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CUBIC NON-POLYNOMIAL SPLINE APPROACH TO THE SOLUTION OF LANE-EMDEN EQUATIONS

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Abstract. *The aim of this paper is converting the nonlinear initial value problems Lane-Emden equations of the first and second kinds to Volterra integrodifferential equations. Uniqueness theorem is proved by iteration method. We propose a technique to remove the singular behavior of the problems and provide the high precision approximate solution by cubic non-polynomial spline method. For the validity and applicability of the suggested method, two illustrative examples are presented with MathCad 15 computation for numerical results.*

Keywords: Lane-Emden equations, Cubic non-polynomial spline, Volterra integrodifferential equations

1. **Introduction.** Singular initial value problems for Lane-Emden (L-E) equations occur in several models of mathematical physics which are modeled by means of the following equation:

$$y''(x) + \frac{\alpha}{x}y'(x) + f(x,y) = g(x), \quad 0 < x \le 1,\ \alpha \ge 0 \tag{1}$$

with the initial conditions

$$y(0) = a, \quad y'(0) = b \tag{2}$$

where a, b are constants, $f(x,y)$ is a continuous function and $g(x) \in C[0,1]$.

Many methods proposed solutions of L-E equations. For instance, in [1] Sasan and Hossein used moving least squares method (MLS), Volterra integral form overcomes their singular behavior at $x = 0$, and the MLS method leads to a satisfactory solution for the equation. In [2] He and Ji solved L-E by Taylors series approximately. In [3] Kycia presented the solution for the existence of analytic solutions for some generalized L-E equation in which endpoints are singularities. In [4] Ezquerro and Hernández-Verón obtained domains of existence and uniqueness solution of nonlinear Fredholm integral equations by Newton's method. In [5] Abdou et al. proposed Banach's fixed point theorem used to prove the existence and uniqueness solution of Volterra-Fredholm integral equation. In [6] Ameri and Nezhad proposed least squares approximation to approximate the solution of fuzzy linear Volterra integral equations.

For non-polynomial spline functions (NPSF), in [7] Kumar et al. developed numerical algorithm to compute the solution of second order boundary value problems. In [8] Salh and Mohammed generalized the non-polynomial spline function by fractional order to solve fractional boundary value problems numerically. In [9] Bushra and Khlefha concerned with the approximate solution of linear two-points boundary value problem using

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NPSF. In [10] Salh and Mohammed compared the results of the performance NPSF for the Bagley-Torvik equation with polynomial spline method. In [11], a mid-knot cubic NPSF was applied to obtaining the numerical solution of a system of second-order boundary value problems. In [12] Amin et al. presented finite difference solution of fractional order partial differential equations.

This paper applies the cubic non-polynomial spline method to the L-E Equation (1) with initial conditions (2) and is organized as follows. Section 2 presents some preliminaries of L-E equations which will be used later. Section 3 illustrates uniqueness theorem by using Laplace transform to convert L-E equation to Volterra integral equation. Section 4 is focused on the proposed method. Then in Section 5 applications for first kind and second kind L-E equations are implemented, and in Section 6 the results are computed. Finally, conclusion is drawn in Section 7.

2. Preliminaries [13].

The L-E equation comes in two kinds, namely first kind L-E equation:

$$y'' + \frac{k}{x}y' + y^m = 0, \quad y(0) = 1,\ y'(0) = 0,\ k > 1 \tag{3}$$

where there is m degree of non-polynomial part, which appears in astrophysics and used for computing the structure of interiors of polytropic stars, where exact solutions exist for $m = 0, 1, 5$, and second kind L-E equation:

$$y'' + \frac{k}{x}y' + e^{y(x)} = 0, \quad y(0) = y'(0) = 0,\ k > 1 \tag{4}$$

The generalized L-E equation of factor $k > 1$ is

$$y'' + \frac{k}{x}y' + f(y) = 0, \quad y(0) = \alpha,\ y'(0) = 0,\ k > 1 \tag{5}$$

where $f(y)$ can take y^m or $e^{y(x)}$ as given earlier.

Remark 2.1. *The main difficulty of these two equations is the singular behavior that occurs at $x = 0$ where the L-E was converted to an equivalent Volterra integral form. To remove this singularity, we find y'' depending on y''' as will illustrate in Section 5.*

3. Uniqueness Theorem.

Convert the initial value problem Equation (5) to an equivalent Volterra integral equation by Laplace transform:

$$\mathcal{L}(y'') + \mathcal{L}\left(\frac{k}{x}y'\right) + \mathcal{L}(f(y)) = \mathcal{L}(0)$$

$$s^2Y(s) - sy(0) - y'(0) + \mathcal{L}\left(\frac{k}{x}y'\right) + \mathcal{L}(f(y)) = \mathcal{L}(0)$$

$$Y(s) = \frac{\alpha}{s} - \frac{1}{s^2}\mathcal{L}\left(\frac{k}{x}y'\right) - \frac{1}{s^2}\mathcal{L}(f(y))$$

by convolution property of Laplace transform, hence

$$y(x) = \alpha - k\int_0^x (x-t)\frac{y'(t)}{t}dt - \int_0^x (x-t)f(y(t))dt \tag{6}$$

Let $k = 1$

$$\begin{aligned} y(x) &= \alpha - \int_0^x (x-t)\frac{y'(t)}{t}dt - \int_0^x \left(\frac{x-t}{t}\right) tf(y(t))dt \\ &= \alpha - \int_0^x \left(\frac{x-t}{t}\right)(y'(t) - tf(y(t)))dt \end{aligned}$$

Let $\delta(x,t,y(t)) = y'(t) - tf(y(t))$

$$y(x) = \alpha - \int_0^x \left(\frac{x-t}{t}\right)\delta(x,t,y(t))dt \tag{7}$$

Theorem 3.1. *Consider the integral equation (7) and assume that $\delta(x,t,y(t))$ is a continuous function for $0 < t < x \le T$, and satisfies the Lipschitz condition. Then Equation (7) has a unique continuous solution.*

Proof: In Equation (7), replace $y(x)$ with $y_n(x)$ and another solution $\hat{y}(x)$ with $y_{n-1}(x)$

$$y_n(x) = \alpha - \int_0^x \left(\frac{x-t}{t}\right)\delta(x,t,y_{n-1}(t))dt, \quad n > 1$$

Subtracting from Equation (7) a similar equation with n replaced by $n-1$, leads to the following

$$y_n(x) - y_{n-1}(x) = \int_0^x \left(\frac{x-t}{t}\right)\delta(x,t,y_{n-1}(t)) - \delta(x,t,y_{n-2}(t))dt$$

Since $\delta(x,t,y(t))$ satisfies Lipschitz condition

$$|y_n(x) - y_{n-1}(x)| \le L\int_0^x |y_{n-1}(t) - y_{n-2}(t)|\left(\frac{x-t}{t}\right)dt$$

Let $\varphi_n(x) = y_n(x) - y_{n-1}(x)$, $n = 1,2,3,\dots$. Then

$$|\varphi_n(x)| \le L\int_0^x |\varphi_{n-1}(x)|\left(\frac{x-t}{t}\right)dt \tag{8}$$

Since $|\varphi_n(x)|$ is bounded, say by β, relation (8) implies that

$$|\varphi_n(x)| \le \beta L\int_0^x \left(\frac{x-t}{t}\right)dt \le \beta L(x\ln(x) - x)$$

By repeating the argument n-times, it would be obtained

$$|\varphi_n(x)| \le \beta L^n \int_0^x\int_0^x \dots \int_0^x \left(\frac{x-t}{t}\right)dtdt\dots dt \le \beta L^n\left(\frac{x^n}{n!}\ln(x) - \frac{a_1x^n}{a_2n!}\right)$$

where a_1, a_2 are constants, such that $a_1 < a_2$.

For any n, obviously:

$$\lim_{n\to\infty}\beta L^n\left(\frac{x^n}{n!}\ln(x) - \frac{a_1x^n}{a_2n!}\right) \to 0$$

which implies that $y(x) = \hat{y}(x)$, $0 \le x \le T$. This completes the proof.

4. **Cubic Non-Polynomial Spline Method [14].**

Cubic NPSF method has been proposed depending on a parameter $k > 0$, when $k \to 0$ reduces to polynomial cubic spline. Due to the parameter k, the numerical solutions obtained by non-polynomial splines in the literature are observed to be more accurate than that computed by polynomial splines.

Definition 4.1. *The cubic non-polynomial spline $s(x)$ with parameter $k > 0$ has the span$\{1, x, \sin kx, \cos kx\}$ defined on $\Omega : a = x_0 < x_1 < \dots < x_n = b$, with $x_j = a + jh$, $0 \le j \le n$, where $h = \frac{b-a}{n}$.*

Restriction $s_j(x)$ on $[x_{j-1}, x_j]$, $1 \le j \le n-1$ satisfies

$$s_j(x) = a_j \sin k(x - x_j) + b_j \cos k(x - x_j) + c_j(x - x_j) + d_j \tag{9}$$

where a_j, b_j, c_j, and d_j are constants and k is frequency of the trigonometric functions which will be used to raise the accuracy of the method.

Differentiate (9) three times with respect to x, and then we get

$$s'_j(x) = a_j k \cos k(x - x_j) - b_j k \sin k(x - x_j) + c_j \tag{10}$$

$$s''_j(x) = -a_j k^2 \sin k(x - x_j) - b_j k^2 \cos k(x - x_j) \tag{11}$$

$$s'''_j(x) = -a_j k^3 \cos k(x - x_j) + b_j k^3 \sin k(x - x_j) \tag{12}$$

for $x = x_j$, in Equations (9)-(12) we have

$$s_j(x_j) = b_j + d_j \approx y(x_j) \tag{13}$$

$$s'_j(x_j) = a_j k + c_j \approx y'(x_j) \tag{14}$$

$$s''_j(x_j) = -b_j k^2 \approx y''(x_j) \tag{15}$$

$$s'''_j(x_j) = -a_j k^3 \approx y'''(x_j) \tag{16}$$

where the values of $y(x_j)$, $y'(x_j)$, $y''(x_j)$ and $y'''(x_j)$ will be computed in Section 5 for first kind L-E equation and second kind L-E equation respectively. Then we can obtain the values of a_j, b_j, c_j, and d_j, for $j = 0, 1, \ldots, n$ as follows:

$$a_j = -\frac{1}{k^3} s'''_j(x_j) \approx -\frac{1}{k^3} y'''(x_j) \tag{17}$$

$$b_j = -\frac{1}{k^2} s''_j(x_j) \approx -\frac{1}{k^2} y''(x_j) \tag{18}$$

$$c_j = s'_j(x_j) + \frac{1}{k^2} s'''_j(x_j) \approx y'(x_j) + \frac{1}{k^2} y'''(x_j) \tag{19}$$

$$d_j = s_j(x_j) + \frac{1}{k^2} s''_j(x_j) \approx y(x_j) + \frac{1}{k^2} y''(x_j) \tag{20}$$

Then, we approximate the solution of the L-E Equations (3) and (4) using cubic nonpolynomial spline (9) with its coefficients (17)-(20).

5. **Applications.**

The results are computed and compared with the exact solutions for Lane-Emden equation of the first kind and with Adomian decomposition method for Lane-Emden equation of the second kind at grade points. All calculations are implemented by MathCad 15.

5.1. **First kind L-E equation.**

The first kind L-E equation of index $m = 5$, has the form:

$$y'' + \frac{2}{x} y' + y^5 = 0, \quad y(0) = 1,\ y'(0) = 0 \tag{21}$$

with exact solution [13]

$$e(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}$$

Let $k = 2$, $\alpha = 1$, and $f(y(t)) = (y(t))^5$, substitute in (6), then

$$y(x) = 1 - 2\int_0^x (x - t)\frac{y'(t)}{t} dt - \int_0^x (x - t)(y(t))^5 dt \tag{22}$$

The derivatives of $y(x)$ by Libenze formula [15], we have

$$y'(x) = -2\int_0^x \frac{y'(t)}{t} dt - \int_0^x (y(t))^5 dt \tag{23}$$

$$y''(x) = \frac{-2}{x} y'(x) - (y(x))^5$$

To get $y''(x)$ without singular at $x = 0$ find $y'''(x)$

$xy''(x) = -2y'(x) - x(y(x))^5$

$$xy'''(x) + y''(x) = -2y''(x) - 5x\left((y(x))^4 y'(x)\right) - (y(x))^5$$

$$y''(x) = \frac{-1}{3}\left(xy'''(x) + 5x\left((y(x))^4 y'(x)\right) + (y(x))^5\right) \tag{24}$$

$$y'''(x) = \frac{-1}{4}\left(xy^{(4)}(x) + 5x\left((y(x))^4 y''(x) + 4(y'(x))^2 (y(x))^3\right) + 10(y(x))^4 y'(x)\right) \tag{25}$$

Compute (22)-(25) at $x_j = a + jh$ and $h = 0.2$, with intervals $[x_{j-1}, x_j]$, $1 \le j \le n-1$ to find $s(x)$ with coefficients a_j, b_j, c_j, and d_j illustrated in (17)-(20) then the approximate solution of (22) is

$$s(x) = \begin{cases} 0.33334\cos x + 0.66667 \\ -0.06622\sin(x-0.2) + 0.32669\cos(x-0.2) + 0.66667 \\ -0.12981\sin(x-0.4) + 0.30702\cos(x-0.4) + 0.66667 \\ -0.18821\sin(x-0.6) + 0.27511\cos(x-0.6) + 0.66667 \\ -0.23912\sin(x-0.8) + 0.23224\cos(x-0.8) + 0.66667 \end{cases}$$

5.2. **Second kind L-E equation.** The second kind L-E equation has the form:

$$y'' + \frac{2}{x}y' + e^{y(x)} = 0, \quad y(0) = y'(0) = 0 \tag{26}$$

with Adomian decomposition solution, [13]:

$$ADM(x) = \frac{1}{5!}x^4 - \frac{1}{3!}x^2 - \frac{8}{3(7!)}x^6 + \frac{122}{9(9!)}x^8 - \frac{5032}{45(11!)}x^{10}$$

Let $k = 2$, $\alpha = 1$ and $f(y) = e^{y(x)}$, substitute in (6), then

$$y(x) = -2\int_0^x (x-t)\frac{y'(t)}{t}dt - \int_0^x (x-t)e^{y(t)}dt \tag{27}$$

The derivatives of $y(x)$ by Libenze formula [15], we have

$$y'(x) = -2\int_0^x \frac{y'(t)}{t}dt - \int_0^x e^{y(x)}dt \tag{28}$$

$$y''(x) = \frac{-2}{x}y'(x) - e^{y(x)}$$

To get $y''(x)$ without singular at $x = 0$ find $y'''(x)$

$$xy''(x) = -2y'(x) - xe^{y(x)}$$

$$xy'''(x) + y''(x) = -2y''(x) - xe^{y(x)}y'(x) - e^{y(x)}$$

$$y''(x) = \frac{-1}{3}\left(xy'''(x) + xy'(x)e^{y(x)} + e^{y(x)}\right) \tag{29}$$

$$y'''(x) = \frac{-1}{3}\left(xy^{(4)}(x) + y'''(x) + y'(x)e^{y(x)} + x\left[e^{y(x)}y''(x) + (y'(x))^2 e^{y(x)}\right] + e^{y(x)}y'(x)\right)$$

$$y'''(x) = \frac{-1}{4}\left(xy^{(4)}(x) + x\left[e^{y(x)}y''(x) + (y'(x))^2 e^{y(x)}\right] + 2e^{y(x)}y'(x)\right) \tag{30}$$

Compute (27)-(30) at $x_j = a + jh$ and $h = 0.2$, with $[x_{j-1}, x_j]$, $1 \le j \le n-1$ to find $s(x)$ with coefficients a_j, b_j, c_j, and d_j illustrated in (17)-(20) then the approximate solution of (27) is

$$s(x) = \begin{cases} 0.33334\cos(x-0) + 0.33334 \\ -0.06622\sin(x-0.2) + 0.32669\cos(x-0.2) + 0.33334 \\ -0.12981\sin(x-0.4) + 0.30702\cos(x-0.4) + 0.33334 \\ -0.18821\sin(x-0.6) + 0.27511\cos(x-0.6) + 0.33334 \\ -0.23912\sin(x-0.8) + 0.23224\cos(x-0.8) + 0.33334 \end{cases}$$

6. **The Results.** In Table 1 a comparison between the exact solution $e(x)$ and the approximate non-polynomial spline solution $s(x)$ for first kind L-E equation is presented with absolute error $|e(x) - s(x)|$. In Table 2 the comparison between Adomian decomposition method (ADM) and $s(x)$ for second kind L-E equation is found, with absolute error $|ADM(x) - s(x)|$. In Table 3 for different step size h and n given in Definition 4.1, we show the root mean square (RMS) error which is obtained by the following [16]

$$RMS = \sqrt{\frac{1}{n}\sum_{j=0}^{n}[e(x_j) - s(x_j)]^2}$$

To illustrate the results graphically, this method is efficient that it gives approximations of high accuracy. In Figure 1 we plot $e(x)$ and $s(x)$ for first kind L-E equation. In Figure 2, we plot ADM and $s(x)$ for second kind L-E equation. Figures 3 and 4 plot RMS error.

From tables and figures we can find that the numerical solutions are approached to the exact solutions with higher accuracy and much smaller absolute error.

TABLE 1. Results for first kind L-E equation

x	$e(x)$	$s(x)$	$\|e(x) - s(x)\|$
0.0	1	1	0
0.1	0.998	0.998	2.767×10^{-6}
0.2	0.993	0.993	2.767×10^{-5}
0.3	0.985	0.985	2.171×10^{-4}
0.4	0.974	0.974	6.677×10^{-4}
0.5	0.961	0.959	1.575×10^{-3}
0.6	0.945	0.942	3.133×10^{-3}
0.7	0.922	0.922	5.531×10^{-3}
0.8	0.908	0.899	8.939×10^{-3}
0.9	0.887	0.874	0.013
1.0	0.866	0.847	0.019

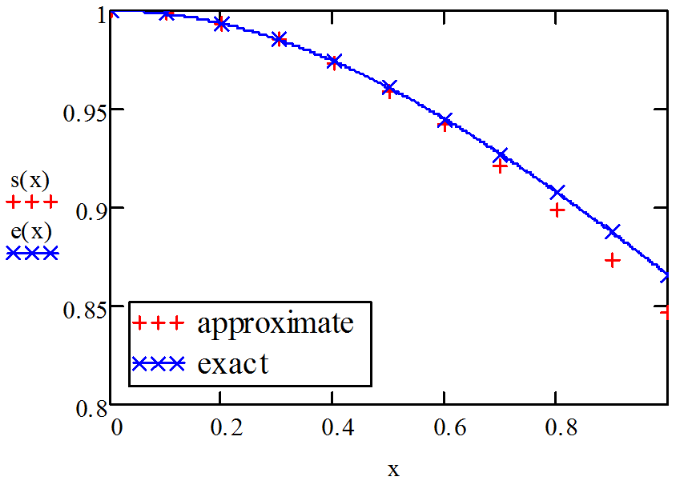

FIGURE 1. Exact and approximate solution for first kind L-E equation

TABLE 2. Results for second kind L-E equation

x	$ADM(x)$	$s(x)$	$\|ADM(x) - s(x)\|$
0.0	0	0	0
0.1	-1.666×10^{-3}	-1.666×10^{-3}	5.556×10^{-7}
0.2	-6.653×10^{-3}	-6.644×10^{-3}	8.893×10^{-6}
0.3	-0.015	-0.015	4.505×10^{-5}
0.4	-0.026	-0.026	1.425×10^{-4}
0.5	-0.041	-0.041	3.481×10^{-4}
0.6	-0.059	-0.058	7.226×10^{-4}
0.7	-0.08	-0.078	1.34×10^{-3}
0.8	-0.103	-0.101	2.288×10^{-3}
0.9	-0.13	-0.126	3.669×10^{-3}
1.0	-0.159	-0.153	5.595×10^{-3}

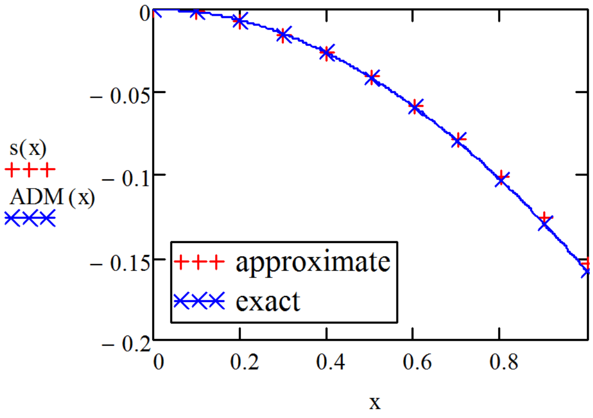

FIGURE 2. ADM and approximate solution for second kind L-E equation

TABLE 3. The RMS error for different values of n and step size h

n	h	RMS error of first kind L-E equation	RMS error of second kind L-E equation
4	1/3	0.5	0.272
6	0.2	8.766×10^{-3}	2.852×10^{-3}
10	0.1	7.84×10^{-3}	2.184×10^{-3}

7. Conclusion.

Laplace transform is used to convert singularly initial value L-E equation (5) to an equivalent Volterra integral form. Equation (7) has unique solution under some conditions. Numerical technique for singularly initial value first and second kinds L-E problem using cubic non-polynomial spline functions is derived. The method obtains acceptable solution as in Tables 1 and 2 and Figures 1 and 2. Compared with Adomain method, cubic non-polynomial spline is straightforward with simple solution process and accurate results. RMS error estimated for different steps size h and n is illustrated in Table 3 and Figures 3 and 4.

As future work the nonlinear polynomial spline can be used for singular fractional differential equations.

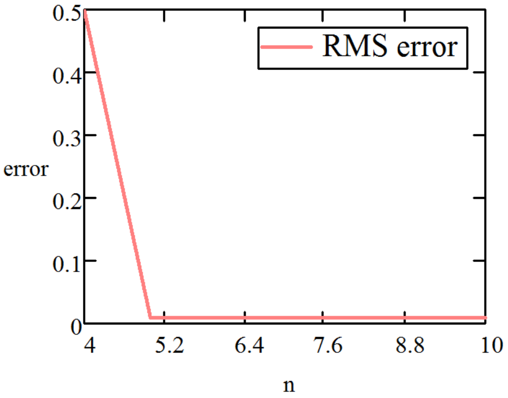

FIGURE 3. RMS error of first kind

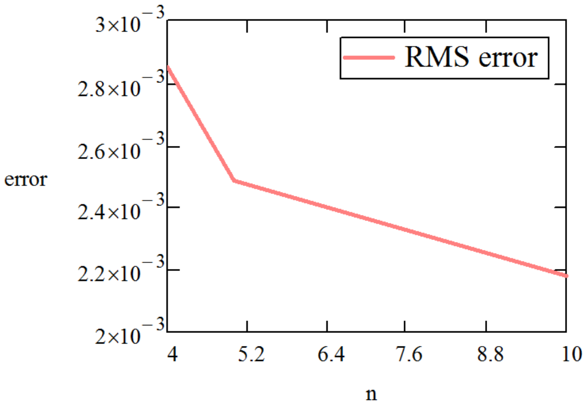

FIGURE 4. RMS error of second kind

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