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SOME NEW EXISTENCE AND UNIQUENESS RESULTS OF CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

Khawlah Hashim Hussain

Department of Mechanical Technology
Basra Technical Institute
Southern Technical University
AL-Basrah 61004, Iraq
khawlah.hussain@stu.edu.iq

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Abstract. *Over recent years, researchers have been paying attention to the newly discovered fractional operators involving nonsingular kernels. The Caputo fractional derivative is one of these operators which has captured the interest of scientists the most because of the many interesting results reported when this derivative is used in modelling some real-world phenomena. However, the theory of these operators is still to be addressed. In this paper, we establish some new conditions for the existence and uniqueness results of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with fractional integral boundary conditions. The desired results are proved by using fixed point theorems due to Banach and Krasnoselskii in Banach spaces. Finally, an example is given to illustrate the results, the example given on this work establishes the precision and efficiency of the proposed technique and shows that the problem has a solution and this solution is a unique solution.*

Keywords: Volterra-Fredholm integro-differential equation, Caputo fractional derivative, Fractional integral boundary conditions, Fixed point method

1. **Introduction.** Fractional calculus is originated form classical mathematics and more accurate for real life phenomena. These phenomena are applied in media, mechanics, electrical engineering, viscoelasticity, control, electromagnetism, etc., as well as in finance, rheology, geology and biology. Fractional systems have the advantages to describe the processes associated with previous history [1, 2, 3, 4, 5].

Several researchers described that the fractional integral and derivative are suitable for modelling to define the memory and hereditary properties of different substances or system and other real world problem. Many types of fractional operators and definition are obtained. This fact enables the researches to pick up the most convenient fractional derivative for the sake of achieving better results in modeling the real world problem under consideration. Integro-differential equations with nonlocal conditions have attracted the attention of many researchers in the last decades [6, 7, 8, 9, 10], because of their applications in numerous fields of science, engineering, physics, economy and so on.

In the last years, with the development of theorems of fractional integro-differential equations, many authors investigated the existence of solutions of abstract fractional integro-differential equations with nonlocal conditions by using semigroups theorems, solution operator theorems and the relation between solution operators and semigroups constructing by probability density functions as well as fixed point techniques [11, 12, 13, 14].

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Recently, Furati et al. [15] established the existence and uniqueness of solutions for the problem:

$$
\begin{aligned}
&{}^{c}D_{a^+}^{\nu,\rho;\psi}u(t) = f(t,u(t)),\ t \in J = (a,b],\ 0<\nu<1,\ 0\le\rho\le 1,\\
&I_{a^+}^{1-\gamma}u(a) = d,\ \nu\le\gamma=\nu+\rho-\nu\rho.
\end{aligned}
$$

Vivek et al. [16] studied that some new existence and uniqueness results are derived by means of the contraction mapping principle and Schaefer's fixed point theorem. Further, they discussed the Ulam-Hyers stability for a ψ-fractional boundary value problem given by

$$
\begin{aligned}
&{}^{c}D_{0^+}^{\nu;\psi}u(t) = f(t,u(t)),\ t\in J=[0,T],\ 0<\nu<1,\\
&au(0)+bu(T)=d,\ a+b\neq 0,\ a,b,d\in\mathbb{R}.
\end{aligned}
$$

Sousa et al. [17] investigated the Ulam-Hyers stability of the solution by means of the Banach fixed-point theorem in the Sobolev space. Besides that, they introduced the concept of α-resolvent and presented, as an example, Ulam-Hyers stability for nonlinear fractional differential and Volterra integro-differential equations involving ψ-Hilfer derivative given by

$$
\begin{aligned}
&{}^{H}D^{\nu,\rho;\psi}u(t) = f(t,u(t)) + \int_0^t k(t,s,u(t))ds,\\
&I_{0^+}^{1-\gamma}u(0) = d.
\end{aligned}
$$

Duraisamy et al. [18] established the subsistence and uniqueness of solutions for fractional integro-differential equations with fractional integral boundary conditions of the form

$$
\begin{aligned}
&{}^{c}D^{\nu}u(t) = f\left(t,u(t),\int_0^t g(t,s,u(s))ds\right),\ t\in J:=[0,1],\ 1<\nu\le 2,\\
&\gamma_1 u(0)+\eta_1 I^{\varsigma}u(t)|_{t=0}=\delta_1,\\
&\gamma_2 u(1)+\eta_2 I^{\varsigma}u(t)|_{t=1}=\delta_2,\ 0<\varsigma<1.
\end{aligned}
$$

Motivated by the above works, we will study a more general problem of fractional integro-differential equations which is called Caputo fractional Volterra-Fredholm integro-differential equations with fractional integral boundary conditions of the form

$$
{}^{c}D^{\nu}u(t) = f(t,u(t),Ku(t),Hu(t)),\ t\in J:=[0,1],\ 1<\nu\le 2, \tag{1}
$$

$$
\begin{aligned}
&\gamma_1 u(0)+\eta_1 I^{\varsigma}u(t)|_{t=0}=\delta_1,\\
&\gamma_2 u(1)+\eta_2 I^{\varsigma}u(t)|_{t=1}=\delta_2,\ 0<\varsigma<1,
\end{aligned} \tag{2}
$$

where ${}^{c}D^{\nu}$ is the Caputo fractional imitative of regulate ν, I^{ς} is the R-L fractional integral of order ς, and γ_i, η_i, δ_i $(i=1,2)$ are authentic constants. The functions $f: J\times X\times X\times X$, $Ku(t)=\int_0^t k(t,s,u(s))ds$ and $Hu(t)=\int_0^1 h(t,s,u(s))ds$, $k,h: J\times J\times X$ are uninterrupted functions. At this juncture, $(X,\|.\|)$ is a Banach space and $C(J,X)$ designates the Banach space of all continuous utility from $[0,1]\longrightarrow X$ artistic with a topology of identical junction with the norm denoted by $\|u\|=\sup_{t\in J}|u(t)|$.

In this paper, we establish some new conditions for the existence and uniqueness results of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with fractional integral boundary conditions. The desired results are proved by using fixed point theorems due to Banach and Krasnoselskii in Banach spaces. The results presented in this paper extend the main results in [17, 18].

The paper is organized as follows. Section 2 presents, as preliminaries, the definition of the fractional derivative, the fractional integral of Riemann-Liouville with respect to another function, and some important results, given as theorems, as well as the spaces in which such operators and theorems are defined. In Section 3, we use the fixed point theorems due to Banach and Krasnoselskii to prove the existence and uniqueness results for the problem (1)-(2). The application of our main results is established in Section 4. Concluding remarks close the paper in Section 5.

2. **Preliminaries.** The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [4, 5, 19, 20, 21, 22, 23, 24, 25, 26, 27].

Definition 2.1. *(Riemann-Liouville fractional integral [4]). The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $u \in C([0,T])$ is defined as*

$$J^{\nu}_{0^+}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds,$$

where Γ denotes the Gamma function.

Definition 2.2. *(Caputo fractional derivative [4]). The fractional derivative of $u(t)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD^{\nu}_{0^+}u(t) &= J^{m-\nu}_{0^+}D^m u(t) \\ &= \begin{cases} \dfrac{1}{\Gamma(m-\nu)}\displaystyle\int_0^t (t-s)^{m-\nu-1}\frac{\partial^m u(s)}{\partial s^m}ds, & m-1<\nu<m, \\ \dfrac{\partial^m u(t)}{\partial t^m}, & \nu = m,\ m \in N, \end{cases} \end{aligned} \tag{3}$$

where the parameter ν is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive ν will be considered.

Hence, we have the following properties:

1) $J^{\nu}_{0^+}J^{v}u = J^{\nu+v}_{0^+}u$, $\nu, v > 0$.

2) $J^{\nu}_{0^+}u^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)}u^{\beta+\nu}$.

3) $D^{\nu}_{0^+}u^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)}u^{\beta-\nu}$, $\nu > 0$, $\beta > -1$.

4) $J^{\nu}_{0^+}D^{\nu}_{0^+}u(t) = u(t) - u(a)$, $0 < \nu < 1$.

5) $J^{\nu}_{0^+}D^{\nu}_{0^+}u(t) = u(t) - \displaystyle\sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{(t-a)^k}{k!}$, $t > 0$.

Definition 2.3. *(Riemann-Liouville fractional derivative [4]). The Riemann-Liouville fractional derivative of order $\nu > 0$ is normally defined as*

$$D^{\nu}_{0^+}u(t) = D^m_{0^+}J^{m-\nu}_{0^+}u(t), \quad m-1<\nu\le m, \quad m \in \mathbb{N}. \tag{4}$$

Theorem 2.1. *(Banach's fixed point theorem [5]). Let (X,d) be a nonempty complete metric space with $T : X \longrightarrow X$ as a contraction mapping. Then map T has a fixed point $x^* \in X$ such that $Tx^* = x^*$.*

Theorem 2.2. *(Krasnoselskii fixed point theorem [5]). Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*

1) $Ax + By \in M$ whenever $x, y \in M$.

2) A is compact and continuous.

3) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.1. *Assume that f is a continuous function. If $u \in C(J, X)$, then u satisfies the problem*

$$\begin{aligned} &{}^cD^{\nu}u(t) = f(t),\ 1 < \nu \leq 2,\ t \in [0,1],\\ &\gamma_1 u(0) + \eta_1 I^{\varsigma}u(t)|_{t=0} = \delta_1,\\ &\gamma_2 u(1) + \eta_2 I^{\varsigma}u(t)|_{t=1} = \delta_2,\ 0 < \varsigma < 1, \end{aligned}$$

if and only if u satisfies the integral equation

$$u(t) = I^{\nu}f(t) + \frac{\delta_1}{\gamma_1} + w_1 t\left(\left[\delta_2 - \frac{\delta_1}{w_2\gamma_1}\right] - \left[\gamma_2 I^{\nu}f(1) + \eta_2 I^{\nu+\varsigma}f(1)\right]\right).$$

3. **Existence and Uniqueness of Solutions.** In this section, we shall give existence and uniqueness results of Equation (1), with the conditions (2). Before starting and proving the main results, we introduce the following hypotheses.

(A1) There exist constants L_f, L_k and $L_h > 0$, $\forall t \in J$, $u_i, v_i, y_i \in C(J, X)$, $i = 1, 2$, such that

$$\begin{aligned} &|f(t,u_1,v_1,y_1) - f(t,u_2,v_2,y_2)| \leq L_f[|u_1-u_2| + |v_1-v_2| + |y_1-y_2|],\\ &|k(t,s,u_1) - k(t,s,u_2)| \leq L_k|u_1-u_2|,\\ &|h(t,s,u_1) - h(t,s,u_2)| \leq L_h|u_1-u_2|. \end{aligned}$$

(A2) There exist functions $N_1 \in L^1(J, X^+)$ such that

$$|f(t,u,v,y)| \leq N_1(t)\Psi(\|u\|),\ (t,u,v,y) \in J \times X^3,$$

$\Psi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous non-decreasing function.

Lemma 3.1. *Let $1 < \nu \leq 2$. Assume that $f(., u(.), Ku(.), Hu(.)) \in C(J, X)$. If $u \in C(J, X)$, then u satisfies the problem (1)-(2) if and only if u satisfies the equation*

$$\begin{aligned} u(t) = {}& I^{\nu}f(s,u(s),Ku(s),Hu(s))(t) + \frac{\delta_1}{\gamma_1} + w_1 t\Bigg(\left[\delta_2 - \frac{\delta_1}{w_2\gamma_1}\right] \qquad (5)\\ &- \left[\gamma_2 I^{\nu}f(s,u(s),Ku(s),Hu(s))(1) + \eta_2 I^{\nu+\varsigma}f(s,u(s),Ku(s),Hu(s))(1)\right]\Bigg),\ t \in J. \end{aligned}$$

In the consequence, we exploit the subsequent assertion:

$$I^{\nu}f(s,u(s))(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s,u(s))ds,\ t \in J.$$

Theorem 3.1. *Assume that the hypotheses (A1) and (A2) are fulfilled, and if*

$$\Omega := L_f\left[\frac{|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}\right] + (L_k + L_h)\left[\frac{|w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)}\right] < 1,$$

then the problem (1)-(2) has at least one solution on J.

Proof: Consider the operator $T : C(J, X) \longrightarrow C(J, X)$ defined by

$$\begin{aligned} (Tu)(t) = {}& I^{\nu}f(s,u(s),Ku(s),Hu(s))(t) + \frac{\delta_1}{\gamma_1} + w_1 t\Bigg(\left[\delta_2 - \frac{\delta_1}{w_2\gamma_1}\right] \qquad (6)\\ &- \left[\gamma_2 I^{\nu}f(s,u(s),Ku(s),Hu(s))(1) + \eta_2 I^{\nu+\varsigma}f(s,u(s),Ku(s),Hu(s))(1)\right]\Bigg). \end{aligned}$$

Now, we define two operators T_1 and T_2. From Equation (6) we have

$$\begin{aligned}(T_1u)(t) &= I^\nu f(s,u(s),Ku(s),Hu(s))(t),\\ (T_2u)(t) &= \frac{\delta_1}{\gamma_1} + w_1 t - \big[\gamma_2 I^\nu f(s,u(s),Ku(s),Hu(s))(1)\\ &\qquad + \eta_2 I^{\nu+\varsigma} f(s,u(s),Ku(s),Hu(s))(1)\big],\ t\in J.\end{aligned}$$

We fix

$$\tau \geq \|l\|\Psi(\tau)\left[\frac{1+|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}\right] + \frac{|\delta_1|}{|\gamma_1|} + |w_1|\left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|}\right].$$

$B_\tau = \{u \in C(J,X) : \|u\| \leq \tau\}$. For $u, v \in B_\tau$, we have

$$\begin{aligned}&|(T_1u)(t) + (T_2v)(t)|\\ &\leq \sup_{t\in J}\Bigg[I^\nu|f(s,u(s),Ku(s),Hu(s))|(t) + \frac{|\delta_1|}{|\gamma_1|} + |w_1|\left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|}\right]\\ &\quad + |w_1|\big[|\gamma_2| I^\nu|f(s,v(s),Kv(s),Hv(s))|(1)\\ &\quad + |\eta_2| I^{\nu+\varsigma}|f(s,v(s),Kv(s),Hv(s))|(1)\big]\Bigg]\\ &\leq \|l\|\Psi(\tau)\left[\frac{1+|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}\right] + \frac{|\delta_1|}{|\gamma_1|} + |w_1|\left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|}\right]\\ &\leq \tau.\end{aligned}$$

Hence, $T_1u + T_2v \in B_\tau$. Now we verify that T_2 is a contraction map. For $u, v \in B_\tau$, we have

$$\begin{aligned}&|(T_2u)(t) - (T_2v)(t)|\\ &\leq \sup_{t\in J}\big[|w_1|\big[|\gamma_2| I^\nu|f(s,u(s),Ku(s),Hu(s)) - f(s,v(s),Kv(s),Hv(s))|(1)\\ &\quad + |\eta_2| I^{\nu+\varsigma}|f(s,u(s),Ku(s),Hu(s)) - f(s,v(s),Kv(s),Hv(s))|(1)\big]\big]\\ &\leq L_f\left[\frac{|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}\right] + (L_k + L_h)\left[\frac{|w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)}\right]\|u-v\|\\ &\leq \Omega\|u-v\|.\end{aligned}$$

Since $\Omega < 1$, then T_2 is contraction.

Moreover, continuousness of f, k and h indicates that T_1 is continuous. Likewise, T_1 is uniformly bounded on B_τ by means of

$$|(T_1u)(t)| \leq \sup_{t\in J}\left[I^\nu|f(s,u(s),Ku(s),Hu(s))|(t)\right] \leq \frac{\|l\|\Psi(\tau)}{\Gamma(\nu+1)}.$$

At present we demonstrate the compactness of the operator T_1. We make sure that for any $t_1, t_2 \in J$, $t_1 < t_2$, and $u \in B_\tau$,

$$\begin{aligned}&|(T_1u)(t_2) - (T_1u)(t_1)|\\ &= |I^\nu f(s,u(s),Ku(s),Hu(s))(t_2) - I^\nu f(s,u(s),Ku(s),Hu(s))(t_1)|\\ &\leq \frac{\|l\|\Psi(\tau)}{\Gamma(\nu)}\left|\int_0^{t_1}\left[(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}\right]ds + \int_{t_1}^{t_2}(t_2-s)^{\nu-1}ds\right|\\ &\leq \frac{\|l\|\Psi(\tau)}{\Gamma(\nu+1)}\left[2|t_2-t_1|^\nu + |t_2^\nu - t_1^\nu|\right]\\ &\quad \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.\end{aligned}$$

Consequently T_1 is equicontinuous. Henceforth, by the Arzela-Ascoli Theorem, T_1 is compact on B_τ. Hence, by Krasnoselkii fixed point theorem, there exists a $u \in C(J, X)$ such that $Tu = u$ which stands the result of the problem (1)-(2) has solution u on J. □

Theorem 3.2. *Assume that the hypotheses (A1) and (A2) are fulfilled, and let*

$$\Upsilon := L_f \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)} \right] + (L_k + L_h) \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)} \right] < 1.$$

Then, the problem (1)-(2) has a unique solution on J.

Proof: Consider $B_\beta = \{u \in C(J, X) : \|u\| \leq \beta\}$, let $M_f = \sup_{t \in J} |f(t, 0, 0, 0)|$, $M_k = \sup_{t \in J} |k(t, 0, 0)|$, $M_h = \sup_{t \in J} |h(t, 0, 0)|$, and $\Theta_1 := \frac{1+|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}$, $\Theta_2 := \frac{1+|w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)}$, $\Theta_3 := \frac{|\delta_1|}{|\gamma_1|} + |w_1| \left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|} \right]$.

We fix $\beta \geq \frac{M_f \Theta_1 + L_f(M_k + M_h)\Theta_2 + \Theta_3}{1 - L_f(\Theta_1 + (L_k + L_h)\Theta_2)}$.

Now, we show that $TB_\beta \subset B_\beta$. For any $u \in B_\beta$, we have

$$\begin{aligned}
&|(Tu)(t)| \\
&\leq \sup_{t \in J} \Bigg[I^\nu |f(s, u(s), Ku(s), Hu(s))|(t) + \frac{|\delta_1|}{|\gamma_1|} + |w_1| \left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|} \right] \\
&\quad + |w_1| \big[|\gamma_2| I^\nu |f(s, u(s), Ku(s), Hu(s))|(1) \\
&\quad + |\eta_2| I^{\nu+\varsigma} |f(s, u(s), Ku(s), Hu(s))|(1) \big] \Bigg] \\
&\leq I^\nu [L_f(\|u\| + \|Ku\| + \|Hu\|) + M_f](1) \\
&\quad + |w_1| [[|\gamma_2| I^\nu [L_f(\|u\| + \|Ku\| + \|Hu\|) + M_f](1)] \\
&\quad + |\eta_2| I^{\nu+\varsigma} [L_f(\|u\| + \|Ku\| + \|Hu\|) + M_f](1) \\
&\leq (L_f \beta + M_f) \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)} \right] + (L_f L_k \beta + L_f L_h \beta + L_f(M_k + M_h)) \\
&\quad \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)} \right] + \frac{|\delta_1|}{|\gamma_1|} + |w_1| \left[|\delta_2| + \frac{|\delta_1|}{|w_2||\gamma_1|} \right] \\
&\leq \beta L_f(\Theta_1 + (L_k + L_h)\Theta_2) + M_f \Theta_1 + L_f(M_k + M_h)\Theta_2 + \Theta_3 \\
&\leq \beta.
\end{aligned}$$

Then, $TB_\beta \subset B_\beta$.

Now we verify that T is a contraction map. For $u, v \in B_\beta$, we have

$$\begin{aligned}
&|(Tu)(t) - (Tv)(t)| \\
&\leq \sup_{t \in J} \big[I^\nu |f(s, u(s), Ku(s), Hu(s)) - f(s, v(s), Kv(s), Hv(s))|(t) \\
&\quad + |w_1| \big[|\gamma_2| I^\nu |f(s, u(s), Ku(s), Hu(s)) - f(s, v(s), Kv(s), Hv(s))|(1) \\
&\quad + |\eta_2| I^{\nu+\varsigma} |f(s, u(s), Ku(s), Hu(s)) - f(s, v(s), Kv(s), Hv(s))|(1) \big] \big] \\
&\leq L_f \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)} \right] + (L_k + L_h) \left[\frac{1 + |w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)} \right] \|u - v\| \\
&\leq \Upsilon \|u - v\|.
\end{aligned}$$

Therefore, we conclude that the operator T is contraction on J. By applying Banach fixed point theorem, there exists a unique solution of the problem (1)-(2). This proof is completed. □

4. **Application.** The solution of integro-differential equations has a major role in the fields of science and engineering. For example, digital filters are a very important part of Digital Signal Processing (DSP). In fact, their extraordinary performance is one of the key reasons that DSP has become so popular. Filters have two uses: signal separation and signal restoration. Signal separation is needed when a signal has been contaminated with interference, noise, or other signals. For example, imagine a device for measuring the electrical activity of a baby's heart (EKG) while still in the womb. The raw signal will likely be corrupted by the breathing and heartbeat of the mother. A filter might be used to separate these signals so that they can be individually analyzed. Signal restoration is used when a signal has been distorted in some way. For example, an audio recording made with poor equipment may be filtered to better represent the sound as it actually occurred. Another example is the deblurring of an image acquired with an improperly focused lens, or a shaky camera [31]. There is also another way to make digital filters, called recursion. When a filter is implemented by convolution, each sample in the output is calculated by weighing the samples in the input, and adding them together. Recursive filters are an extension of this, using previously calculated values from the output, besides points from the input. In [28, 29], methodology for an upgraded framework of an FIR digital filter from software level to the hardware level on the basis of the FIR filter structure is detailed. Also in [30], the coupling between a non-uniform sampling scheme and an asynchronous designs is discussed in order to implement a digital filter.

Let us here consider the Caputo fractional Volterra-Fredholm integro-differential equations with fractional integral boundary conditions

$$\begin{aligned} &{}^{c}D^{\frac{5}{4}}u(t) = \frac{1}{(t+6)}\frac{|u(t)|}{1+|u(t)|} + \frac{1}{6}\int_0^t \frac{e^{-5s}}{3}\frac{|u(t)|}{1+|u(t)|}ds + \frac{1}{6}\int_0^1 \frac{e^{-3s}}{4}\cos^2 s ds, \\ &\frac{1}{3}u(0) + \frac{1}{2}I^{\frac{1}{4}}u(t)|_{t=0} = \frac{3}{2},\ t \in J = [0,1], \\ &u(1) + I^{\frac{1}{4}}u(t)|_{t=1} = 2. \end{aligned} \tag{7}$$

Here $\nu = \frac{5}{4}$, $\varsigma = \frac{1}{4}$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = 1$, $\eta_1 = \frac{1}{2}$, $\eta_2 = 1$, $\delta_1 = \frac{3}{2}$, $\delta_2 = 2$, $L_f = \frac{1}{6}$, $L_k = \frac{1}{3}$, $L_h = \frac{1}{4}$, $w_1 = 0.5312$, $w_2 = 0.4755$. Then

$$\begin{aligned} \Omega &= L_f\left(\frac{|w_1||\gamma_2|}{\Gamma(\nu+1)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+1)}\right) + (L_k + L_h)\left(\frac{|w_1||\gamma_2|}{\Gamma(\nu+2)} + \frac{|w_1||\eta_2|}{\Gamma(\nu+\varsigma+2)}\right) \\ &= 0.35969 \\ &< 1. \end{aligned}$$

The conditions (A1) and (A2) are satisfied. Consequently, by Theorem 3.1, the problem (7) takes at least one solution on J. Also, it is easy to see that $\Upsilon < 1$, and consequently, by Theorem 3.2, the problem (7) has a unique solution on J.

5. **Conclusions.** In this paper, we establish some new conditions for the existence and uniqueness results of solutions for a class of nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with fractional integral boundary conditions. The desired results are proved by using generalized Gronwall inequality, aid of fixed point theorems due to Banach and Krasnoselskii in Banach spaces.

Our results extend and unify many existing results in the literature. This paper contributes to the growth of the fractional calculus, especially in the case fractional differential equations involving a general formulation of Caputo fractional derivative with respect to another function.

The problem considered in this paper can be generalized to a higher dimension involving a general formulation of Hilfer fractional derivative with respect to another function. Also, studying nonlinear fractional systems of Volterra-Fredholm integro-differential equations with nonlocal conditions is a direction which we are working on.

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Author Biography

Khawlah Hashim Hussain obtained her BSc degree in Mathematics from the College of Science, University of Basra, Iraq, 1988; MSc degree in Applied Mathematics from Dr. Babasaheb Ambedkar Marathwada University, India, 2013.

She received the gold medal in excellence from the university because she was the first in her department in studying the MSc. She is currently working as an assistant lecturer at Southern Technical University, Technical Institute in Basra. She has 7 research positions published in Scopus containers and 7 in international and Arab magazines.