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OPTIMIZING SALES AND PROFIT PROCESSES USING IMPULSE CONTROL

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Abstract. *The target mathematical model comprises a sales process and a profit process. The profit process uses the throughput function derived from the sales process as an input into the profit process. The model of sales process is finally described by a log-normal stochastic differential equation. We present that the theory of impulse control can be used for the problem of optimization for the maximum benefit of the whole sales process and profit process. We are introducing a utility function that takes account of the costs involved throughout the process. We use dynamic programming as a method of optimizing this utility function by the smooth pasting technique in finance. Moreover, numerical examples are provided for verification of this paper. One is the value of optimal impulse control, and the other is the comparison of sales for the production flow systems (Testrun1 and Testrun2, 3).*
Keywords: Throughput function, Optimum impulse control theory, Smooth pasting technique, Utility function

1. **Introduction.** A motive of the present research that has led to promote such research during many years of experience in general industrial machine control equipment manufacturing business is as follows. There has been no discussion about the theory of optimum control that combines the sales process and the profit process in the equipment manufacturing company.

We proposed mathematical modeling, system condition estimation and optimization limited to the production process. One of the concerns in production systems is the delay issue. Delays are due to lead times, inventory shortages, logistics. In the production business, cooperation with external companies is essential. What is often discussed is how to maintain good cooperation. SCM (supply chain management, hereinafter referred to as SCM) has been applied to production systems to solve these issues [1, 2, 3, 4, 5]. SCM studies commonly develop mathematical models based on game theory. In addition, many studies explore SCM from the perspective of management engineering [6]. SCM systems address many aspects of development, production, distribution, and sales, depending on the particular nature of the industry.

In our previous study, we proposed a mathematical model for a thermal reaction process of external heating equipment. The new control system design for this process, which treats a heat source flowing model for an externally attached device is proposed. The

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equation of a distributed parameter system as a coupled system with the heat reaction process is presented [8]. We also proposed that the target control system can be configured using the control parameter of the overall heat exchange coefficient (OHEC), which is given using a linear approximation from BPDE to an ordinary differential equation (ODE). The numerical simulation results ware represented for the optimal control system, and the gradient method is used in this simulation. Our findings showed that this study is suitable for possible practical systems [9].

In our previous study, we simulate a small-to-midsize firm without sufficient working capital to continue operations. Therefore, we need to raise working capital from financial institutions. Here, we call this cash flow. In essence, the rate of return (RoR) is at least proportional to the production lead time. In other words, if RoR forms a log-normal distribution, it is realistic to assume that the cash flow will also have the same log-normal distribution [10]. With regard to equipment manufacturing, a small-to-midsize firm is required company president's guarantee for a borrowing inevitably. Therefore, whether a value of manufacturing equipment after a repayment of a loan varies relative to a repayment period or not is reported. Therefore, the value of the manufacturing equipment after repayment of the loan fluctuates depending on the repayment period, so a prior simulation is needed.

The previous research applying Fluid mechanics that the trial production of a new concept vertical take-off and landing rotorcraft of flexible kite wing attached multicopter is very interesting [11]. Regarding system identification and predictive model control, they noted with interest that there are two distinct disciplines, but there is a gap between these two topics. To fill this gap between them, they also reported in this paper that system identification and model predictive control are combined into a single iterative identification and model predictive control strategy [12].

Previously, we have reported that by creating a state in which the production density of each process corresponds to physical propagation, the production process is most appropriately described using a diffusion equation [13]. In other words, if the potential of the production field (stochastic field) is minimized, the equation is defined by the production density function $S_i(t, x)$ and the constraint is described using an advective diffusion equation to determine the transportation speed ρ [13].

To enable efficient application to a production system, we have proposed a mathematical model that focuses on the selection process and production lead time adaptation mechanism. To model the throughput time for a production demand/production system in the production stage, the dynamic behavior is derived using a lognormal stochastic differential equation. Using this model, the evaluation equation for the compatibility condition production lead time is defined using the risk-neutral integral, and the evaluation formula for the above conditions is calculated. Furthermore, by performing the synchronization process, the throughput for the production process is reduced [5].

This study employs a stochastic BPDE (bilinear partial differential equation) and verifies the existence of a unique solution to the model by using BPDE [7]. This study is an extension of a previous work that applied deadline scheduling in the production process to external supplier companies (hereafter referred to as suppliers) [5]. The goal of our study is to increase production throughput and consequently reduce production costs and increase profit. As the sales process and the profit process become increasingly complex and are subjected to various dynamic forces, suppliers also become more complex. Thus, it is important to consider these factors when developing a supplier system.

The summary of this paper is as follows. After the exit from the selling process, which is the production process based on raw materials, it is an input to the profit process. Throughput in the production system is generated as a control function. The model is expressed as a stochastic first-order bilinear PDE. The general solution of this model has just one solution, and we have confirmed the ordinary differentiation in the presence of ordinary orthogonal functions. In addition, we develop a profit generation model on the outlet side that is looked for in a certain engineering system. At this time, it is also found that if the above throughput is used as a control function, a lognormal stochastic differential equation can be obtained by a simple transformation. It is pointed out that the theory of optimum impulse control can be applied to this model. Moreover, constants are used as parameters of the throughput function, which is a control function. Given the general time function, we believe that non-linear theory or optimum bilinear theory can be applied. Based on mathematical and physical understandings of production engineering, we are conducting research aimed at establishing an academic area called mathematical production engineering. As our business size is a small-to-medium-sized enterprise, human intervention constitutes a significant part of the production process, and profit can sometimes be greatly affected by human behavior. Therefore, when considering human intervention from outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of such intervention.

The most important points are as follows.

1) The input in the sales process comes from the function which generates the end profit under the impulse control.
2) Profit is generated in the profit process by considering the cost of the output of the sales process. Moreover, on the basis of this profit, it contributes raw materials to the sales process.
3) We introduce a utility function for the entire sales and profit process. The entire system is optimized using impulse control through the utility function.
4) To verify the proposal presented in this paper, numerical examples of the sales process and profit process are provided.

To the best of our knowledge, applying a stochastic BPDE to a supplier model has not yet been proposed.

If financial institutions guarantee repayment of a small-to-midsize firm with an equipment manufacturing period, the company can avoid a default risk completely if it pays amount of payment. Because all money is paid back to the debt guarantor of a company (the company president in the case of a small-to-midsize firm), the appraisal value of such credit obligation is equal to the amount of the loan. On the other hand, asset value of manufacturing equipment means a remaining value after subtracting a repaid money amount from a cash flow.

2. **Production Firm of a Small or Medium Enterprise and Production System.**

The company in this study is the "supplier" in Figure 1 and "factory" here. Companies assume that N (number of) suppliers exist; however, this study focuses on one company since no data is published for the rest of the company $(N-1)$. Then a production process which is called a production flow process is shown in Figure 2. The production process, which produces small quantities of a wide variety of products, goes through several stages of the production process. In Figure 2, the processes consist of six stages. Every S1-S6 step of the production process produces materials.

S1 to S6 perform work stages 1 to 6 on the production line in Figure 2. They are S1-S6 in Tables 4, 6 and 8 in Appendix A. K1-K9 in the table are nine workers. Figure 2 represents a production process called a production flow system, which is a production method used in the production of control equipment. The production flow system, which

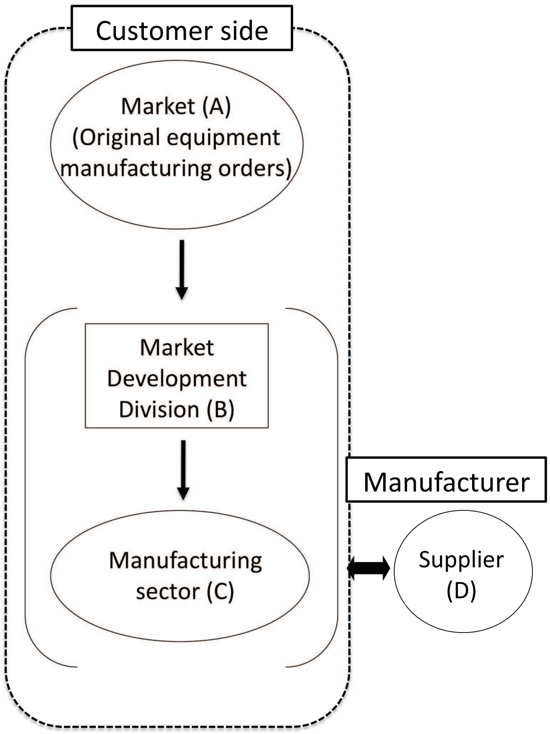

FIGURE 1. Business structure of company of research target

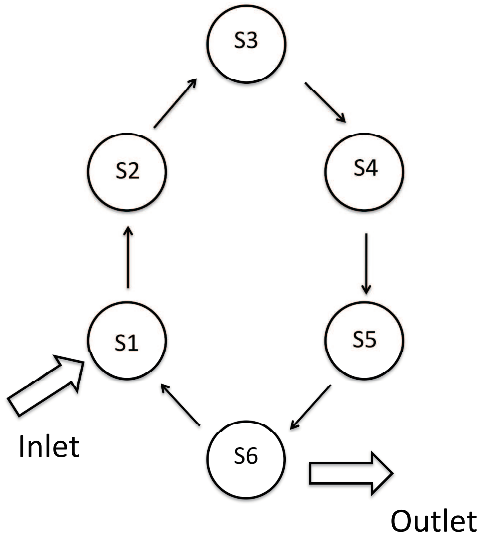

FIGURE 2. Production flow process

in this case has six stages, is commercialized by the production of material in steps S1-S6 of the production process.

The direction of the deflection is the direction of the production flow. With this system, production materials are supplied from the inlet and the final product will be shipped from the outlet.

Assumption 2.1. *The production structure is nonlinear.*

Assumption 2.2. *The production structure is a closed structure; that is, the production is driven by a cyclic system (production flow system).*

- Reasonability of Assumption 2.1. Assumption 2.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the rate of return generation structure in a stochastic production process (hereafter called the production field). Because such a structure is at least dependent on demand, it is considered to have a non-linear structure. Since the value of such a commodity is dependent on the rate of return, its production structure is non-linear. As a result, Assumption 2.1 reflects the realistic production structure and is somewhat valid.
- Reasonability of Assumption 2.2. Assumption 2.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption 2.2 is reasonable because production starts from S1. For more detailed analysis, please refer to our Appendix A.

3. Impulse Control to Evaluate the Entire Sales and Profit Process.

3.1. Mathematics model of the sales and profit process.

We report that the production process is defined by a lognormal stochastic differential equation by continuing from our previous studies. We have mapped the evaluation equation for optimal impulse control of the reference by Professor Ohnishi to the sales process and the profit process [17]. This verification is verified by the validity in numerical simulation. Here is a new idea we have proposed.

Now, the one-dimensional stochastic filtration probability space is defined as follows.

$$\left[\hat{\Omega}, \hat{\mathcal{F}}, \hat{\boldsymbol{P}}, \left(\hat{\mathcal{F}}(t); t \in \mathbf{R}_+\right)\right] \tag{1}$$

We bring in the variable $\hat{X}^x(t)$, which is the total amount of the production capital of the enterprise that has already been accumulated, and this is the next stochastic differential equation.

$$\hat{X}^x(t); \quad t \in \mathbf{R}_+ \tag{2}$$

$$d\hat{X}^x(t) = \mu dt + \sigma dB(t), \quad 0 \leq t \leq T^x \tag{3}$$

$$\hat{X}^x(t) = 0, \quad t > T^x \tag{4}$$

where $x \in \mathbf{R}_+$, μ, σ and $B(t)$ represent the original production capital (production equipment, labour), a drift factor, volatility and standard Brownian movement respectively.

T^x is given as follows:

$$T^x = \inf\{t \in \mathbf{R}_+ : X^x(t) \in \mathbf{R}_- := (-\infty, 0]\} \tag{5}$$

where T^x is the default time. The default is that profits continue to be negative and go bankrupt. If financial institutions guarantee repayment of a small-to-midsize firm with an equipment manufacturing period, the company can avoid a default risk completely if it pays amount of payment. Because all money is paid back to the debt guarantor of a company (the company president in the case of a small-to-midsize firm), the appraisal value of such credit obligation is equal to the amount of the loan. Next, Figure 3 consists of a sales process and a profit generation process. The sales process is a bilinear stochastic differential equation model with a delay [7, 8]. Another is the process of generating profits. The major expenditures here are variable costs except fixed costs. Figure 3 shows the structure that generates profits using the capital $\hat{X}^x(t)$ of such a company. The delay distribution system is the system in which a two-variable $X^{x,\delta}(t, d)$ function with a delay element $d \in [0, L]$ and a time element $t \in [0, T^x]$ is used as the state amount.

$$\frac{\partial X^{x,\delta}(t, d)}{\partial t} + \rho \frac{\partial X^{x,\delta}(t, d)}{\partial d} = W(t)\left\{X^{x,\delta}(t, d) - X^0(t)\right\} + \xi(t, d) \tag{6}$$

where ρ, $X^0(t)$ and $\xi(t, d)$ are a velocity, an initial value and the time delay term of supplier respectively. Equation (6) is defined as the following $2D$ filtration probability space.

$$\{\Omega, \mathcal{F}, P(\mathcal{F}(t, d));\ (t, d) \in \mathbf{R}_{++}\} \tag{7}$$

where $X^{x,\delta}(t, d)$ is the following.

$$\left[X^{x,\delta}(t, d);\ (t, d) \in \mathbf{R}_{++}\right] \tag{8}$$

where $x \in \mathbf{R}_+$ is the initial distribution volume.

This is the initial and boundary state for Equation (6) respectively.

$$X^{x,\delta}(t, \cdot) = h_i(\cdot) \tag{9}$$

$$\left.\frac{\partial X^{x,\delta}(t, d)}{\partial d}\right|_{d=L} = 0 \tag{10}$$

where $h_i(\cdot)$ is the amount of capital input for production.

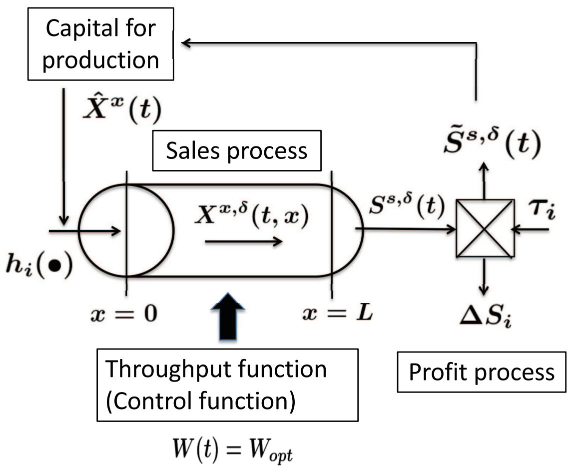

FIGURE 3. Sales and profit processes

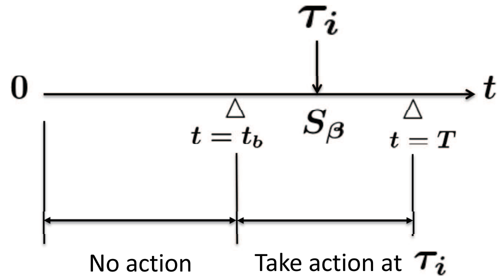

FIGURE 4. Timing to perform impulse control

At that moment, when $\xi(t,d)=0$ in Equation (6), the following equation is derived from the theory of first-order ordinary derivative equations.

$$\frac{dX^{x,\delta}(t,L)}{dt} = W(t)\left[X^{x,\delta}(t,L) - X^0(t)\right] \tag{11}$$

where $\{W(t), t \in \mathbf{R}_+\}$ is the throughput function.

In this case, if $S^{s,\delta}(t)$ is defined by Equation (12), $S^{s,\delta}(t)$ has a stochastic structure as in Equations (1) to (5).

Definition 3.1.

$$S^{s,\delta}(t) = X^{x,\delta}(t,L) - h_i(\cdot) \tag{12}$$

Here, when considering the dependency of the state as a mathematical model of $S^{s,\delta}$, the following equation is obtained [7]. That is, $X^{x,\delta}(t,L)$ and $S^{s,\delta}(t)$ can be treated by "Equivalence by strict linearization" in control theory from Equation (12).

$$\frac{dS^{s,\delta}(t)}{S^{s,\delta}(t)} = W_{opt}dt + \sigma d\xi^s(t) \tag{13}$$

where W_{opt}, σ and $\xi^s(t)$ are the optimum value of production throughput, a volatility and a standard Brownian movement respectively.

3.2. **Optimization of the utility function using impulsive control.** The temporal sequence of $S^{s,\delta}$ is indicated in Figure 4. $\{\tau_i, i=1,2,\ldots\}$ in Figure 4 is an arbitrary time point up to $T^x(<\infty)$ and is the time at which the profit is generated. In other words, $\Delta S_i \in [0, S^{s,\delta}(\tau_i^-)]$, which is a part of $S^{s,\delta}$, is the profit. Therefore, the model of Equation (13) is shown in Figure 3.

Definition 3.2. *Optimal strategy* δ

$$\delta = (\tau_i, \Delta S_i), \quad i=1,2,\ldots \tag{14}$$

δ is the pair of an interference time τ_i during profit generation and its quantity ΔS_i.

Here, with the aid of the process $\tilde{S}^{s,\delta}(t)$, the next equation is obtained according to Equation (13).

$$d\tilde{S}^{s,\delta}(t) = W_{opt}\tilde{S}^{s,\delta}(t)dt + \sigma\tilde{S}^{s,\delta}(t)dB^s(t) - dZ^{s,\delta}(t) \tag{15}$$

where $Z^{s,\delta}(t)$, σ and $B^s(t)$ are a strategy for profit generation, a volatility and a standard Brownian movement respectively. $Z^{s,\delta}(t)$ is derived as follows.

$$Z^{s,\delta}(t) = \sum_{i=1}^{+\infty} \Delta_i \cdot \mathbf{1}_{(\tau_i<t)}, \quad t \in \mathbf{R}_+ \tag{16}$$

That is, $Z^{s,\delta}(t)$ is the cumulative amount of profit.

$$\tilde{S}^{s,\delta}(t) = S^{s,\delta}(\tau_i^-) - \Delta S_i, \quad \tau_i \leqq t \leqq \tau_{i+1} \tag{17}$$

Currently, the total output capital of the enterprise is given by the following equation.

$$\hat{X}^x(t) \equiv X^{x,\delta}(\tau_i) - \left(\tilde{S}^{s,\delta}(t) - h_i(\cdot)\right) \tag{18}$$

Equation (3) replaces the following.

$$dX^{x,\delta}(t) = \mu dt + \sigma dB(t) + \tilde{S}^{s,\delta}(t)dt - h_i(\cdot)dt \tag{19}$$

where $0 \leqq t \leqq T^x$.

Assumption 3.1. *Constraints for profit generation*

$$K(\Delta S_i) = m\Delta S_i - c_i \tag{20}$$

where m, $m\Delta S_i$ and c_i are respectively a parameter related profit, a profit and a cost argument.

The status variable $X^{x,\delta}(t)$ running through the sales process is as follows.

$$\begin{aligned} dX^{x,\delta}(t) &= \mu X^{x,\delta}dt + \sigma X^{x,\delta}dW(t) - h_i(\cdot) + [S^{x,\delta}(t) - \Delta S_i] \\ &= \mu X^{x,\delta}dt + \sigma X^{x,\delta}dW(t) - h_i(\cdot) + \tilde{S}^{x,\delta}(t) \end{aligned} \tag{21}$$

The business strategy consists of maximizing the cumulative amount of corporate sales. This paper is designed to locate the business strategy δ. The strategy δ is referred to as "Impulse control". Here, Equation (21) is related by the following equation.

$$0 \leqq t \leqq T^{x,\delta} \tag{22}$$

$$h^{x,\delta}(t) = 0, \quad T^{x,\delta} < t \tag{23}$$

$$T^{x,\delta} = \inf\{t \in \mathbf{R}_+ : \quad h^{x,\delta}(t) \in \mathbf{R}_-\} \tag{24}$$

Definition 3.3. *Optimal value function for the expected present value*

$$v(x) = \inf_{\delta\in\Delta} v^\delta(x), \quad x \in \mathbf{R}_+ \tag{25}$$

Now, in the interest of simplicity, we define the following equation as the process of accumulating profit (Figure 3) [17].

Definition 3.4. *Utility function $v^\delta(x)$ of the entire production process and profit process*

$$v^\delta(x) = E\left[\int_0^\infty e^{-rt}c\left(\hat{X}^{x,\delta}(t)\right)dt + \sum_{i=1}^{\infty} e^{-r\tau_i}K\left(X^{x,\delta}(\tau_i), \Delta X_i\right)\mathbf{1}_{(\tau_i<\infty)}\right] \tag{26}$$

where $c(x)$ is the operating cost rate to achieve $\hat{X}^{x,\delta}$, and K is the cost rate to generate profit. Further, $\mu > r$.

In other words, $X^{x,\delta}$ and $S^{s,\delta}$ can be dealt with as equivalent values in Equation (12). Therefore, the cost function of Equation (26) is significant due to the constraint of Equation (21). For an arbitrary function $u : \mathbf{R}_+ \to \mathbf{R}$, two types of operators are defined below.

Definition 3.5. *Utility function $v^{\delta}(x)$*

$$[Mu](x) := \inf_{\Delta x \in \mathbf{R} | x+\Delta x \in \mathbf{R}_+} \{(k|\Delta x| + K) + u(x + \Delta x)\} \tag{27}$$

$$[Nu](x) := \inf_{\tau \in \{\mathcal{F}(t)-stoppingtime\}} E\left[\int_0^t e^{-rt} c(X^x(t))dt + e^{-rt}[Mu](X^x(\tau^-))\right] \tag{28}$$

- The operator M of Equation (29) is the determination of the best interference action, assuming that interference is in progress.
- The operator N of Equation (30) corresponds to determining the optimum time for the next interference.

Assume that the optimum value function ($u : \mathbf{R}_+ \to \mathbf{R}$) is a class C^2 function. From the principle of optimum in Bellman's dynamic programming, the next equation holds [17].

$$\begin{aligned}
v(x) &= [Nu](x) && (29)\\
&= \inf_{\tau \in \Xi} E\left[\int_0^{\tau} e^{-rt} c(X^x(t))dt + e^{-r\tau}[Mv](X^x(\tau^-))\right] && (30)\\
&\leqq E\left[\int_0^{s} e^{-rt} c(X^x(t))dt + e^{-rs} v(X^x(s^-))\right] && (31)\\
&= v(x) + E\left[\int_0^{s} e^{-rt}\{[Lu](X^x(t))dt + c(X^x(t))\}dt\right.\\
&\quad \left. + \int_0^{s} e^{-rs} v'(X^x(t))\sigma(X^x(t))dW(t)\right] && (32)\\
&= v(x) + E\left[\int_0^{s} e^{-rs}\{[Lu](X^x(t)) + c(X^x(t))\}dt\right], \quad x \in \mathbf{R}_+, s \in \mathbf{R}_+ && (33)
\end{aligned}$$

where L is a differential operator defined by the following equation, and the fourth equation above takes the lemma of Ito.

Definition 3.6. *Operator L*

$$[Lu](x) := \lim_{t \downarrow 0} \frac{E[e^{-rt}u(X^x(t))] - u(x)}{t} = \frac{1}{2}\sigma^2 x^2 \frac{d^2 u(x)}{dt^2} + \mu x \frac{du(x)}{dt} - ru(x) \tag{34}$$

Starting at Inequality (31), if the optimal value function $u : \mathbf{R}_+ \to \mathbf{R}$ is assumed to be a class C^2 function, the following equation applies.

$$[Lu](x) + c(x) \geqq 0, \quad x \in \mathbf{R}_+ \tag{35}$$

Definition 3.7. *Quasi-Variational Inequality (QVI)*

$$u(x) \leqq [Mu](x), \quad x \in \mathbf{R}_+ \tag{36}$$

$$[Lu](x) + c(x) \geqq 0, \quad x \in \mathbf{R}_+ \tag{37}$$

$$\{u(x) - [Mu](x)\}\{[Lu](x) + c(x)\} = 0, \quad x \in \mathbf{R}_+ \tag{38}$$

All three conditions above are a quasi-variational inequality (QVI) for the optimal impulse control problem. In terms of Equation (38), one of Inequality (36) or (37) relates to the equation.

3.3. **Derivation of profits as a whole.** In general, analytically resolve QVI (Equations (36), (37) and (38)). It is difficult to resolve, even numerically. It is therefore useful to clarify a class which can find a nearly explicit solution using the structure of the problem. Hence, one effective principle or method is "Smooth pasting technique" [16, 17]. In Figure

4, when $t = (0, t_b)$ and $0 \leqq x \leqq S_b$, the impulse control is not executed and the system is the establishment that is earning. When $t = \tau_i$ and $x = S_\beta$, the impulse control is executed and the system is the establishment of profit $(0 < S_\beta < S_b)$.

It is described that the optimal impulse control δ^* is described by the following control rules using two parameters (t_b, S_b).

Here we define the following pre-conditions [16, 17]:

- The open interval $[0, t_b)$ and $0 \leqq x \leqq S_b$ is defined as the non-interfering region. That is, the impulse control is not executed and the system is the establishment that is earning.
- The half straight line $[t_b, \infty)$ is set as the interference region. That is, when $t = \tau_i$ and $x = S_\beta$, the impulse control is executed and the system is the establishment of profit.

The above-mentioned optimum impulse control function v satisfies the following conditions.

Lemma 3.1. *v satisfies the following second-order ODE.*

$$([Lv](x) + c(x)) = \frac{1}{2}\sigma^2 x^2 \frac{d^2 v(x)}{dx^2} + \mu x \frac{dv(x)}{dx} - rv(x) + c(x) = 0, \quad x \in (0, S_b) \tag{39}$$

Lemma 3.2. *Value matching conditions: Continuity of v*

$$v(S_b) = k|S_\beta - S_b| - c + v(S_\beta) = k(S_b - S_\beta) - c + v(S_\beta) \tag{40}$$

$$v'(S_\beta) = k \tag{41}$$

Lemma 3.3. *Smooth pasting conditions; A continuation of $dv(x)/dx$*

$$v'(S_b) = k \tag{42}$$

From the right hand of Equation (39), we get this characteristic equation.

$$\frac{1}{2}\sigma^2\lambda^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\lambda - r = 0 \tag{43}$$

Equation (43) has two real solutions.

$$\lambda_\pm := \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2} \tag{44}$$

If we request that the function value does not diverge as $x \downarrow 0_+$, then the general solution of the right hand of Equation (39) is

$$v(x; a) := a_+ e^{\lambda_+ x} + a_- e^{\lambda_- x}, \quad x \in [0, S_b] \tag{45}$$

where $a \in \mathbf{R}_{++}$ is the positive parameter to be determined.

When $x = 0$, $v(x) = 0$, the following equation is obtained from Equation (45).

$$a_+ + a_- = 0 \tag{46}$$

Therefore,

$$v(x : a) = a_+\left(e^{\lambda_+ x} - e^{\lambda_- x}\right) \tag{47}$$

Equation (47) becomes the following equation with the help of Equation (44).

$$v(x : a) = a_+ \left\{ \exp\left(-\frac{\hat{\mu}}{\sigma^2} x\right) \times \exp\left(\frac{\sqrt{\hat{\mu}^2 + 2\sigma^2 r}}{\sigma^2} x\right) - \exp\left(-\frac{\hat{\mu}}{\sigma^2} x\right) \times \exp\left(\frac{\sqrt{\hat{\mu}^2 + 2\sigma^2 r}}{\sigma^2} x\right) \right\}$$

$$= a_+ \exp\left(-\frac{\hat{\mu}}{\sigma^2}x\right)\left\{\exp\left(\frac{\sqrt{\hat{\mu}^2+2\sigma^2 r}}{\sigma^2}x\right) - \exp\left(-\frac{\sqrt{\hat{\mu}^2+2\sigma^2 r}}{\sigma^2}x\right)\right\}$$

$$= 2a_+ \exp(-\beta x)\left\{\frac{e^{\gamma x}-e^{-\gamma x}}{2}\right\}$$

$$= 2a_+ \exp(-\beta x)\sinh(\gamma x) \tag{48}$$

where β, γ and $\hat{\mu}$ are as follows:

$$\beta = \frac{\hat{\mu}}{\sigma^2}, \quad \gamma = \exp\left(\frac{\sqrt{\hat{\mu}^2+2\sigma^2 r}}{\sigma^2}\right), \quad \hat{\mu} = \mu - \frac{1}{2}\sigma^2 \tag{49}$$

From the above, given that S_b is the profit, when this value is reached, S_b can be executed at an appropriate time. However, the condition at the moment is $\mu > r$, that is, the trend coefficient of sales is larger than the interest rate r. The condition of this $\mu > r$ is that the above-mentioned strategy is a necessary condition for business continuity. Similarly, it is a break-even analysis for a new approach. Currently, the pairing requirement is as follows.

$$v(S_b) = k(S_b - S_\beta) - c + v(S_\beta) \tag{50}$$

where $v(S_b)$, $v(S_\beta)$, c and k are a cost in S_b, a cost in S_β, expenses and a parameter respectively. $k = 2a_+ \cosh(\gamma x)$

$$S_\beta = S_b - \frac{v(S_b) - v(S_\beta) + c}{k} \tag{51}$$

4. Numerical Examples.

4.1. Numerical example of sales, profit process and environmental parameters of management.

Figure 5 shows the model of sales process (Equation (13)). Figure 6 shows the model of profit process (Equation (16)). Equation (15) combines Equations (13) and (16). Equation (15) is the model which takes account of the entire sales and profit process. The following is the numerical example of these processes. The diagram used in Figures 7 to 16 is described. $v(S_\beta)$ is – – ■ – – (Sales). S_β is – – ▲ – – (Profit). $v(S_b) = 2a_+e^{-\beta x}\sinh(\gamma x)$ (Equation (48)) is – – ♦ – – (Cost in S_b).

Table 1. Graph symbol of Figures 7 to 16

Graph symbol	Meaning of graph
– – ■ – –	$v(S_\beta)$ (Cost in S_β)
– – ▲ – –	Profit of the entire system S_β (Profit) (Equation (51))
– – ♦ – –	$v(S_b) = 2a_+e^{-\beta x}\sinh(\gamma x)$ (Equation (48)) (Cost in S_b, $x \in [0, S_b]$)
x-axis	S_b (Sales)

Figures 7 to 9 are examples focusing on the combination of volatility (σ) and interest rate (r). Figures 10 to 12 are basically the same as Figures 7 to 9, and are examples focusing on the combination of volatility and interest rate. Figures 13 to 15 are examples focusing on trend value (μ) and interest rates. Figure 16 is an example of $\mu < r$. Please refer to below for details.

- With respect to Figures 7 to 9:

  Figures 7 to 9 are the graphs of sales generated in the sales process. Figure 7 is great volatility and Figure 9 is fixed at low volatility. The greater the volatility, the greater the sales to generate profit. In other word, we need to increase sales. In addition, the value of $v(S_b) = S_\beta$, that is, the gap between the cost of generating the sales and profit becomes large.

- With respect to Figures 10 to 12:

  Similar to Figures 7 to 9, Figures 10 to 12 are another graphs associated with the sale. As the interest rate (r) increases, the value of $v_b = S_\beta$ increases as in above item.
- With respect to Figures 13 to 15:

  Figures 13 to 15 are graphs of the profit generated from the profit process. When $t = \tau_i$ is achieved, it enters the process of profit and makes a profit. The profit is given from Equation (16). It shows that the value of $V_\beta = S_\beta$, that is, the cost of the genealogy of a profit increase like $\mu \approx r$. $v(S_b) = 2a_+ \exp(-\beta x)\sinh(\gamma x)$, however, from Equation (16), $v(S_\beta) = 2a_+ \exp(-\beta x)$. In addition, the larger the volatility, the higher the cost.
- With respect to Figure 16:

  If $\mu < r$ (does not meet business continuity requirements), there is no point in increasing the profit if the volatility is significant and the cost is high [$v(S_b) \neq S_\beta$]. The cost of generating sales is always higher than the cost of generating profit.

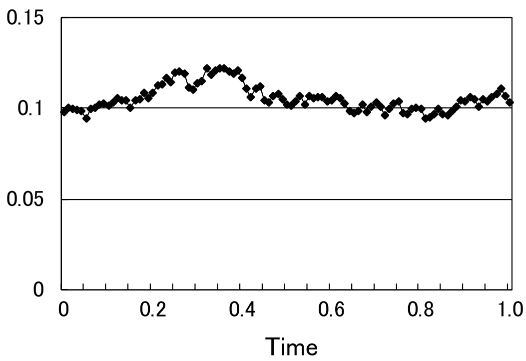

FIGURE 5. Process of sales for production

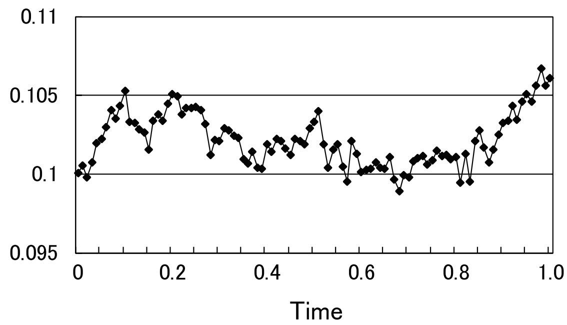

FIGURE 6. Process of profit for production

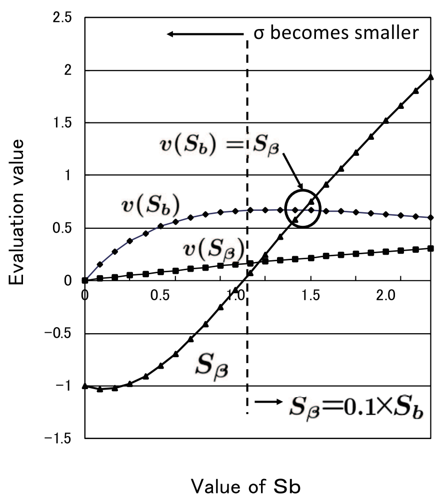

FIGURE 7. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

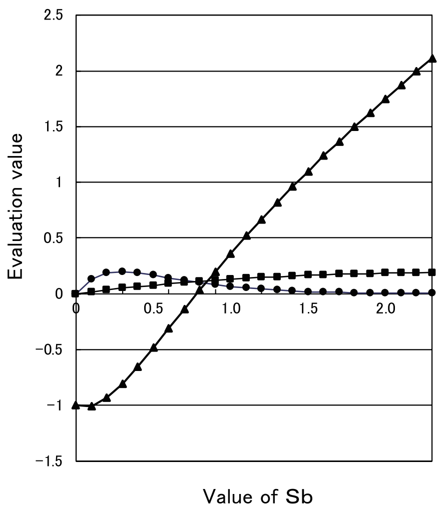

FIGURE 8. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

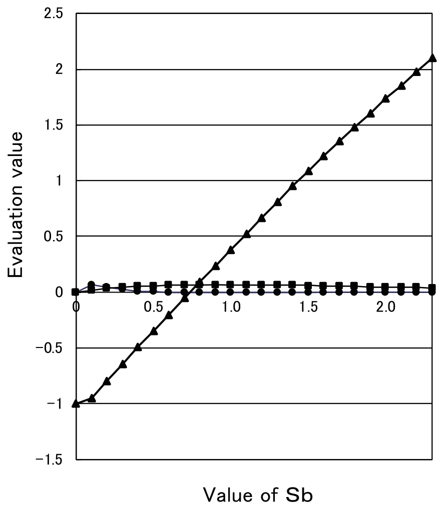

FIGURE 9. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

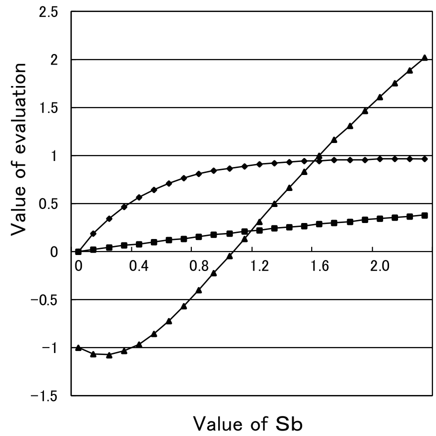

FIGURE 10. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

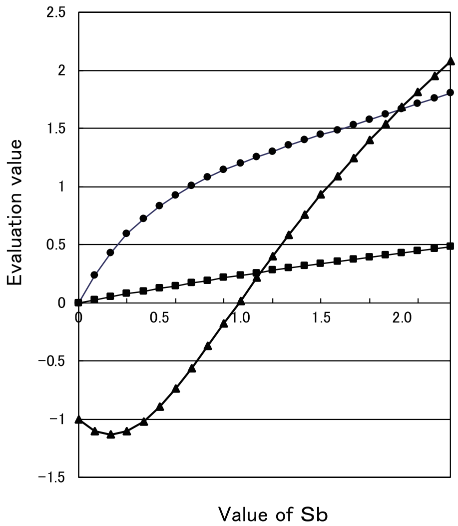

FIGURE 11. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

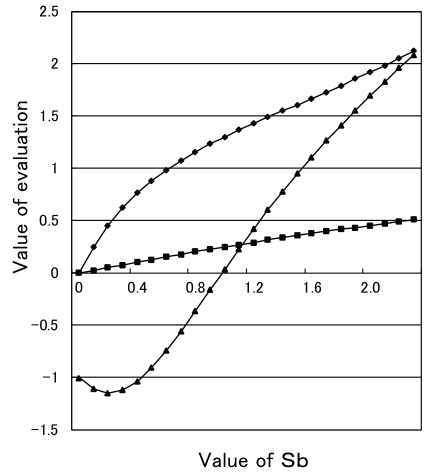

FIGURE 12. Profit evaluation under impulse control where $t \in [0, t_b]$ and $x \in [0, S_b]$

Based on the above findings, the relationship between the sales and the profit has been clarified. In addition, the relationship between interest rates (r), volatility (σ), business trends (μ) and spending (c), which are environmental parameters of management, and the associated costs, has been clarified.

As described above, after the exit from the selling process, which is the production process based on raw materials, it is an input to the profit process. Throughput in the sales process is generated as a control function. The mathematical model is expressed as a stochastic first-order bilinear PDE. The general solution of this model has just one solution, and we have confirmed the ordinary differentiation in the presence of ordinary orthogonal functions. In addition, we developed a profit process on the outlet side that was looked for in a certain engineering system. At this time, it was also found that if the above throughput is used as a control function, a lognormal stochastic differential

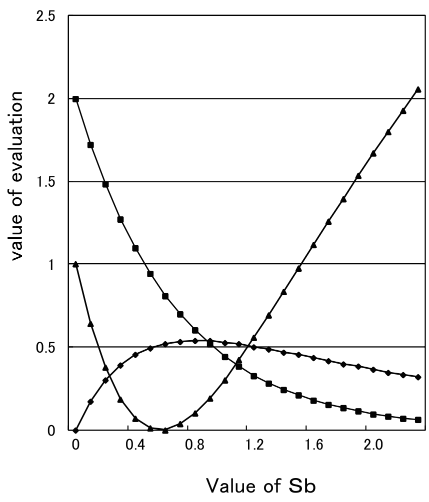

FIGURE 13. Profit evaluation under impulse control where $t = \tau_i$, $i = 1, 2, \ldots$ and $x = S_\beta$

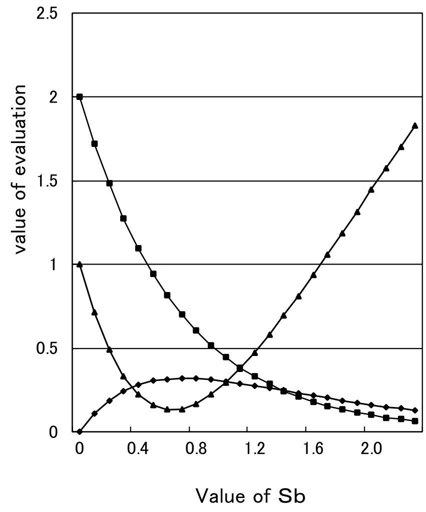

FIGURE 14. Profit evaluation under impulse control where $t = \tau_i$, $i = 1, 2, \ldots$ and $x = S_\beta$

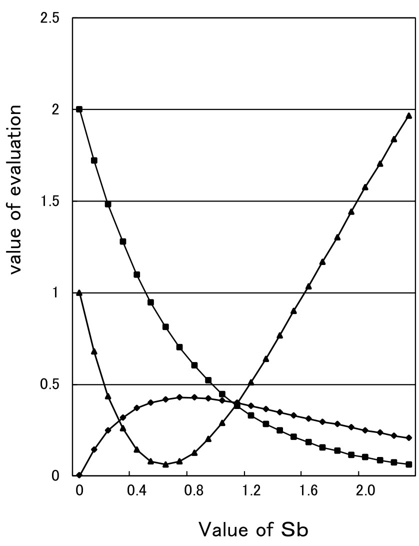

FIGURE 15. Profit evaluation under impulse control where $t = \tau_i$, $i = 1, 2, \ldots$ and $x = S_\beta$

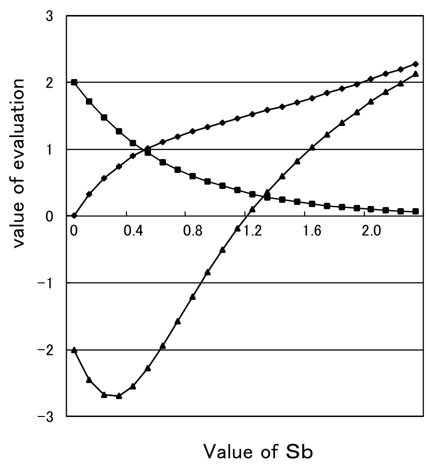

FIGURE 16. Cost exceed sales in the event of $\mu < r$

equation can be obtained by a simple transformation. It was pointed out that the theory of optimum impulse control can be applied to this model.

In this study, constants are used as parameters of the throughput function, which is a control function. Given the general time function, we believe that non-linear theory or optimum bilinear theory can be applied.

4.2. **Numerical example based on the production flow system data.** Here, the following data are adopted. Figure 17 is the cost evaluation curve of Testrun1. The total cumulative cost is 0.172. Each of Testrun2 and Testrun3 has a combined cost of 0.00003, which is near zero, so the figure is omitted.

The cost evaluation value (Figure 17) for $x \in [0, S_b]$ is calculated from Equation (48). In other word, $v(x : a) \equiv v(S_b)$ is an optimal value function that minimizes cost. That

TABLE 2. Value of sales and profit evaluation parameters

Figure number	μ	r	σ	c
Figure 7	1	0.2	0.8	1
Figure 8	1	0.2	0.5	1
Figure 9	1	0.5	0.8	1
Figure 10	1	0.3	0.3	1
Figure 11	1	1	0.8	1
Figure 12	1	1.15	0.8	1
Figure 13	1	0.5	0.5	1
Figure 14	0.51	0.5	0.5	1
Figure 15	0.51	0.5	0.8	1
Figure 16	1	1.5	1.04	4

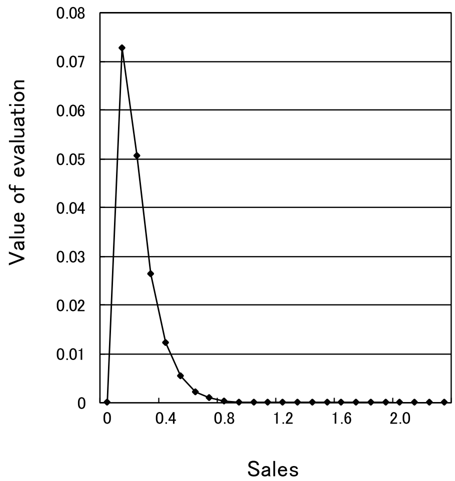

FIGURE 17. Evaluation curve of costs ($x \in S_b$)

is, in Figure 17, it can be seen that the larger μ and the smaller σ, the smaller the cost $v(S_b)$.

Figures 18 to 19 correspond to $- - \blacktriangle - -$ mark diagrams of Figure 7 to Figure 12, and these figures correspond to the trend value and the volatility value corresponding to Testrun1 and Testrun2, 3 (Table 3). In Figure 18 of Testrun1, sales $S_b = 1.6$ are required to achieve the evaluation value $S_\beta = 1$. In Figure 19 of Testrun2, 3, sales $S_b = 1.5$ are required to achieve the evaluation value $S_\beta = 1$. In other word, Testrun2 and 3 may make more profit with fewer sales than Testrun1. This makes Testrun2 and 3 more effective than Testrun1.

5. **Conclusion.** The materials entering the sales process are entered into the profit process. Throughput in the sales process is generated as a control function. We introduced the utility function to obtain a profit in its entirety. The entire process is optimized using impulse control under the utility function. At this time, the sales process is defined through a stochastic differential equation with lognormal type. In this study, constants are used as parameters of the throughput function, which is a control function. Given the general time function, we believe that non-linear theory or optimum bilinear theory can be applied. Using this paper, the following can be learned from the actual system.

- Business strategy can formulate a strategy (impulse control) to maximize the cumulative amount of business profits.

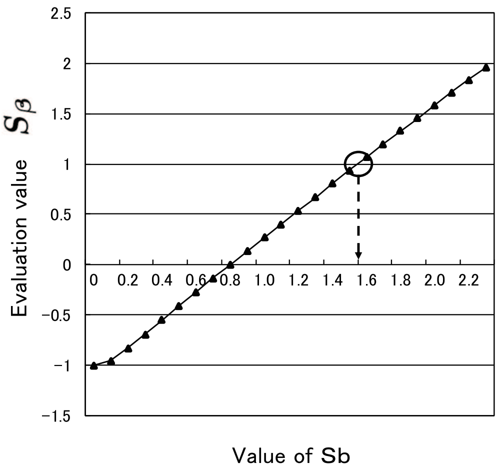

FIGURE 18. Relationship between valuation value S_β and sales S_b in Testrun1

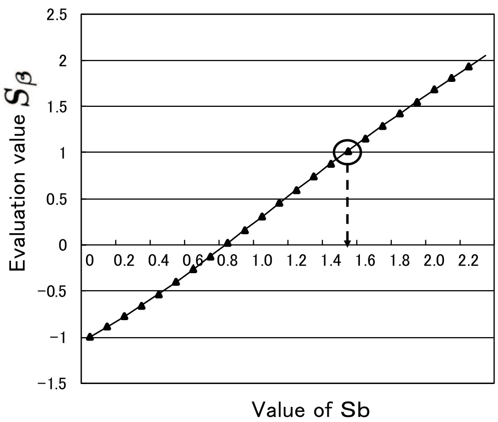

FIGURE 19. Relationship between valuation value S_β and sales S_b in Testrun2, 3

TABLE 3. Relationship of sales with profit

Name of Testrun	Average	Volatility	Total cumulative cost	Figure (− − ▲ − −)
Testrun1	0.7	0.3	0.172	Figure 18
Testrun2 and 3	0.9	0.1	0.00003	Figure 19

- The required workflow costs such as production costs can be captured, and a strict quantification of the business becomes possible.
- Business manager can get an entry point to know the business continuity requirements.

In the future, we used a predefined constant value as the throughput function, but we would like to simulate it as a general time function. In addition, we would like to examine whether non-linear theory in control theory or optimal bilinear theory can be applied.

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Appendix A. **Analysis of Actual Data in the Production Flow System.** Based on the control equipment, the product can be manufactured in one cycle. The production throughput required to maintain 6 pieces of equipment/day is as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25\,(\text{min}) \tag{52}$$

where the throughput of the previous process is set as 20 (min). In Equation (52), "28" represents the throughput of the previous process plus the idle time for synchronization. "8" is the number of processes and the total number of all processes is "8" plus the previous process. "60" is given by 20 (min) × 3 (cycles).

One process throughput (20 min) in full synchronization is

$$T_s = 3 \times 120 + 40 = 400\,(\text{min}) \tag{53}$$

Therefore, a throughput reduction of about 10% can be achieved. However, the time between processes involves some asynchronous idle time.

As a result, the above Testrun is as follows.

- (Testrun1): Each throughput in every process (S1-S6) is asynchronous, and its process throughput is asynchronous. Table 4 represents the production time (min) in each process. Table 5 represents the variance in each process performed by workers. Table 4 represents the target time, and the theoretical throughput is given by $3 \times 199 + 2 \times 15 = 627$ (min).

  In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. Figure 20 is a graph illustrating the measurement data in Table 4, and it represents the total working time for each worker (K1-K9). The graph in Figure 21 represents the variance data for each working time in Table 4.

- (Testrun2): Set to synchronously process the throughput.

  The target time in Table 6 is 500 (min), and the theoretical throughput (not including the synchronized idle time) is 400 (min). Table 7 represents the variance data of each working process (S1-S6) for each worker (K1-K9).

- (Testrun3): The process throughput is performed synchronously with the reclassification of the process. The theoretical throughput (not including the synchronized idle time) is 400 (min) in Table 8.

  Table 9 represents the variance data of Table 8. "WS" in the measurement tables represents the standard working time. This is an empirical value obtained from long-term experiments.

TABLE 4. Total production time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

TABLE 5. Volatility of Table 4

	S1	S2	S3	S4	S5	S6
K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

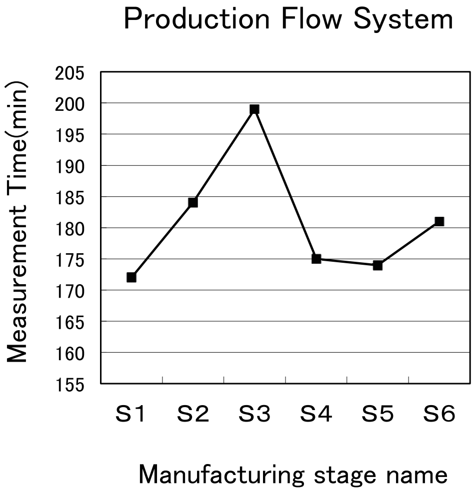

FIGURE 20. Total work time for each stage (S1-S6) in Table 4

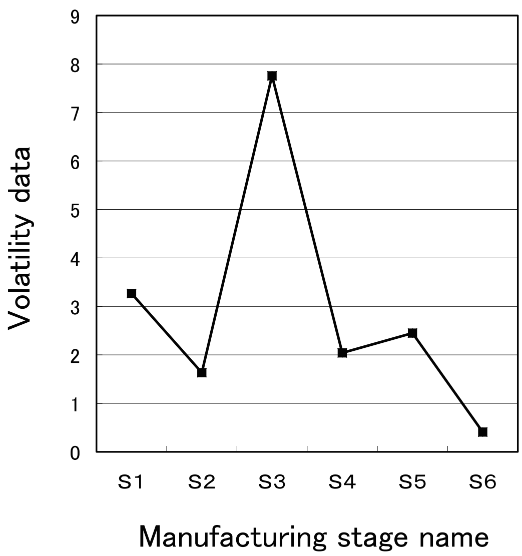

FIGURE 21. Volatility data for each stage (S1-S6) in Table 4

Table 6. Total production time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

Table 7. Volatility of Table 6

	S1	S2	S3	S4	S5	S6
K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

Table 8. Total production time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	21	21	21	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	22	22	20	20	20	20
K9	20	25	25	25	20	20	20
Total	180	165	164	161	180	180	180

Table 9. Variance of Table 8

	S1	S2	S3	S4	S5	S6
K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.33	3	3	0	0	0
K5	1.33	1.33	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.33	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0

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