $\tilde{\mathbf{V}}_k^{C\text{-}KP}$ results in

$$\begin{aligned}
&\tilde{\mathbf{V}}_k^{C\text{-}KP} - \tilde{\mathbf{V}}_k^{C\text{-}MCCV} \\
&= \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{P}}_k \\
&\quad - \hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{P}}_k \\
&\quad + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k
\end{aligned} \tag{43}$$

Define $\boldsymbol{G}_k = \mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}$ and $\boldsymbol{J}_k = \mathbf{D}_k\hat{\mathbf{P}}_k$. The above equation can be given by

$$\begin{aligned}
&\tilde{\mathbf{V}}_k^{C\text{-}KP} - \tilde{\mathbf{V}}_k^{C\text{-}MCCV} \\
&= \boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{G}_k^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\boldsymbol{G}_k^{-1}\boldsymbol{J}_k - \boldsymbol{J}_k\boldsymbol{G}_k^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\boldsymbol{G}_k^{-1}\boldsymbol{J}_k \\
&\quad + \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1}\mathbf{D}_k\hat{\mathbf{V}}_k
\end{aligned} \tag{44}$$

If $\boldsymbol{S}_k = \boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{G}_k^{-1}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{\frac{1}{2}} - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-\frac{1}{2}}$, the above equation can be written as

$$\tilde{\mathbf{V}}_k^{C\text{-}KP} - \tilde{\mathbf{V}}_k^{C\text{-}MCCV} = \boldsymbol{S}_k\boldsymbol{S}_k^{\mathrm{T}} \tag{45}$$

Therefore, we conclude that $\tilde{\mathbf{V}}_k^{C\text{-}MCCV} \leq \tilde{\mathbf{V}}_k^{C\text{-}KP}$, the equation holds if and only if $\boldsymbol{S}_k = \mathbf{0}$.

If $\boldsymbol{S}_k = \mathbf{0}$, after several lines of algebraic manipulation we obtain

$$\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{P}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1} - \hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\left(\mathbf{D}_k\hat{\mathbf{V}}_k\mathbf{D}_k^{\mathrm{T}}\right)^{-1} = \mathbf{0} \tag{46}$$

Since $\mathbf{D}_k \in \mathbb{R}^{s\times n}$, the equation holds if and only if $\hat{\mathbf{P}}_k = \hat{\mathbf{V}}_k$. If $\hat{\mathbf{P}}_k = \hat{\mathbf{V}}_k$, the system is an unconstrained system, which contradicts the equation constraints on the system. Therefore, when the state estimate finally converges to the constraint region $\mathbf{D}_k\mathbf{X}_k = \boldsymbol{d}_k$, the constraint condition loses its effect. And the constrained system can be approximated as an unconstrained system at this time, which means $\hat{\mathbf{P}}_k = \hat{\mathbf{V}}_k$ and $\tilde{\mathbf{V}}_k^{C\text{-}MCCV} = \tilde{\mathbf{V}}_k^{C\text{-}KP}$. As the iteration of the C-MCCV algorithm proceeds, although the covariance of the state estimate is smaller than that of C-KP, the difference gradually decreases. When the state estimation finally converges to the constrained region, the covariance of C-MCCV is equal to that of C-KP, and the two estimation methods are equivalent.

5. **Simulation Results.** Consider the most widely used example under the linear equation state constraint, which dynamically locates a car driving on land, assuming that the

road is limited to a straight line in a deviated northeast direction. Then the motion model of the car can be described as

$$\mathbf{X}_k = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{X}_{k-1} + \boldsymbol{w}_k \tag{47}$$

$$\mathbf{Z}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{X}_k + \boldsymbol{v}_k \tag{48}$$

For instance, if it is known that the vehicle is traveling on the road with a heading of θ then the constraint equation of the system can be given by

$$\begin{bmatrix} 1 & -\tan\theta & 0 & 0 \\ 0 & 0 & 1 & -\tan\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$

The heading of the vehicle travel is $\theta = 60°$ and the sample period T is 3s. The initial conditions are set to $\hat{\mathbf{X}}_0 = \begin{bmatrix} 0 & 0 & 10\tan\theta & 10 \end{bmatrix}^{\mathrm{T}}$.

The system was simulated by Matlab for 300s, and the simulation results are shown in the following figures. Figure 1 shows the east position estimation, Figure 2 shows the east position estimation error variance, and Figure 3 shows the east position estimation variance.

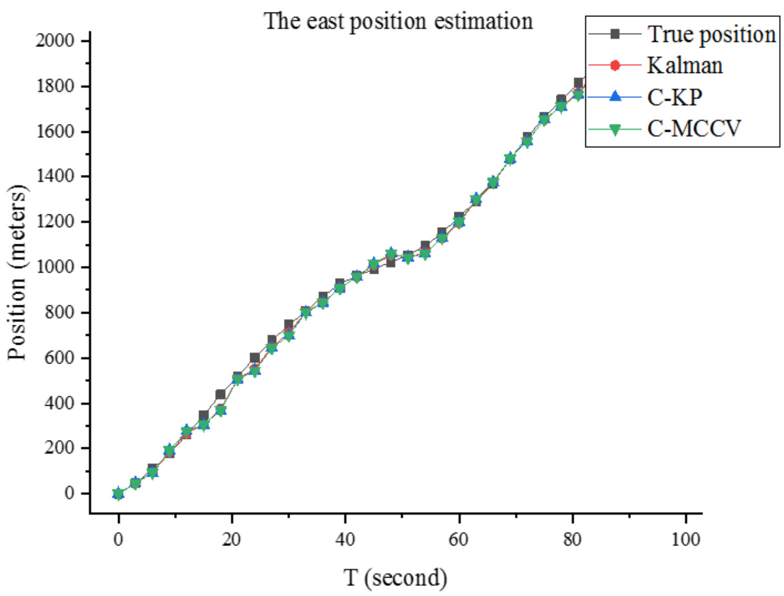

FIGURE 1. The east position estimation

It can be seen that the error variance of C-MCCV is greater than the error covariance of C-KP, and the variance of C-MCCV is less than the covariance of C-KP in the beginning 10 seconds. However, as the algorithm converges, the difference in variance gradually decreases and eventually becomes equal. At this point, the state constraint estimation converges to the constraint region, and the constraint condition is invalid. The state covariance of the two algorithms is consistent. The equality constrained system can be approximated as an unconstrained system when the state estimation is in the constrained region.

C-MCCV and C-KP are equivalent to each other when the constrained estimates converge to the constrained region, which is determined by the property that the constrained

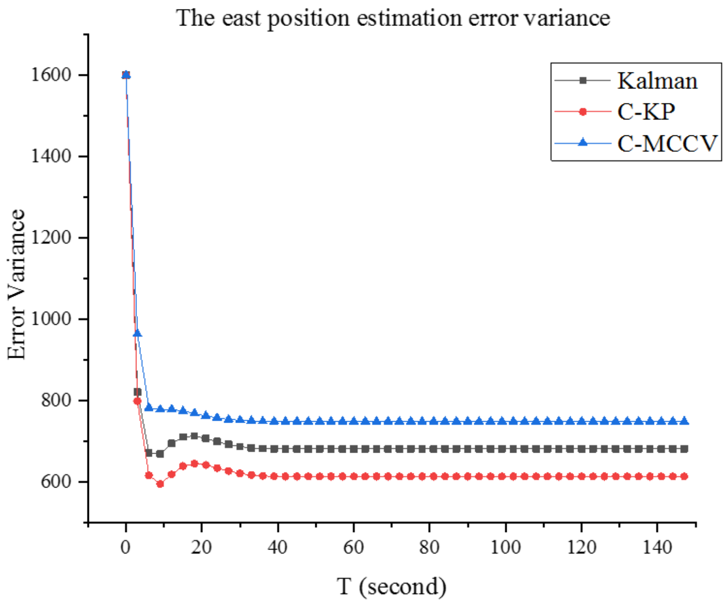

FIGURE 2. The east position estimation error variance

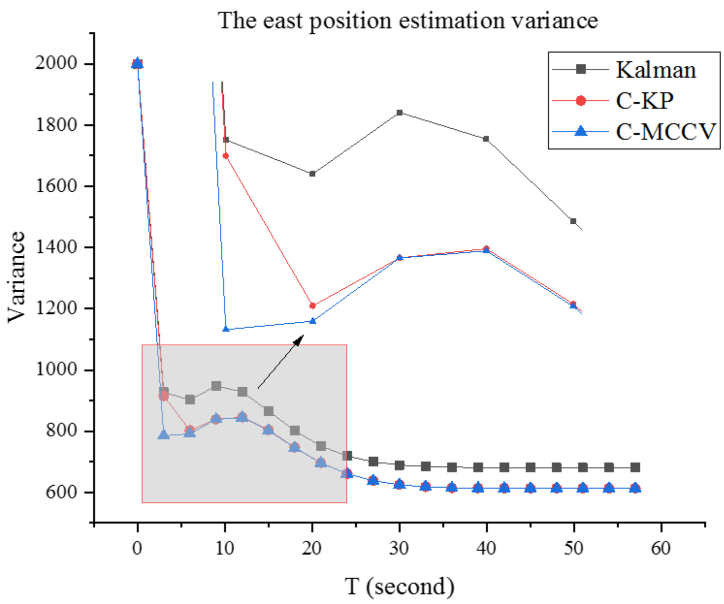

FIGURE 3. The east position estimation variance

state estimates eventually tend to be unconstrained. However, the error covariance of the constrained state estimate is $E\left[\left(\tilde{\mathbf{X}}-\mathbf{X}\right)\left(\tilde{\mathbf{X}}-\mathbf{X}\right)^{\mathrm{T}}\right]$, that is, the covariance of the error between $\tilde{\mathbf{X}}$ and $\mathbf{X}$. It indicates how close the estimated value under the constraint is to the true value. The variance is $E\left[\left(\tilde{\mathbf{X}}-\bar{\tilde{\mathbf{X}}}\right)\left(\tilde{\mathbf{X}}-\bar{\tilde{\mathbf{X}}}\right)^{\mathrm{T}}\right]$, the covariance between $\tilde{\mathbf{X}}$ and $\bar{\tilde{\mathbf{X}}}$. Regarding $\bar{\tilde{\mathbf{X}}}$ as the ideal value of the constraint condition, the variance of the constraint state estimation can be regarded as the error covariance, that is, the degree to which the constraint estimation value approaches the constraint condition. Compared with C-KP, C-MCCV reduces the variance of the state estimation in the early stage of the algorithm, which makes the constraint estimation closer to the constraint conditions and achieves better results.

We also consider that the car is subject to nonlinear constraints. A ground vehicle is assumed to travel along a circular road segment with the turn center chosen as the origin of the X-Y coordinates. And the nonlinear constraint equation is

$$g(\mathbf{X}_k) = (x_1)^2 + (x_2)^2 = R^2 \tag{50}$$

where x_1 denotes the position in the x-direction and x_2 denotes the position in the y-direction. The sample period T is 3s and $R = 100$. The initial conditions are set to $\hat{\mathbf{X}}_0 = \left[\begin{array}{cccc} 0 & 0 & 10\tan\theta & 10 \end{array}\right]^{\mathrm{T}}$.

The system was also simulated by Matlab for 300s, and the simulation results are shown in the following figures. Figure 4 shows the position estimation. Figure 5 shows the traces of error variance, and Figure 6 shows the traces of variance.

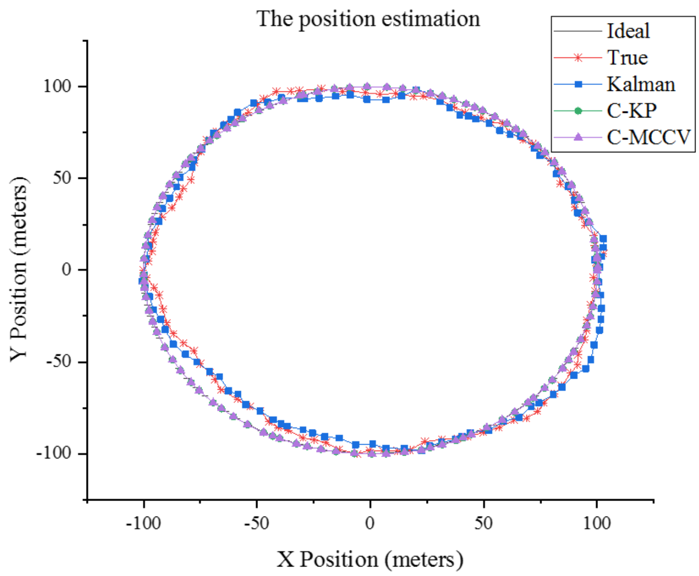

FIGURE 4. The position estimation

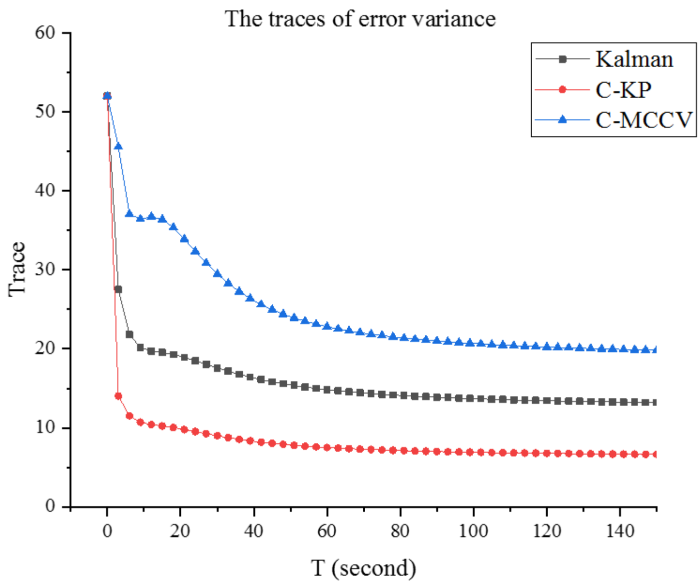

FIGURE 5. The traces of error variance

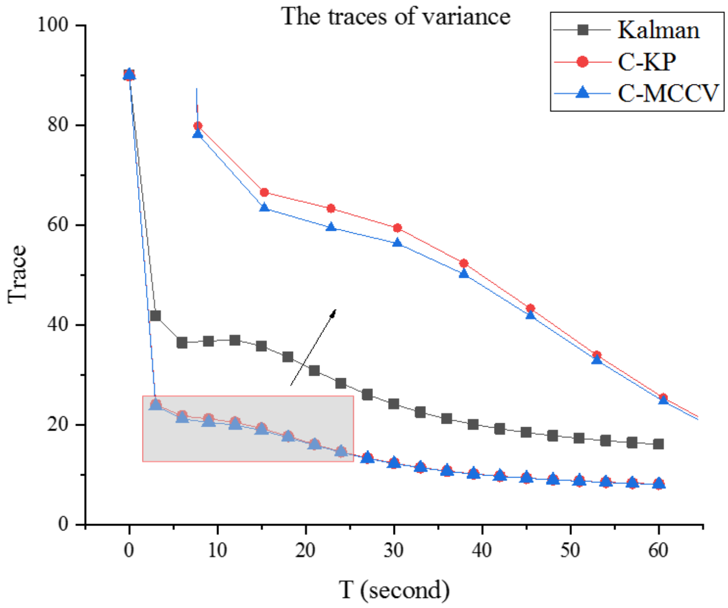

FIGURE 6. The traces of variance

It can be seen that C-MCCV shows the same property under nonlinear equality constraints as that under linear equality constraints. Compared with the traditional constraint estimation method, the C-MCCV proposed in this paper reduces the variance of state estimation, the estimated value is closer to the constraint region, and the effect is better.

6. **Conclusions.** In this paper, for linear equality constrained systems, different from the existing estimation methods to seek the optimal error covariance, from the perspective of optimal covariance, the minimum variance estimate with equality constraints based on matrix-convex-combination is proposed. Then we compare the statistical properties of C-MCCV to that of C-KP. Finally, we conclude that compared with C-KP, C-MCCV reduces the covariance of state estimate but increases the error covariance. Meanwhile, it is verified that when the state estimate of linear equality constrained system eventually converges to the constrained region, the equality constrained system can be approximated as an unconstrained system in the constrained region, and the C-MCCV is equivalent to C-KP. The minimum variance estimate with equality constraints based on matrix-convex-combination proposed in this paper provides a new direction for the state estimation of constrained systems. In practical engineering, it can solve the problem that there are two or more optimal estimations simultaneously and has important application in multi-sensor data fusion and system identification [24].

The estimation algorithm proposed in this paper is based on linear systems and equality state constraints, while nonlinear systems and inequality constraints are not studied. In practical applications, nonlinear systems and inequality constraints are widespread [25], and there are some cases where multiple nonlinear constraints or hybrid linear and nonlinear constraints are imposed. Meanwhile, multi-sensor data fusion is widely used in various fields, and multi-sensor can further improve the accuracy of constraint estimation in theory. It is of interest to further extend the algorithm in this paper to nonlinear constraints and multi-sensor dynamic systems.

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Appendix A. The Proof of Theorem 2.1. Considering that the expectation of $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ satisfy $E\left(\hat{\mathbf{X}}_1\right) = E\left(\hat{\mathbf{X}}_2\right) = E(\mathbf{X})$, then the expectation of $\hat{\mathbf{X}}$ satisfies

$$E\left(\hat{\mathbf{X}}\right) = \mathbf{A}E\left(\hat{\mathbf{X}}_1\right) + (\mathbf{I} - \mathbf{A})\,E\left(\hat{\mathbf{X}}_2\right) = \mathbf{A}E(\mathbf{X}) + (\mathbf{I} - \mathbf{A})\,E(\mathbf{X}) = E(\mathbf{X}) \tag{51}$$

So $\hat{\mathbf{X}}$ is an unbiased estimate of state variable $\mathbf{X}$.

The variance of $\hat{\mathbf{X}}$ can be given by

$$Cov\left(\hat{\mathbf{X}}\right) = \mathbf{A}Cov\left(\hat{\mathbf{X}}_1\right)\mathbf{A}^{\mathrm{T}} + (\mathbf{I} - \mathbf{A})\,Cov\left(\hat{\mathbf{X}}_2\right)\left(\mathbf{I} - \mathbf{A}^{\mathrm{T}}\right) \tag{52}$$

Derivative of matrix $\mathbf{A}$ on both sides gives

$$\frac{d\left[tra\left(Cov\left(\hat{\mathbf{X}}\right)\right)\right]}{d\mathbf{A}} = 2\mathbf{A}\left[Cov\left(\hat{\mathbf{X}}_1\right) + Cov\left(\hat{\mathbf{X}}_2\right)\right] - 2Cov\left(\hat{\mathbf{X}}_2\right) \tag{53}$$

When $\mathbf{A} = Cov\left(\hat{\mathbf{X}}_2\right)\left[Cov\left(\hat{\mathbf{X}}_1\right) + Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1}$, the variance of fusion estimation is minimum. If we replace $\mathbf{A}$ with $Cov\left(\hat{\mathbf{X}}_2\right)\left[Cov\left(\hat{\mathbf{X}}_1\right) + Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1}$ in (52), after several lines of algebraic manipulation we obtain

$$\left[Cov\left(\hat{\mathbf{X}}\right)\right]^{-1} = \left[Cov\left(\hat{\mathbf{X}}_1\right)\right]^{-1} + \left[Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1} \tag{54}$$

Since $Cov\left(\hat{\mathbf{X}}_1\right)$ and $Cov\left(\hat{\mathbf{X}}_2\right)$ are positive definite matrices, we can see from Lemma 2.1 that

$$\begin{aligned} \left[\left[Cov\left(\hat{\mathbf{X}}_1\right)\right]^{-1} + \left[Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1}\right]^{-1} &< Cov\left(\hat{\mathbf{X}}_1\right) \\ \left[\left[Cov\left(\hat{\mathbf{X}}_1\right)\right]^{-1} + \left[Cov\left(\hat{\mathbf{X}}_2\right)\right]^{-1}\right]^{-1} &< Cov\left(\hat{\mathbf{X}}_2\right) \end{aligned} \tag{55}$$

Therefore, we conclude that $Cov\left(\hat{\mathbf{X}}\right) < \min\left[Cov\left(\hat{\mathbf{X}}_1\right), Cov\left(\hat{\mathbf{X}}_2\right)\right]$.